SPECIAL PROPERTIES OF THE SEQUENCE $W_n(a,b;p,q)$

A. F. HORADAM
University of New England, Armidale, N.S.W., Australia

1. INTRODUCTION

Elsewhere in this journal [1] the sequence $\{w_n(a,b;p,q)\}$ has been introduced and its basic properties exhibited. Here we investigate three special properties of the sequence, namely, the "Pythagorean" property (2), the geometrical-paradox property (3), and the complex case (4). These are generalizations of results earlier published for the sequence $\{h_n(r,s)\} \equiv \{w_n(r, r+s;1, -1)\}$ which may be consulted in [3], [4], [5] respectively.

But observe that with reference to $\{h_n(r,s)\}$ the notation in this paper varies slightly from that used in [2], [3], [4] and [5]. Our properties in this paper form the second of the proposed series of articles envisaged in [1]. Notation and content of [1] are assumed, when required.

Some interesting special cases of $\{w_n(a,b;p,q)\}$ occur which we record for later reference (2):

(1.1)	integers	$a=1$,	$b=2$,	$p=2$,	$q=1$
(1.2)	odd numbers	1	3	2	1
(1.3)	arithmetic progression (common difference)	a	a + d	2	1
(1.4)	geometric progression (common ratio q)	a	q	q + 1	q
(1.5)	Fermat's sequence u_n (3, 2)	1	3	3	2
(1.6)	Fermat's sequence v_n (3, 2)	2	3	3	2
(1.7)	Pell's sequence u_n (2, -1)	1	2	2	-1
(1.8)	Pell's sequence v_n (2, -1)	2	2	2	-1

Sequence (1.1) has already been noted in [1], while sequences (1.5) — (1.8) were mentioned in [6]. However, sequences (1.2) — (1.4) have not been previously recorded in this series of papers.

2. THE "PYTHAGOREAN" PROPERTY

Any w_n at all may be substituted in the known formula for Pythagorean triples: $(u^2 - v^2)^2 + (2uv)^2 = (u^2+v^2)^2$. Writing $u = w_{n+2}$, $v = w_{n+1}$, we obtain

$$(2.1)\qquad (w_{n+2}^2 - w_{n+1}^2)^2 + (2w_{n+2}w_{n+1})^2 = (w_{n+2}^2 + w_{n+1}^2)^2 .$$

Next, using the recurrence relation $w_{n+2} = pw_{n+1} - qw_n$ [1], we may express (2.1) in a variety of ways, some of them quite complicated. Generally, we have

$$(2.2)\qquad \left[(pw_{n+1} - qw_n)^2 - w_{n+1}^2\right]^2 + \left[2w_{n+1}(pw_{n+1} - qw_n)\right]^2 = \left[(pw_{n+1} - qw_n)^2 + w_{n+1}^2\right]^2 .$$

Assigned values of n, p, q (and a, b) may be inserted in this formula to yield various Pythagorean triples. For example, $n = 0$ with $a = 1$ $(=w_0)$, $b = 2$ $(=w_1)$, $p = 5$, $q = -1$ (a fairly random choice) produces the Pythagorean set 117, 4 4, 125.

More particularly, for the special sequences described in paragraph 1, we deduce, with $n = 0$ for simplicity, the following Pythagorean triples:

(1.1)	5	12	13
(1.2)	16	30	34
(1.3)	$2ad + 3d^2$	$2a^2 + 6ad + 4d^2$	$2a^2 + 6ad + 5d^2$
(1.4)	$a^2q^2(q^2 - 1)$	$2a^2q^3$	$a^2q^2(q^2 + 1)$
(1.5)	40	4 2	58
(1.6)	16	30	34
(1.7)	21	20	29
(1.8)	32	24	40

Triples for (1.2) and (1.6) just happen to coincide with $n = 0$ since $w_1 = 3$, $w_2 = 5$ for both sequences. No other values of n reproduce this coincidence for these two sequences.

Our concern here is not so much with the general Pythagorean formula (2.2) as with the cases arising when $p = 1$, $q = -1$ since these restrictions lead to $\{h_n(r,s)\}$, $\{f_n\}$ and $\{a_n\}$. In this respect, observe that, in (2.1), $w_{n+2}^2 - w_{n+1}^2 = (w_{n+2} + w_{n+1})(w_{n+2} - w_{n+1})$.

