THREE DIOPHANTINE EQUATIONS – PART II*

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6. THE PELL EQUATIONS

Equation (3) is the special case $d = 2$ of the equation

$$s^2 - dt^2 = 1 , \tag{18}$$

where d is positive and is not a perfect square. Equation (18) is known as the Pell equation. Another way of solving Eq. (3) is provided by the following theorem concerning the Pell equations found in most text books on the theory of numbers. (For a proof of the theorem, see [2].)

Theorem: If (s_1, t_1) is the minimal positive solution of Eq. (18), then every positive solution is given by

$$s_n + t_n\sqrt{d} = (s_1 + t_1\sqrt{d})^n, \quad n > 0. \tag{19}$$

(A solution (s,t) is called positive if $s > 0$, $t > 0$.) The minimal positive solution of Eq. (3) is $(3,2)$. Then, according to this theorem, all positive solutions are given by

$$s_n + t_n\sqrt{2} = (3 + 2\sqrt{2})^n, \quad n = 1,2,3,\cdots . \tag{20}$$

Equations (15) and (16) are easily derived from Eq. (20) as follows:

$$\begin{aligned} s_n + t_n\sqrt{2} = (3 + 2\sqrt{2})^n &= (3 + 2\sqrt{2})^{n-1}(3 + 2\sqrt{2}) = (s_{n-1} + t_{n-1}\sqrt{2})(3 + 2\sqrt{2}) \\ &= (3s_{n-1} + 4t_{n-1}) + (2s_{n-1} + 3t_{n-1})\sqrt{2} . \end{aligned}$$

Therefore $s_n = 3s_{n-1} + 4t_{n-1}$, and $t_n = 2s_{n-1} + 3t_{n-1}$.

7. RECURRENCE RELATIONS

If (x_n, z_n) is one of the sequence of non-negative solutions of Eq. (1) with $n \geq 2$, we can derive from Eqs. (7) and (8) a formula that expresses x_n

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as a linear function of x_{n-1} and x_{n-2}, and a formula that expresses z_n as a linear function of z_{n-1} and z_{n-2}. If we replace n by n - 1 in Eqs. (7) and (8), we get

$$x_{n-1} = 3x_{n-2} + 2z_{n-2} + 1 , \tag{21}$$

$$z_{n-1} = 4x_{n-2} + 3z_{n-2} + 2 . \tag{22}$$

From (21) and (22) we get

$$2z_{n-2} = x_{n-1} - 3x_{n-2} - 1 , \tag{23}$$

$$4x_{n-2} = z_{n-1} - 3z_{n-2} - 2 . \tag{24}$$

Then, from Eqs. (7), (22) and (23),

$$x_n = 3x_{n-1} + 2z_{n-1} + 1 .$$

$$x_n = 3x_{n-1} + 2(4x_{n-2} + 3z_{n-2} + 2) + 1 .$$

$$x_n = 3x_{n-1} + 8x_{n-2} + 6z_{n-2} + 5.$$

$$x_n = 3x_{n-1} + 8x_{n-2} + 3(x_{n-1} - 3x_{n-2} - 1) + 5 .$$

$$x_n = 6x_{n-1} - x_{n-2} + 2 . \tag{25}$$

Similarly, from Eqs. (8), (21) and (24),

$$z_n = 4x_{n-1} + 3z_{n-1} + 2 .$$

$$z_n = 4(3x_{n-2} + 2z_{n-2} + 1) + 3z_{n-1} + 2 .$$

$$z_n = 12x_{n-2} + 8z_{n-2} + 3z_{n-1} + 6.$$

$$z_n = 3(z_{n-1} - 3z_{n-2} - 2) + 8z_{n-2} + 3z_{n-1} + 6 .$$

$$z_n = 6z_{n-1} - z_{n-2} \;. \tag{26}$$

EXERCISES

5. Let (u_n, v_n) be the n^{th} solution in positive integers of Eq. (2), $n \geq 2$. Use Eqs. (12) and (13) to derive the recurrence relations

