RECURRENT SEQUENCES IN THE EQUATION $DQ^2 = R^2 + N$

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INTRODUCTION

The recreational exploration of numbers by the amateur can lead to discovery, or to a different way of looking at problems, because he often does not know the conventional approaches. Sometimes, as a form of amusement, I picked a quadratic expression at random, set it equal to a square and then tried to solve the resulting equation in positive integers. Whenever I was able to solve the problem I noticed that recurrency was evident. One of the most satisfying results came from the solution of $5x^2 \pm 6y + 1 = y^2$ where the recurrent relationships involved Fibonacci and Lucas sequences. However, the method reported [1] for this solution is not general. An improvement in the method resulted from exploring the Pell and Lagrange* equations. As experimental data accumulated I was able to make some conjectures and when I discussed the results with my friend, Professor Burton W. Jones, he urged me to try to prove them. For his encouragement, I am grateful.

The following are some of these conjectures:

a) For any recurrent equation such as $U_{n+1} = cU_n + U_{n-1}$ or $U_{n+1} = cU_n - U_{n-1}$, c constant and even, there exists at least one Pell equation such that the sequence of X's and of Y's follow the given recurrent law.

b) In a Pell equation if $DY_1^2 = X_1^2 + 1$ then the recurrent law for the sequence of X's or Y's is $U_{n+1} = cU_n + U_{n-1}$ and if $DY_1^2 = X_1^2 - 1$ then the governing law is $U_{n+1} = cU_n - U_{n-1}$.

c) In Lagrange equations having the same D as a Pell equation, there exists a recurrent law common to both. (Proof to be offered in another communication.)

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*The Lagrange equations $Dy^2 = x^2 \pm N$, $N > 1$ will be discussed in another communication.

Since a method of developing the sequence of one of the variables, in a Pell equation, independent of the other is so easy and since the proof justifying such treatment uses only elementary algebra, without the use of continued fractions or convergents, I thought that the method might be of interest. As will be demonstrated, problems, relating to the Pell equations which seem difficult, are solved in an almost trivial fashion by means of the theorems to be developed here. (Before continuing the reader is invited to try solving problems 1-5.)

PART I — THE PELL EQUATIONS $DY_n^2 = X_n^2 - (-1)^n$ and $DY_n^2 = X_n^2 - 1$

For a given $D > 1$ and not a square the complete* Pell equations are either of the forms

$$DY_n^2 = X_n^2 - (-1)^n \tag{1}$$

or

$$DY_n^2 = X_n^2 - 1, \quad n = 0, 1, 2, 3, \cdots \quad . \tag{2}$$

While both of these equations have the trivial solution $X_0 = 1$, $Y_0 = 0$, the key to the general solution is in finding X_1, Y_1, either by inspection or otherwise. How this may be done by convergents is explained by Burton W. Jones [2], C. D. Olds [3], R. Kortum and G. McNeil [4] and others in books on number theory.

The least positive, non-trivial solution (X_1, Y_1) is variously called the minimal or fundamental or generating solution. Once this solution is found, the general solution is given by

$$X_n + Y_n\sqrt{D} = (X_1 + Y_1\sqrt{D})^n \quad . \tag{3}$$

*The equation $DY_n^2 = X_n^2 - (-1)^n$, $n = 0,1,2,3,\cdots$, is complete. However, it is commonly treated as two equations, e.g., $DY_{2k}^2 = X_{2k}^2 - 1$ and $DY_{2k+1}^2 = X_{2k+1}^2 + 1$, $k = 0,1,2,3,\cdots$. Unless otherwise stated, we will assume that for the given D, the Pell equation is complete and we are dealing with all possible solutions.

The sum of the rational terms in the binomial expansion of $(X_1 + Y_1\sqrt{D})^n$ is X_n and the sum of the irrational terms is $Y_n\sqrt{D}$. That equation (3) gives all of the possible solutions was first shown by Robert D. Carmichael and later explained in his book Diophantine Analysis [5].