Substitution of $p = 1$, $q = -1$ in (2.2) yields

$$(2.2)'\qquad (w_n w_{n+3})^2 + (2w_{n+2}w_{n+1})^2 = (w_{n+2}^2 + w_{n+1}^2)^2$$

with a similar result for the case $p = -1$, $q = -1$. No other values of p, q produce the term $(w_n w_{n+3})^2$.

Thus we have the sequences whose n^{th} terms are

$$w_n(a, b; 1, -1) \equiv af_{n-2} + bf_{n-1} \equiv h_n(a, b - a) \tag{2.3}$$

and

$$w_n(a, b; -1, -1) \equiv (-1)^n(af_{n-2} - bf_{n-1}) \equiv g_n(a, b - a) \quad \text{(say)} \tag{2.4}$$

where the g- and h-notation are introduced for convenience.

Putting $a = r$, $b = r + s$ in (2.2)', we derive the Pythagorean generalization for $\{h_n(r, s)\}$ determined in [2] and [3], namely,

$$(h_n h_{n+3})^2 + (2h_{n+1} h_{n+2})^2 = (2h_{n+1} h_{n+2} + h_n^2)^2 \tag{2.5}$$

in which the right-hand side is merely an alternative expression for the sum of the squares in the right-hand side of (2.2)'.

Examples of (2.2)' are, with (say) $n = 0$, $a = 5$, $b = 2$, from (2.3), $45^2 + 28^2 = 55^2$, and, from (2.4), $5^2 + 12^2 = 13^2$. Illustrations of the Pythagorean formula (2.5) have been given in [3]. More especially, for the Fibonacci and Lucas sequences $\{f_n\}$, $\{a_n\}$ the Pythagorean triples are, for $n = 0$, 3, 4, 5 and 8, 6, 10, respectively, while for $n = 1$ (say) they are 5, 12, 13 and 7, 24, 25, respectively.

As the properties of $\{h_n(r, s)\}$ have been developed in [2], it is thought worthwhile to examine some similar properties of the companion g-sequence relating to Pythagorean number triples. To this purpose we now direct our attention.

Just as it was shown in [3], with reference to (2.3), that all Pythagorean number triples are Fibonacci number triples, so may we likewise demonstrate the same for (2.4). Instead of putting

$$a = x - y, \quad b = y \tag{2.6}$$

in (2.3), we substitute

$$a = x + y, \quad b = y \tag{2.7}$$

in (2.4). In some of the concrete cases of (2.3) and (2.4), some part of the number triples will be negative; for instance, in the second case quoted above, the actual triple is $-5, -12, 13$.

Many different, but related, sequences give the same triple, but for different values of n. First, take the case $p = 1$, $q = -1$. Write $x = w_{n+2}$, $y = w_{n+1}$ as in [3]. Then by (2.3)

$$\begin{cases} x = af_n + bf_{n+1} \\ y = af_{n-1} + bf_n \end{cases} \tag{2.8}$$

Solve (2.6). Hence

$$\begin{cases} a = (-1)^n(xf_n - yf_{n+1}) \\ b = (-1)^{n+1}(xf_{n-1} - yf_n) \end{cases} , \tag{2.9}$$

where we have used the fundamental Fibonacci formula [2]

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^{n+1} .$$

Giving n all possible integral values, we obtain an infinite sequence of sequences of which a selected few are

$$\begin{cases} h_n(y, x - y), \quad h_n(x - y, -x + 2y) , \\ h_n(-x + 2y, 2x - 3y), \quad h_n(2x - 3y, -3x + 5y) , \end{cases} \tag{2.10}$$

corresponding to $n = -1, 0, 1, 2$, respectively.

The second of the sequences (2.10) already occurs in (2.6). A given Pythagorean triple may be derived from any of these sequences if the correct value of n is associated with it (since we are operating on the same 4 numbers $x - y$, y, x, $x + y$ in each sequence). Examples are (i), if $x = 3$, $y = 2$, the triple 5, 12, 13 is obtained from the sequences $h_n(2,1)$, $h_n(1,1)$, $h_n(1,0)$ and $h_n(0,1)$ when $n = -1, 0, 1, 2$ respectively: (ii) if $x = 4$, $y = 3$, the triple

7, 24, 25 is obtained from the sequences $h_n(3,1)$, $h_n(1,2)$, $h_n(2,-1)$, $h_n(-1,3)$ when $n = -1, 0, 1, 2$ respectively.

