A LUCAS TRIANGLE

MARK FEINBERG
Student, University of Pennsylvania, Philadelphia, Pennsylvania

It is well known that the Fibonacci Sequence can be derived by summing diagonals of Pascal's Triangle. How about the Lucas Sequence? Is there an arithmetical triangle whose diagonals sum to give the Lucas Sequence?

One such triangle is generated by the coefficients of the expansion $(a + b)^{n-1}(a + 2b)$:

		1	3	4	7	11	18	29	47	76	123 ...
row n											
1	1	2									
2	1	3	2								
3	1	4	5	2							
4	1	5	9	7	2						
5	1	6	14	16	9	2					
6	1	7	20	30	25	11	2				
7	1	8	27	50	55	36	13	2			
8	1	9	35	77	105	91	49	15	2		
9	1	10	44	112	182	196	140	64	17	2	
10	1	11	54	156	294	378	336	204	81	19	2
column r	1	2	3	4	5	6	7	8	9	10	11

The sum of the numbers on row n is $3 \times 2^{n-1}$. For row $n = 5$:

$$1 + 6 + 14 + 16 + 9 + 2 = 48 = 3 \times 2^4 .$$

The n^{th} row has $n + 1$ terms. Each number of this Lucas Triangle is the sum of the number above it and the number to the left of that one. Except for the first column, the sum of the first R numbers in column r equals the R^{th} number of column $r + 1$:

$$2 + 5 + 9 + 14 = 30$$

For $r > 1$, any number of the triangle can be expressed as

$$\frac{n!}{(r-1)!\,(n-r+1)!} + \frac{(n-1)!}{(r-2)!\,(n-r+1)!}$$

For example, the 5th number of row 7 is 55:

$$\frac{7!}{4!\,3!} + \frac{6!}{3!\,3!} = 35 + 20 = 55 \;.$$

Actually, one can find row n of this Lucas Triangle by adding row n of Pascal's Triangle to row $n - 1$ of Pascal's Triangle:

1	1 1	1	2						
1	2 1	1 1	1	3	2				
1	3 1	3 2	1 1	1	4	5	2		
1	4 1	6 3	4 3	1 1	1	5	9	7	2

Figure 1

This fact is not extremely surprising. By summing the n^{th} diagonal of the Lucas Triangle, one is actually simultaneously adding the $(n+1)^{st}$ and the $(n-1)^{st}$ diagonals of Pascal's Triangle. The $(n+1)^{st}$ diagonal of Pascal's Triangle adds up to F_{n+1}; the $(n-1)^{st}$ diagonal sums to F_{n-1}, and one Fibonacci-Lucas identity is:

$$F_{n+1} + F_{n-1} = L_n \;.$$

The vertical columns of the Lucas Triangle are of interest. Notice that the second column, $r = 2$, is equivalent to enumeration. From the general expression, any number in this column is

$$\frac{n!}{1!\,(n-1)!} + \frac{(n-1)!}{0!\,(n-1)!}$$

The R^{th} number of this column is on row $n = R$. Thus the R^{th} number can be expressed as

$$\frac{R!}{1!\,(R-1)!} + \frac{(R-1)!}{0!\,(R-1)!} = R + 1$$

Any number in the third column, $r = 3$, is given by

$$\frac{n!}{2!\,(n-2)!} + \frac{(n-1)!}{1!\,(n-2)!}$$

The R^{th} number of this column is on row $n = R + 1$. The R^{th} number is then given by

$$\frac{(R+1)!}{2!\,(R-1)!} + \frac{R!}{1!\,(R-1)!}$$

$$= \frac{(R+1)! + 2(R!)}{2!\,(R-1)!} = \frac{R!\,(R+1+2)}{2!\,(R-1)!} = \frac{R(R+3)}{2} = \frac{R^2 + 3R}{2}$$