When the minimal solution (X_1, Y_1) is substituted in equations (1) and (2) we have respectively the minimal equations

$$DY_1^2 = X_1^2 + 1^* \tag{4}$$

and

$$DY_1^2 = X_1^2 - 1 . \tag{5}$$

In either case, and irrespective of the sign preceding 1, the general solution is given by the single equation (3).

PROOF OF THREE THEOREMS ON RECURRENCY IN THE PELL EQUATIONS

Theorem 1. In the integer solution of a Pell equation, the sequence of X's is recurrent as is the sequence of Y's according to the recurrent law, $U_{n+1} = cU_n \pm U_{n-1}$, $c = 2X_1$. The + sign is used if the minimal equation is $DY_1^2 = X_1^2 + 1$ and the - sign is used if $DY_1^2 = X_1^2 - 1$.

To prove this theorem we combine the minimal equations (4) and (5) so that

$$DY_1^2 = X_1^2 \pm 1 . \tag{6}$$

Then for reference we prepare, from the general solution (3), the following set of equations:

$$(X_1 + Y_1\sqrt{D})^{n-1} = X_{n-1} + Y_{n-1}\sqrt{D} \tag{7a}$$

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* If the minimal equation for a certain D is $DY_1^2 = X_1^2 + 1$ then there are solutions for $DY^2 = X^2 \pm 1$. If the minimal equation is $DY_1^2 = X_1^2 - 1$ then the only solutions are for $DY^2 = X^2 - 1$. Thus $DY^2 = X^2 + 1$ is not solvable for all D's nor does it have a trivial solution.

(7b) $$(X_1 + Y_1\sqrt{D})^n = X_n + Y_n\sqrt{D}$$

(7c) $$(X_1 + Y_1\sqrt{D})^{n+1} = X_{n+1} + Y_{n+1}\sqrt{D} .$$

When $X_1^2 + 2X_1Y_1\sqrt{D}$ is added to both sides of $DY_1^2 = X_1^2 \pm 1$ we obtain $X_1^2 + 2X_1Y_1\sqrt{D} + DY_1^2 = 2X_1^2 + 2X_1Y_1\sqrt{D} \pm 1$ or

(8) $$(X_1 + Y_1\sqrt{D})^2 = 2X_1(X_1 + Y_1\sqrt{D}) \pm 1 .$$

Multiplying both sides of this equation by $(X_1 + Y_1\sqrt{D})^{n-1}$ we derive

(9) $$(X_1 + Y_1\sqrt{D})^{n+1} = 2X_1(X_1 + Y_1\sqrt{D})^n \pm (X_1 + Y_1\sqrt{D})^{n-1} .$$

Now when the appropriate substitutions are made in this equation from set (7) we get

$$X_{n+1} + Y_{n+1}\sqrt{D} = 2X_1(X_n + Y_n\sqrt{D}) \pm (X_{n-1} + Y_{n-1}\sqrt{D})$$

and rearranging this equation we have

(10) $$X_{n+1} + Y_{n+1}\sqrt{D} = (2X_1X_n \pm X_{n-1}) + (2X_1Y_n \pm Y_{n-1})\sqrt{D} .$$

After equating the rational and then the irrational terms in (10) we finally derive

(11) $$X_{n+1} = 2X_1X_n \pm X_{n-1}$$

and

(12) $$Y_{n+1} = 2X_1Y_n \pm Y_{n-1} .$$

Thus the proof of Theorem 1 is complete and equations (11) and (12) are the equations of the Theorem.*

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*The equations of the Theorem seem similar to expressions found for the convergents of continued fractions. For instance, the numerator of the n^{th} convergent is $p_n = a_np_{n-1} + p_{n-2}$. This equation seems similar to $X_n = cX_{n-1} \pm X_{n-2}$ but in the equations of Theorem 1, + or - signs are used whereas in the convergent expression only the + sign appears.

As a consequence of Theorem 1 we have

Theorem 2. For every recurrent equation, $U_{n+1} = cU_n + U_{n-1}$ or $U_{n+1} = cU_n - U_{n-1}$ where c is even, there exists at least one Pell equation for which the sequence of either variable is governed by the given recurrent law.