Correspondingly, in the case $p = -1$, $q = -1$, write $x = w_{n+2}$, $y = -w_{n+1}$ so that by (2.4)

$$\text{(2.11)} \qquad \begin{cases} x = (-1)^n (af_n - bf_{n+1}) \\ y = (-1)^n (-af_{n-1} + bf_n) \end{cases}$$

whence, solving with the aid of the fundamental Fibonacci formula quoted above, we have

$$\text{(2.12)} \qquad \begin{cases} a = xf_n + yf_{n+1} \\ b = xf_{n+1} + yf_n \end{cases}$$

leading to an infinite sequence of sequences of which a selected few are, for $n = -1, 0, 1, 2$,

$$\text{(2.13)} \qquad \begin{cases} g_n(y,\ x - y), & g_n(x + y,\ -x)\ , \\ g_n(x + 2y,\ -y), & g_n(2x + 3y,\ -x - y)\ , \end{cases}$$

respectively. With $x = 3$, $y = 2$, for instance, the triple -5, -12, 13 arises from $g_n(2,1)$, $g_n(5,-3)$, $g_n(7,-2)$, $g_n(12,-5)$ when $n = -1, 0, 1, 2$ respectively. Observe that the second sequence in (2.13) already occurs in (2.7). Had we written $x = -w_{n+2}$, $y = w_{n+1}$ above, then of course we would have obtained the negatives of the values of a, b given in (2.12).

Remarks similar to the other remarks for $h_n(a,b,-a)$ in [3] may be paralleled for $g_n(a,b-a)$.

3. THE GEOMETRICAL PARADOX

A well-known geometrical problem requires a given square to be sub-divided in a specified manner and re-arranged so as to form a rectangle of certain dimensions. In the process of re-arrangement, it appears as though a small area of one square unit has been gained or lost. This illusion is due to inaccurate re-assembling of the sub-divided parts. Precise re-arrangement reveals the existance of a very small parallelogram of unit area included in the rectangle. Mathematically, the secret of the paradox lies with the Fibonacci formula quoted in Section 2.

Previously in [4] I generalized this paradox to the sequence $\{h_n(r,s)\}$. Our basic generalized formula now is 1, with n replaced by $n+1$, $w_n w_{n+2} - w_{n+1}^2 = eq^n$. As in [4], the construction guarantees two cases, n even and n odd. See Figs. 1, 2, 3. Clearly, the spirit of the standard construction is preserved only if $q < 0$. Write $q_1 = -q$ $(q_1 > 0)$. From the figures, we see that the exigencies of the constructions impose the restriction $p = q_1 = 1$, so that the defining recurrence relation [1] is now $w_{n+2} = w_{n+1} + w_n$, the fundamental formula [1] is $w_n w_{n+2} - w_{n+1}^2 = (-1)^n e$, and the area of the parallelogram [4] is e. Consequently, the only sequences for which the standard construction is applicable are $w_n(a,b;1,-1) = h_n(a, b-a)$ by (2.3).

Briefly repeating the basic results proved in [4], we have, after calculations:

(3.1) $$\lambda_n = \sqrt{w_{n+1}^2 + w_{n-1}^2}\,, \quad \mu_n = \sqrt{w_n^2 + w_{n-2}^2} \quad ;$$

(3.2) $$\lim_{n\to\infty}\left(\frac{\lambda_n}{\mu_n}\right) = \alpha_1$$

(3.3) $$\begin{cases} \tan\theta_n = \tan\left(\dfrac{\pi}{2} - \gamma_n - \delta_n\right), \left[\tan\gamma_n = \dfrac{w_{n-1}}{w_{n+1}}\,,\ \tan\delta_n = \dfrac{w_n}{w_{n-2}}\right] \\ \qquad = \dfrac{e_1}{e_1 + 3w_n w_{n-1}} = t_n \end{cases}$$