$$u_n = 6u_{n-1} - u_{n-2} + 2 \,, \tag{27}$$

$$v_n = 6v_{n-1} - v_{n-2} \;. \tag{28}$$

6. Let (s_n, t_n) be the n^{th} solution in positive integers of Eq. (3), $n \geq 2$. Use Eqs. (15) and (16) to derive the recurrence relations

$$s_n = 6s_{n-1} - s_{n-2} \,, \tag{29}$$

$$t_n = 6t_{n-1} - t_{n-2} \;. \tag{30}$$

8. CLOSED FORMULAS

If a sequence $y_0, y_1, y_2, \cdots, y_7, \cdots$ is defined by specifying the values of the first few terms and determining the values of the rest inductively by means of a linear recurrence relation, then there is a standard technique for finding a formula that expresses y_n in terms of n. For example, it can be shown that if the recurrence relation is the equation

$$y_{n+2} - 6y_{n+1} + y_n = 0 \;, \tag{31}$$

then

$$y_n = c_1 r_1^n + c_2 r_2^n \;, \tag{32}$$

where r_1 and r_2 are the roots of the characteristic equation

$$E^2 - 6E + 1 = 0 \;, \tag{33}$$

and the constants c_1 and c_2 are determined by the values of y_1 and y_2. (See [3] for a proof of this assertion.) The roots of (33) are $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$. So in this case

$$y_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n \tag{34}$$

The recurrence relations for z_n, v_n, s_n and t_n all have the form (31) with characteristic equation (33). Hence the closed formulas for z_n, v_n, s_n and t_n all have the form of Eq. (34), and differ only in the values of the constants c_1 and c_2. To determine the constants in the formula

$$z_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n ,$$

we make use of the fact that $z_0 = 1$ and $z_1 = 5$. Then

$$1 = c_1(3 + 2\sqrt{2})^0 + c_2(3 - 2\sqrt{2})^0 ,$$
$$5 = c_1(3 + 2\sqrt{2})^1 + c_2(3 - 2\sqrt{2})^1 .$$

Therefore $c_1 + c_2 = 1$ and $c_1 - c_2 = \frac{1}{2}\sqrt{2}$. Consequently, $c_1 = \frac{1}{4}(2 + \sqrt{2}), c_2 = \frac{1}{4}(2 - \sqrt{2})$, and

$$a = \frac{1}{4}\left[(2 + \sqrt{2})\ (3 + 2\sqrt{2})^n + (2 - \sqrt{2})\ (3 - 2\sqrt{2})^n\right] . \tag{35}$$

EXERCISES

7. Determine the values of c_1 and c_2 in each of these closed formulas:

$$s_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n ; \tag{36}$$

$$t_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n ; \tag{37}$$

$$v_n = c_1(3 + 2\sqrt{2})^n + c_2(3 - 2\sqrt{2})^n . \tag{38}$$

It can be shown that if the recurrence relation defining a sequence $\{y_n\}$ is the non-homogeneous equation

(39) $$y_{n+2} - 6y_{n+1} + y_n = 2\,,$$

then

(40) $$y_n = c_1 r_1^n + c_2 r_2^n - \tfrac{1}{2}\,,$$

where r_1 and r_2 are the roots of (33), and c_1 and c_2 are determined by the values of y_0 and y_1. The recurrence relations for x_n and u_n have the form of (40). Hence the closed formulas for x_n and u_n, after evaluation of the constants c_1 and c_2, are

(41) $$x_n = \tfrac{1}{4}\left[(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n - 2\right],$$

(42) $$u_n = \tfrac{1}{4}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2\right].$$