To prove this theorem we note from Theorem 1 that $c = 2X_1$ whence $X_1 = c/2$. When this value of X_1 is substituted in the minimal equations $DY_1^2 = X_1^2 \pm 1$ we have

$$DY_1^2 = \left(\frac{c}{2}\right)^2 \pm 1 .$$

Except for a trivial case,

$$\left(\frac{c}{2}\right)^2 \pm 1 \neq \square ,$$

therefore we can let

$$\left(\frac{c}{2}\right)^2 \pm 1 = D$$

whence $Y_1 = 1$ and thus we have proved Theorem 2. If

$$\left(\frac{c}{2}\right)^2 \pm 1$$

contains a square factor >1 there may be other solutions as demonstrated by problem 1.

In equation (1), $DY_n^2 = X_n^2 - (-1)^n$, we notice that when $n = 2k$ then

$$DY_{2k}^2 = X_{2k}^2 - 1 \tag{13}$$

and when $n = 2k + 1$ then

$$DY_{2k+1}^2 = X_{2k+1}^2 + 1, \quad k = 0,1,2,3,\cdots . \tag{14}$$

In order to study the sequence of every other term in a Pell equation we have

Theorem 3. The sequence of every other X or Y in a Pell equation is recurrent. If the recurrent law for the Pell equation is $U_{n+1} = cU_n + U_{n-1}$ then the sequence of every other X or Y is

$$U_{n+3} = (c^2 + 2)U_{n+1} - U_{n-1}$$

and if the recurrent law is $U_{n+1} = cU_n - U_{n-1}$ then the sequence of every other X or Y is governed by

$$U_{n+3} = (c^2 - 2)\, U_n - U_{n-1} \; .$$

We prove the two parts of Theorem 3 together using the ambiguous $\pm$ sign.

$$U_{n+1} = cU_n \pm U_{n-1}$$

then

$$U_{n+2} = cU_{n+1} \pm U_n$$

and

$$U_{n+3} = cU_{n+2} \pm U_{n+1} \; .$$

But

$$U_{n+2} = cU_{n+1} \pm U_n$$

therefore

$$U_{n+3} = c(cU_{n+1} \pm U_n) \pm U_{n+1}$$

or

$$U_{n+3} = c^2U_{n+1} \pm cU_n \pm U_{n+1}$$

and

$$U_{n+3} = (c^2 \pm 1)U_{n+1} \pm cU_n .$$

But

$$\pm cU_n = \pm U_{n+1} - U_{n-1}$$

therefore

$$U_{n+3} = (c^2 \pm 1)U_{n+1} \pm U_{n+1} - U_{n-1}$$

or

$$U_{n+3} = (c^2 \pm 2)U_{n+1} - U_{n-1} . \tag{15}$$

With the derivation of equation (15) we have proved Theorem 3. For convenience we let $c^2 \pm 2 = c_2$ and then the equations of Theorem 3 become

$$U'_{k+1} = c_2U'_k - U'_{k-1}, \quad U'_0 = U_0, U'_1 = U_2 \tag{16}$$

or

$$U'_1 = U_1, \quad U'_2 = U_3 .$$

The method of proof for Theorem 3 demonstrates that the properties of the sequences of X's or of Y's in the Pell equations are simply the properties to be expected from considerations of the recurrent equations $U_{n+1} = cU_n \pm U_{n-1}$.

EXAMPLES

Example 1. When $D = 2$ the minimal solution is $2Y_1^2 = X_1^2 + 1$, $Y_1 = 1$, $X_1 = 1$. From Theorem 1 we know that we must use the recurrent equation with the + sign and that the constant $c = 2X_1 = 2$. Thus, the sequence of X's develops from $X_{n+1} = 2X_n + X_{n-1}$, $X_0 = 1$, $X_1 = 1$.