The 6th number of the column is 27:

$$\frac{36 + 18}{2} = 27 .$$

One can generalize to say that the R^{th} number of the column which begins on row $n = N$ is given by

$$\frac{(R+N-1)!}{N!\,(R-1)!} + \frac{(R+N-2)!}{(N-1)!\,(R-1)!}$$

The 4th number of the column which starts on row $n = 6$ is 140:

$$\frac{(4+6-1)!}{6!\,(4-1)!} + \frac{(4+6-2)!}{5!\,(4-1)!} = \frac{9!}{6!\,3!} + \frac{8!}{5!\,3!} = 84 + 56 = 140 .$$

Pascal's Triangle is symmetric. Flipping the Triangle around doesn't change it. Not so with the Lucas Triangle. Rotating the Lucas Triangle 180^0 gives:

2 3 5 8 13 21 34 55 89 ...

row n										
1	2	1								
2	2	3	1							
3	2	5	4	1						
4	2	7	9	5	1					
5	2	9	16	14	6	1				
6	2	11	25	30	20	7	1			
7	2	13	36	55	50	27	8	1		
8	2	15	49	91	105	77	35	9	1	
9	2	17	64	140	196	182	112	44	10	1
column r	1	2	3	4	5	6	7	8	9	10

Summing diagonals of this arrangement gives the Fibonacci Sequence. This can be explained by referring to Figure 1. The n^{th} diagonal of the rotated Lucas Triangle is the sum of the n^{th} and $n + 1^{th}$ Pascal diagonals. The n^{th} Pascal diagonal of the rotated Lucas Triangle sums to

$$F_n + F_{n+1} = F_{n+2} .$$

The second column of this rotated triangle is composed of the odd numbers. Any number in this column can be expressed as

$$\frac{n!}{(n-1)!\,1!} + \frac{(n-1)!}{(n-2)!\,1!} .$$

Since the R^{th} number of the column is on row $n = R$, the above expression is equivalent to

$$\frac{R!}{(R-1)!\,1!} + \frac{(R-1)!}{(R-2)!\,1!} = R + R - 1 = 2R - 1 .$$

Perhaps the most interesting of all the Lucas Triangle's vertical columns is the third column of the rotated arrangement. Here, the R^{th} number is R^2. The expression for any number of this column is

$$\frac{n!}{(n-2)!\,2!} + \frac{(n-1)!}{(n-3)!\,2!} \;.$$

Since the R^{th} number of the column is on row $n = R + 1$, the R^{th} number is given by

$$\frac{(R+1)!}{(R-1)!\,2!} + \frac{R!}{(R-2)!\,2!}$$

$$= \frac{(R+1)! + (R-1)(R!)}{(R-1)!\,2!} = \frac{R!\,(R+1+R-1)}{(R-1)!\,2!}$$

$$= \frac{R(2R)}{2} = R^2 \;.$$

In general, the R^{th} number of the column which begins on row $n = N$ of the rotated triangle is

$$\frac{(R+N-1)!}{(R-1)!\,N!} + \frac{(R+N-2)!}{(R-2)!\,N!}$$

For example, the 5th number of the column beginning on row $n = 4$ is 105:

$$\frac{(5+4-1)!}{(5-1)!\,4!} + \frac{(5+4-2)!}{(5-2)!\,4!} = \frac{8!}{4!\,4!} + \frac{7!}{3!\,4!} = 70 + 35 = 105 \;.$$

In conclusion, the coefficients of the expansion $(a + b)^{n-1}(a + 2b)$ produce an interesting Lucas Triangle. This triangle is not, however, unique. Quite conceivably, utilization of various other Fibonacci-Lucas identities will lead to different and, perhaps, even more interesting Lucas Triangles.

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Mark's younger brother, Andrew, is also a Science Fair Champion, and we hope soon we'll have the privilege of publishing his first mathematics paper.

The following are Mark's <u>Fibonacci Quarterly</u> papers:

1.	Fibonacci-Tribonacci	Oct. 1963
2.	New Slants	Oct. 1964
3.	Lucas Triangle	Dec. 1967

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