9. HOW EQUATIONS (1), (2), AND (3) ARE RELATED TO EACH OTHER

The sequence of non-negative integers $\{z_n\}$, $\{v_n\}$ and $\{t_n\}$ which arise in the solution of Eqs. (1), (2) and (3), respectively, all satisfy the same recurrence relation (31). This shows that the solutions of Eqs. (1), (2) and (3) are intimately related to each other. We shall now derive the equations that relate them to each other from the closed formulas for x_n, z_n, s_n, t_n, u_n and v_n. The formulas for z_n, x_n and u_n are Eqs. (35), (41) and (42), respectively. The formulas for s_n, t_n and v_n obtained in Exercise 7 are

(36') $$s_n = \tfrac{1}{2}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right],$$

(37') $$t_n = \frac{\sqrt{2}}{4}\left[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\right],$$

(38') $$v_n = \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\right].$$

By solving Eqs. (42) and (38') for $(3+2\sqrt{2})^n$ and $(3-2\sqrt{2})^n$, respectively, we find

(43) $$(3+2\sqrt{2})^n = 2u_n + 2\sqrt{2}v_n + 1\,,$$

$$(3 - 2\sqrt{2})^n = 2u_n - 2\sqrt{2}v_n + 1 . \tag{44}$$

Making these substitutions for $(3 + 2\sqrt{2})^n$ and $(3 - 2\sqrt{2})^n$ in Eqs. (41) and (35), we obtain the following equations relating the solutions (x_n, z_n) of Eq. (1) to the solutions (u_n, v_n) of Eq. (2):

$$x_n = u_n + 2v_n , \tag{45}$$

$$z_n = 2u_n + 2v_n + 1 . \tag{46}$$

If we solve Eqs. (45) and (46) for u_n and v_n, we get these equations:

$$u_n = z_n - x_n - 1 , \tag{47}$$

$$v_n = \tfrac{1}{2}(2x_n - z_n + 1). \tag{48}$$

EXERCISES

8. Use Eqs. (36'), (37'), (43) and (44) to derive these equations relating the solutions (s_n, t_n) of Eq. (3) to the solutions (u_n, v_n) of Eq. (2):

$$s_n = 2u_n + 1, \tag{49}$$

$$t_n = 2v_n , \tag{50}$$

$$u_n = \tfrac{1}{2}(s_n - 1), \tag{51}$$

$$v_n = \tfrac{1}{2} t_n . \tag{52}$$

9. Use the results of Exercise 8 and the paragraph that precedes it to derive these equations relating the solutions (s_n, t_n) of Eq. (3) to the solutions (x_n, z_n) of Eq. (1):

$$s_n = 2z_n - 2x_n - 1 , \tag{53}$$

(54) $$t_n = 2x_n - z_n + 1 ,$$

(55) $$x_n = \tfrac{1}{2}(s_n + 2t_n - 1) ,$$

(56) $$z_n = s_n + t_n .$$

10. Without using the closed formulas (41), (35), (42) and (38') for x_n, z_n, u_n and v_n, respectively, verify that if (x_n, z_n) is a solution of Eq. (1), in non-negative integers, and u_n and v_n are defined by Eqs. (47) and (48), respectively, then u_n and v_n are non-negative integers, and (u_n, v_n) is a solution of Eq. (2). Also verify, conversely, that if (u_n, v_n) is a solution of Eq. (2) in non-negative integers, and x_n and z_n are defined by Eqs. (45) and (46), respectively, then x_n and z_n are non-negative integers, and (x_n, z_n) is a solution of Eq. (1). (See [1], pp. 20-21.)

11. Without using the closed formulas for x_n, z_n, s_n, and t_n, verify that if (x_n, z_n) is a solution of Eq. (1) in non-negative integers, and s_n and t_n are defined by Eqs. (53) and (54), respectively, then s_n and t_n are non-negative integers, and (s_n, t_n) is a solution of Eq. (3). Also verify, conversely, that if (s_n, t_n) is a solution of Eq. (3) in non-negative integers, and x_n and z_n are defined by Eqs. (55) and (56), respectively, then x_n and z_n are non-negative integers, and (x_n, z_n) is a solution of Eq. (1).