(3.4) $$\lim_{n\to\infty}\left(\frac{t_n}{t_{n+1}}\right) = \alpha_1^2 = 1 + \alpha_1\ ,$$

where in (3.3) we have set

(3.5) $$e_1 = ab + a^2 - b^2\ .$$

Initially, in Fig. 3 we have

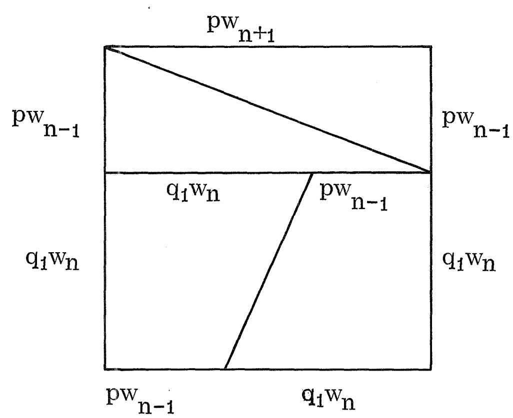

Fig. 1

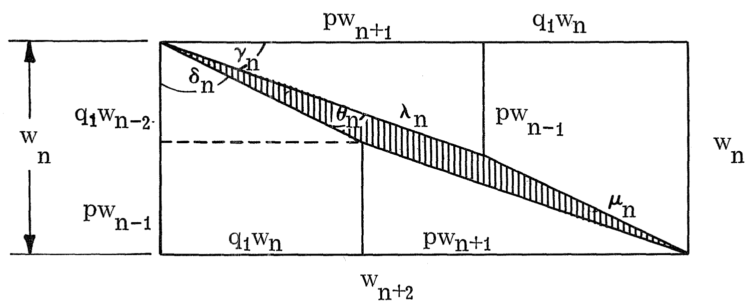

Fig. 2 (n even)

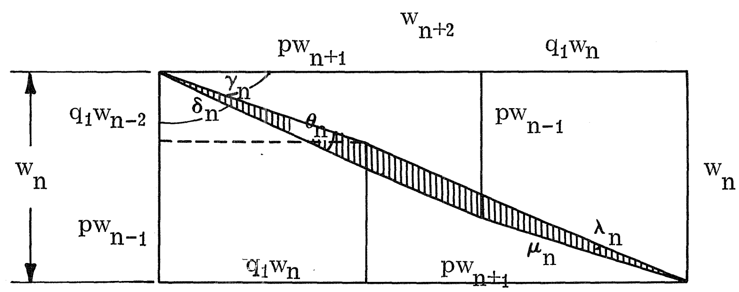

Fig. 3 (n odd)

($p = q_1 = 1$ in Figs. 1-3)

$$\tan\theta_n = \tan(\gamma_n + \delta_n - \pi/2) \quad . \tag{3.6}$$

Eventually, after calculation this leads back to (3.3).

Worth noting is the fact that (3.3) is a considerable simplification of the form for $\tan\theta_n$ given in [4].

Concrete instances of the paradox, with details of specific values for θ_n, λ_n, μ_n, are to be found in [4].

4. THE COMPLEX CASE

Label each of the fundamental constants a, b, p, q, e associated with a sequence different from $\{w_n\}$ by a subscript symbolic of that sequence; that is, for the sequence $\{h_n\}$, for instance, express these constants as a_h, b_h, p_h, q_h, e_h.

Define

$$\begin{cases} d_n = w_n + iw_{n+1} \qquad (i^2 = -1) \\ \quad = bu_{n-1} - qau_{n-2} + i(bu_n - qau_{n-1}) \end{cases} \tag{4.1}$$

using a known expression [1] for w_n. Hence

$$\begin{cases} d_0 = a_d = a + ib \\ d_1 = b_d = b + i(pb - qa) \end{cases} \tag{4.2}$$