$$X_2 = 2X_1 + X_0 = 2 \cdot 1 + 1 = 3$$

$$X_3 = 2X_2 + X_1 = 2 \cdot 3 + 2 = 7$$

$$X_4 = 2X_3 + X_2 = 2 \cdot 7 + 3 = 17 ,$$

etc. Thus

$$X = 1, 1, 3, 7, 17, 41, 99, \cdots .$$

Similarly for Y we have $Y_{n+1} = 2Y_n + Y_{n-1}$, $Y_0 = 0$, $Y_1 = 1$.

$$Y_2 = 2Y_1 + Y_0 = 2 \cdot 1 + 0 = 2$$

$$Y_3 = 2Y_2 + Y_1 = 2 \cdot 2 + 1 = 5$$

$$Y_4 = 2Y_3 + Y_2 = 2 \cdot 5 + 2 = 12 ,$$

etc., and

$$Y = 0, 1, 2, 5, 12, 29, 70, \cdots .$$

Example 2. For $D = 3$ the minimal solution is $X_1 = 2$, $Y_1 = 1$ and the minimal equation is $3Y_1^2 = X_1^2 - 1$, whence the recurrent law for $D = 3$ is

$$U_{n+1} = cU_n - U_{n-1}, \quad c = 2X_1 = 2 \cdot 2 = 4 .$$

Then

$$X_2 = 4X_1 - X_0 = 4 \cdot 2 - 1 = 7$$

$$X_3 = 4X_2 - X_1 = 4 \cdot 7 - 2 = 26$$

$$X_4 = 4X_3 - X_2 = 4 \cdot 26 - 7 = 99,$$

etc., and for the Y's

$$Y_2 = 4Y_1 - Y_0 = 4 \cdot 1 - 0 = 4$$

$$Y_3 = 4Y_2 - Y_1 = 4 \cdot 4 - 1 = 15$$

$$Y_4 = 4Y_3 - Y_2 = 5 \cdot 15 - 4 = 56,$$

etc., and

$$X = 1, 2, 7, 26, 99, \cdots$$

and

$$Y = 0, 1, 4, 15, 56, \cdots .$$

PROBLEMS

The following problems illustrate the use of the theorems developed here. Without knowledge of these theorems, I believe the problems might be difficult to solve.

Problem 1. The numbers 2024 and 32257 are consecutive values of one of the variables in a Pell equation. What are the corresponding values of the other variable? (There are two solutions.)

Problem 2. For $8Y^2 = X^2 - 1$ we have

$$X = 1, 3, 17, 99, \cdots$$

$$Y = 0, 1, 6, 35, \cdots$$

and

$$U_{n+1} = 6U_n - U_{n-1} .$$

Find another Pell equation(s) for which this recurrent law holds.

Problem 3. Prove that

$$X_n = \frac{X_1 Y_n \pm Y_{n-1}}{Y_1}$$

and

$$Y_n = \frac{X_1 X_n \pm X_{n-1}}{Y_1 D} \quad .$$

Use the $+$ sign if $DY_1^2 = X_1^2 + 1$ and the $-$ sign if $DY_1^2 = X_1^2 - 1$. Notice that in this problem the recurrent sequence of one variable is developed in terms of constants and the other variable.

Problem 4. In a Pell equation where $D = a^2 - 1$, $a > 1$, prove that $X_n \pm X_{n-1} \equiv 0 \bmod (X_1 \pm 1)$ using corresponding signs on each side of the congruence.

Problem 5. In Pell equations if $DY_1^2 = X_1^2 + 1$, prove:

$$\sum_{j=1}^{n} X_j = \frac{X_{n+1} + X_n - X_1 - 1}{c}$$

and

$$\sum_{j=1}^{n} Y_j = \frac{Y_{n+1} + Y_n - Y_1}{c} \quad , \quad c = 2X_1 \, .$$

Note that if $c = 1$ and the X's are Lucas numbers and the Y's are Fibonacci numbers then we have the summation equations for the Lucas and Fibonacci sequences. If $DY_1^2 = X_1^2 - 1$, show that the comparable summations are

$$\sum_{j=1}^{n} X_j = \frac{X_{n+1} - X_n - X_1 - 1}{c - 2}$$

$$\sum_{j=1}^{n} Y_j = \frac{Y_{n+1} - Y_n - Y_1}{c - 2} \quad , \; c = 2X_1, \; X_1 \neq 1.$$