If we drop the subscripts in Eqs. (45) through (56), each pair of equations, (45) and (46), (47) and (48), (49) and (50), (51) and (52), (53) and (54), and (55) and (56), defines a linear transformation that converts one of the Eqs. (1), (2) or (3) into one of the other two.

10. FORMULAS FOR GENERATING SIMULTANEOUSLY SUCCESSIVE SOLUTIONS OF EQUATIONS (1), (2), AND (3)

From Eqs. (45) and (50) we get

(57) $$x_n = u_n + t_n .$$

From Eqs. (45), (46), (12) and (13), we get

(58) $$u_{n+1} = x_n + z_n ,$$

(59) $$v_{n+1} = v_n + z_n .$$

Then, starting with $u_0 = 0$, $v_0 = 0$, and applying recursively the sequence of Eqs. (49), (50), (57), (56), (58) and (59), we can generate in succession s_0, t_0, x_0, z_0, u_1, v_1, s_1, t_1, x_1, z_1, u_2, v_2, s_2, t_2, x_2, z_2, and so on. The first ten non-negative solutions to Eqs. (2), (3) and (1), respectively, obtained in this way, are tabulated below.

n	(u_n, v_n)	(s_n, t_n)	(x_n, z_n)
0	(0, 0)	(1, 0)	(0, 1)
1	(1, 1)	(3, 2)	(3, 5)
2	(8, 6)	(17, 12)	(20, 29)
3	(49, 35)	(99, 70)	(119, 169)
4	(288, 204)	(577, 408)	(696, 985)
5	(1681, 1189)	(3363, 2378)	(4059, 5741)
6	(9800, 6930)	(19601, 13860)	(23660, 33461)
7	(57121, 40391)	(114243, 80782)	(137903, 195025)
8	(332928, 235416)	(665857, 470832)	(803760, 1136689)
9	(1940449, 1372105)	(3880899, 2744210)	(4684659, 6625109)

EXERCISE

12. Find (u_{10}, v_{10}), (s_{10}, t_{10}) and (x_{10}, z_{10}).

11. SOLUTIONS WITH EVEN OR ODD INDEX

It is of interest to examine separately the even-numbered solutions (x_{2i}, z_{2i}), (u_{2i}, v_{2i}) and (s_{2i}, t_{2i}) of Eqs. (1), (2) and (3), respectively, and the odd-numbered solutions (x_{2i+1}, z_{2i+1}), (u_{2i+1}, v_{2i+1}) and (s_{2i+1}, t_{2i+1}). These solutions can be expressed in terms of the solutions (x_i, z_i), (u_i, v_i) and (s_i, t_i). For example, we know from Eq. (20) that

$$s_{2i} + t_{2i}\sqrt{2} = (3 + 2\sqrt{2})^{2i} = \left[(3 + 2\sqrt{2})^i\right]^2 = (s_i + t_i\sqrt{2})^2 .$$

That is,

$$s_{2i} + t_{2i}\sqrt{2} = (s_i^2 + 2t_i^2) + 2s_it_i\sqrt{2} \quad .$$

Therefore

$$s_{2i} = s_i^2 + 2t_i^2 = 2s_i^2 - 1 = 1 + 4t_i^2 \quad , \tag{60}$$

and

$$t_{2i} = 2s_it_i \; . \tag{61}$$

By using Eqs. (48), (50), (54), (55), (56), (60) and (61), we can show that

$$x_{2i} = 2t_i(t_i + s_i) = 2t_iz_i = 4z_i v_i = 2z_i (2x_i - z_i + 1) \quad , \tag{62}$$