After substituting $u_n = pu_{n-1} - qu_{n-2}$, we deduce from (4.1), (4.2) that

$$d_n = pd_{n-1} - qd_{n-2} \tag{4.3}$$

and

$$\begin{cases} d_n = \{b + i(pb - qa)\}\, u_{n-1} - q(a + ib)u_{n-2} \\ \quad = (w_1 + iw_2)u_{n-1} - q(w_0 + iw_1)u_{n-2} \\ \quad = d_1 u_{n-1} - qd_0 u_{n-2} \\ \quad = b_d u_{n-1} - qa_d u_{n-2} \end{cases} \tag{4.4}$$

from (4.1), which is a form we could anticipate. Of course, we could have substituted $w_n = au_n + (b - pa)u_{n-1}$ and obtained an equivalent result. Thus

$$\{d_n\} \equiv \{w_n\ (a + ib,\ b + i(pb - qa)\ ;\ p, q)\}. \tag{4.5}$$

Moreover,

$$\begin{cases} e_d = p\, a_d\, b_d - q\, a_d^2 - b_d^2 \\ \quad = (1 - q + ip)e \end{cases} \tag{4.6}$$

after calculation.

Fundamental properties of d_n are deducible in an analogous way to those of w_n [1]. Only the three most interesting general properties are stated for the record:

$$d_{n-1}\, d_{n+1} - d_n^2 = e_d\, q^{n-1} \tag{4.7}$$

$$(d_n\, d_{n+3})^2 + (-2pq\, d_{n+1}\, d_{n+2})^2 = (-2pq\, d_{n+1}\, d_{n+2} + d_n^2)^2 + 2c_1c_2d_n^2 \tag{4.8}$$

$$\frac{d_{n+r} + q^r d_{n-r}}{d_n} = v_r \tag{4.9}$$

(that is, the right-hand side of (4.9) is independent of a, b, n). In the Pythagorean result (4.8), we have written

$$\begin{cases} c_1 = pd_{n+2} - qd_{n+1} - d_n \\ c_2 = c_1 + 2d_n \end{cases} \tag{4.10}$$

All these results are easy to verify using as appropriate (4.3) or (4.1) with $w_n = A\alpha^n + B\beta^n$ [1] being a convenient substitution on (4.7) and (4.9). Be it noted that with this approach we may need to use $w_{n-1}w_{n+2} - w_n w_{n+1} = epq^{n-1}$, which is a special case of [1] (4.18) for which $r = t = 1$.

Particular cases of the above theoretical results lead back to those in [5]. For example $p = -q = 1$ implies $w_n(a,b;1,-1) = h_n(a, b - a)$ by (2.3)

Under these conditions, replace d_n by k_n. Then (4.6), for instance, gives [5].

$$e_k = e_c e_h \, , \tag{4.11}$$

where c is the complex Fibonacci sequence for which $a = b = 1$ and [5], (3.5),

$$e_c = 2 + i, \quad e_h = ab + a^2 - b^2 \, . \tag{4.12}$$

Extending [5] we may define a generalized quaternion as:

$$q_n = w_n + iw_{n+1} + jw_{n+2} + kw_{n+3} \tag{4.13}$$

with conjugate quaternion

$$\overline{q}_n = w_n - iw_{n+1} - jw_{n+2} - kw_{n+3} \, , \tag{4.14}$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji$, $jk = -kj$, $ki = -ik$.

From (4.13), (4.14),

$$w_n = \frac{q_n + \overline{q}_n}{2} \, . \tag{4.15}$$

Finally, for the conjugate $\overline{d}_n$ it follows that

$$\begin{cases} a_{\overline{d}} = \overline{a_d} \\ b_{\overline{d}} = \overline{b_d} \\ e_{\overline{d}} = \overline{e_d} \end{cases} \tag{4.16}$$

(Note: Helpful advice from the referee has been incorporated into the early part of Section 2 and is hereby acknowledged.)

REFERENCES

1. A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," Oct. 1965, pp. 161-176.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math. Monthly, Vol. 68, No. 5, 1961, pp. 455-459.
3. A. F. Horadam, "Fibonacci Number Triples," Amer. Math Monthly, Vol. 68, No. 8, 1961, pp. 751-753.
4. A. F. Horadam, "A Geometric Paradox," Math Magazine, pp. 1-11.
5. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," Amer. Math. Monthly, Vol. 70, No. 3, 1963, pp. 289-291.
6. A. F. Horadam, "Generating Functions for Powers of a Certain Generalized Sequence of Numbers," (to be published in the Duke Mathematical Journal.

* * * * *

* * * * *

* * * * *