Problem 6. In each of the following equations find recurrent sequences of rational x's such that y is integral. The ambiguous sign is used to avoid negative roots.

a) $$3x^2 \pm 4x + 1 = y^2$$

b) $$3x^2 \pm 5x + 2 = y^2$$

c) $$2x^2 \pm 6x + 5 = y^2$$

d) $$6x^2 \pm 5x + 1 = y^2$$

EPILOGUE

In this part of the paper, some terms and notations are introduced which were found to be convenient.

a) In the Pell equation, $DY_n^2 = X_n^2 - (-1)^n$, $n = 0,1,2,3,\cdots$, we notice that as n increases, 1 is alternately subtracted and added to the X^2 term. Thus the equation is referred to as an alternating equation. For the equation $DY_n^2 = X_n^2 - 1$, $n = 0,1,2,3,\cdots$, 1 is always subtracted from X^2 and is referred to as non-alternating. The term alternating Pell equation implies the minimal equation $DY_1^2 = X_1^2 + 1$ and the recurrent law $U_{n+1} = cU_n + U_{n-1}$, whereas the term non-alternating Pell equation implies $DY_1^2 = X_1^2 - 1$ and the recurrent law $U_{n+1} = cU_n - U_{n-1}$. In this connection it is interesting to note that in recurrent equations where the n's are negative, the neighboring terms in the sequence developed from $U_{n-1} = U_{n+1} - cU_n$ have opposite signs and thus the signs in the sequence alternate. If $U_{n-1} = cU_n - U_{n+1}$ and $n < 1$, the neighboring terms of the sequence have the same signs and the sequence is non-alternating.

The use of non-positive n's in the equations of Theorem 1 leads to the conjugate solutions of the Pell equations.

b) In the recurrent equation $U_{n+1} = cU_n + U_{n-1}, c \geq 1$ is associated with the + sign preceding the U_{n-1} term and in the equation $U_{n+1} = cU_n - U_{n-1}$ $c \geq 1$ is associated with the - sign preceding the U_{n-1} term. A convenient notation for these recurrent equations is c^+ and c^-. For example 6^+ implies $U_{n+1} = 6U_n + U_{n-1}$ and 4^- implies $U_{n+1} = 4U_n - U_{n-1}$.

Since c^+ or c^- indicates the manner in which the recurrent sequence is developed they are called the indicator, I, of the sequence.

If α and β are the first two terms of a sequence, then the development of the sequence is completely determined by the indicator and the first two terms as $I(\alpha, \beta)$. For example, if $I = 3^+$, $\alpha = 2$, $\beta = 3$ then 3^+, (2,3) defines the sequence and implies $U_{n+1} = 3U_n + U_{n-1}$, $U_0 = 2$, $U_1 = 3$.

Throughout my notes I have used this notation because of its convenience and brevity.

Since each of the Pell equations, (1) and (2) have a unique recurrent law for a given D then it follows that they have a unique indicator but a given indicator does not necessarily determine a Pell equation uniquely.

c) If a sequence is determined by I, (α, β) and α, β have a common factor, f, then all terms of the sequence contain this factor. Let $\alpha = f\alpha_1$ and $\beta = f\beta_1$ then

$$I,\ (\alpha, \beta) = I,\ (f\alpha_1, f\beta_1) = I,\ f(\alpha_1, \beta_1)\ .$$

The n^{th} term of the sequence can be developed from I, (α_1, β_1) to the n^{th} term which is then multiplied by f and by this procedure we can use smaller numbers.

d) Applying these concepts to the Pell equations we have for the general recurrent solution

$$X = I,\ (1, X_1)$$

$$Y = I,\ Y_1(0, 1)$$

where $I = 2X_1^+$ if $DY_1^2 = X_1^2 + 1$ and $I = 2X_1^-$ if $DY_1^2 = X_1^2 - 1$.

We see that in general for any Pell equation $Y_n \equiv 0 \bmod Y_1$.

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