and

$$z_{2i} = t_i^2 + z_i^2 = (2x_i - z_i + 1)^2 + z_i^2 \quad . \tag{63}$$

By using Eqs. (49), (50), (51), (52), (60) and (61), we can show that

$$u_{2i} = 2t_i^2 = 8v_i^2 \quad , \tag{64}$$

and

$$v_{2i} = s_it_i = 2v_is_i = 2v_i(2u_i + 1) \; . \tag{65}$$

By invoking Eqs. (58) and (59), we can show that

$$u_{2i+1} = (v_i + v_{i+1})^2 = (u_{i+1} - u_i)^2 \; , \tag{66}$$

and

$$v_{2i+1} = z_i (v_i + v_{i+1}) \quad . \tag{67}$$

The following equations are also easily derived:

$$s_{2i+1} = 2z_i^2 + (v_i + v_{i+1})^2 = 2z_i^2 + (z_i + t_i)^2 , \tag{68}$$

$$t_{2i+1} = 2z_i + (v_i + v_{i+1}) = 2z_i(z_i + t_i) , \tag{69}$$

$$x_{2i+1} = (z_i + 2x_i + 1)^2 - z_i^2 , \tag{70}$$

$$z_{2i+1} = (z_i + 2x_i + 1)^2 + z_i^2 , \tag{71}$$

12. SUM AND DIFFERENCE RULES

The following rules are either already included among the equations we have derived so far, or are easily derived from them.

$$s_i + t_i = z_i , \tag{56}$$

$$s_i - t_i = z_{i-1} , \tag{72}$$

$$u_i + v_i = u_{i+1} - v_{i+1} = \tfrac{1}{2}(z_i - 1) , \tag{73}$$

$$u_i - v_i = u_{i-1} + v_{i-1} = \tfrac{1}{2}(z_{i-1} - 1) , \tag{74}$$

$$z_i + x_i = u_{i+1} , \tag{58}$$

$$z_i - x_i = u_i + 1 , \tag{47}$$

$$s_{2i} + t_{2i} = t_i^2 + z_i^2 , \tag{75}$$

$$s_{2i} - t_{2i} = t_i^2 + z_{i-1}^2 , \tag{76}$$

$$u_{2i} + v_{2i} = 2v_i (v_i + v_{i+1}) , \tag{77}$$

$$u_{2i} - v_{2i} = 2v_i (t_i - z_i) , \tag{78}$$

$$z_{2i} + x_{2i} = (z_i + t_i)^2 , \tag{79}$$

(80) $$z_{2i} - x_{2i} = (z_i - t_i)^2 = s_i^2 ,$$

(81) $$s_{2i+1} + t_{2i+1} = 3(t_i + z_i)^2 + 2t_iz_i + 2 ,$$

(82) $$s_{2i+1} - t_{2i+1} = z_{2i} = t_i^2 + z_i^2 ,$$

(83) $$u_{2i+1} + v_{2i+1} = 2v_{i+1}(v_i + v_{i+1}) ,$$

(84) $$u_{2i+1} - v_{2i+1} = 2v_i(v_i + v_{i+1}) ,$$

(85) $$z_{2i+1} + x_{2i+1} = 2(z_i + 2x_i + 1)^2 ,$$

(86) $$z_{2i+1} - x_{2i+1} = 2z_i^2 ,$$

(87) $$z_{2i+1} - (x_{2i+1} + 1) = (u_{i+1} - u_i)^2 .$$

13. HISTORICAL NOTE

Dickson's History of the Theory of Numbers, Vol. II, contains scattered notes about Eqs. (1) and (2), and denotes a sixty-page chapter to the Pell equation, of which Eq. (3) is a special case. (See [4].) Some of the more interesting facts cited by Dickson are reproduced below.

Concerning Eq. (1).

Fermat showed that if (x, z) is a solution of Eq. (1), then so is $(3x + 2z + 1,\ 4x + 3z + 2)$. (See Eqs. (7) and (8).)

C. Hutton (1842) found that if p_r/q_r is the r^{th} convergent of the continued fraction for the square root of 2, then p_rp_{r+1} and $2q_rq_{r+1}$ are consecutive integers, and the sum of their squares is equal to q_{2r+1}^2.

P. Bachmann (1892) proved that the only integral solutions of $x^2 + y^2 = z^2$, $z > 0$, x and y consecutive, are given by

$$x + y + z\sqrt{2} = (1 + \sqrt{2})(3 + 2\sqrt{2})^k, \quad k = 0, 1, 2, \cdots .$$

R. W. D. Christie (1897) expressed the solutions of Eq. (1) in terms of continuants. The continuant $C(a_1, a_2, \cdots, a_r)$ is the r^{th} order determinant

$$\begin{vmatrix} a_1 & 1 & & & \bigcirc \\ -1 & a_2 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & 1 \\ \bigcirc & & & & -1 & a_r \end{vmatrix}$$

in which the term u_{ij} of the principal diagonal is equal to a_i, $(i = 1,\cdots,r)$, each term $u_{i+1,i}$, $(i = 1,\cdots,r-1)$, immediately under the principal diagonal is equal to -1, and each term $u_{i-1,i}$, $(i = 2,\cdots,r)$, immediately above the principal diagonal, is equal to 1, and every other term is equal to 0. Let Q_r stand for the r^{th} order continuant $C(2,\cdots,2)$ in which all the diagonal elements are 2, and define $2_0 = 1$. Christie observed that the positive integral solutions of Eq. (1) are given by

$$x_r = Q_0 + Q_1 + \cdots + Q_{2r-1}, \qquad z_r = Q_{2r}, \qquad r = 1,2,\cdots .$$

This result was proved by T. Muir (1899-1901).

Concerning Eq. (2).

Euler (1732) found solutions to Eq. (2) in the following way: He started with the identity of Plutarch (about 100 AD),

$$\frac{8u(u+1)}{2} + 1 = (2u+1)^2 .$$

By Eq. (2),

$$\frac{u(u+1)}{2} = v^2 .$$

Then $8v^2 + 1 = (2u+1)^2$. Let $s = 2u+1$, and $t = 2v$. Then s and t satisfy Eq. (3), which Euler solved by using his general method for solving the Pell equation.

Euler proved, too, that u and v satisfy Eq. (2) only when

$$u = \frac{\alpha+\beta-2}{4}, \quad v = \frac{\alpha-\beta}{4\sqrt{2}},$$

where

$$\alpha = (3 + 2\sqrt{2})^n, \qquad \beta = (3 - \sqrt{2})^n, \quad n = 0, 1, 2, \cdots.$$

From this result, he derived the recursion formulas given by Eqs. (27) and (28).

E. Lionnet (1881) stated that 0, 1 and 6 are the only triangular numbers whose squares are triangular numbers. This assertion was proved by Moret-Blanc (1882). In the notation of Section 2, Lionnet's result is that $S(T(n)) = T(m)$ only if $n = 0, 1$ or 3. Since $S(T(0)) = 0 = T(0)$, $S(T(1)) = 1 = T(1)$, and $S(T(3)) = 36 = T(8)$, it follows from Lionnet's result that the equation $S(T(n)) = T(S(n))$ has only the trivial solutions $(0,0)$ and $(1,1)$.

Concerning Eq. (3).

Among those who worked on solving equations of the form $S^2 - dt^2 = 1$ were Diophantus (about 250 AD), and Brahmegupta (born 598 AD).

The general problem of solving all equations of this form was proposed by Fermat in February 1657. Hence an equation of this form should be called a Fermat equation. It came to be known as the Pell equation as a result of an error by Euler, who incorrectly attributed to Pell the method of solution given in Wallis' *Opera*.

Lagrange gave the first proof that every Pell equation has integral solutions with $t \neq 0$ if d is not a square.

Others who contributed to the voluminous literature on this equation are Legendre, Dirichlet and Gauss.

14. REFERENCES

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