A LIMITED ARITHMETIC ON SIMPLE CONTINUED FRACTIONS

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1. Introduction. As is well known, a number of remarkable and interesting relationships exist between the golden ratio of the Greeks and the numbers in the Fibonacci sequence. Binet's formula is one example of such a relationship and another is the familiar equation

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

where $\alpha = (1 + \sqrt{5})/2$ and F_n denotes the n^{th} Fibonacci number. In this paper, we derive other interesting relationships involving the Fibonacci numbers and the simple continued fraction expansions of multiples of the golden ratio. We also extend these results to obtain more general theorems about a certain class of quadratic surds.

Specifically we establish necessary and sufficient conditions for integral multiples of the golden ratio to be of period one, obtain sufficient conditions for these multiples to be of period two and establish some partial converses for those of period two. We then generalize by replacing the golden ratio by arbitrary simple continued fractions of period one and then by arbitrary simple continued fractions of period two. Some results are exactly analogous while others are only partial. Some curious side results are also established.

2. Results involving the golden ratio. We begin by considering the following table of simple continued fraction expansions of positive integral multiples of α. Of course, these expansions are periodic and the repeating part of the expansion is indicated by dots in the style of Hardy and Wright [1].

Careful scrutiny of the table reveals a variety of patterns. Some of the patterns are only apparent but others, as indicated by the theorems following the table are generally true.

Note that small Latin letters will always be used to denote positive integers. Also, L_n will always denote the n^{th} Lucas number.

n	Expansion of $n\alpha$
1	$[1, \dot{1}]$
2	$[3, \dot{4}]$
3	$[4, \dot{1}, \dot{5}]$
4	$[6, \dot{2}, \dot{8}]$
5	$[8, \dot{11}]$
6	$[9, \dot{1}, 2, 2, 2, 1, \dot{12}]$
7	$[11, \dot{3}, \dot{15}]$
8	$[12, \dot{1}, \dot{16}]$
9	$[14, 1, 1, 3, 1, 1, \dot{19}]$
10	$[16, \dot{5}, 1, 1, 5, \dot{22}]$
11	$[17, \dot{1}, 3, 1, \dot{23}]$
12	$[19, \dot{2}, 2, 2, \dot{26}]$
13	$[21, \dot{29}]$
14	$[22, \dot{1}, 1, 1, 7, 6, 7, 1, 1, 1, \dot{30}]$
15	$[24, \dot{3}, 1, 2, 3, 2, 1, 3, \dot{33}]$
16	$[25, \dot{1}, 7, 1, \dot{34}]$
17	$[27, \dot{1}, 1, \dot{37}]$
18	$[29, \dot{8}, \dot{40}]$
19	$[30, \dot{1}, 2, 1, 7, 1, 2, 1, \dot{41}]$
20	$[32, \dot{2}, 1, 3, 2, 1, 1, 10, 1, 1, 2, 3, 1, 2, \dot{44}]$
21	$[33, \dot{1}, \dot{45}]$
22	$[35, \dot{1}, 1, 2, 11, 1, 8, 1, 11, 2, 1, 1, \dot{48}]$
29	$[46, \dot{1}, 11, 1, \dot{63}]$
34	$[55, \dot{76}]$
36	$[58, \dot{4}, \dot{80}]$
47	$[76, \dot{21}, \dot{105}]$
55	$[88, \dot{1}, \dot{121}]$
89	$[144, \dot{199}]$

Theorem 1. Let n be a positive integer. Then $n\alpha = [a, \dot{b}]$ if and only if $n = F_{2m-1}$, $a = F_{2m}$, and $b = L_{2m-1}$ for some $m \geq 1$.

Theorem 2. Let n be a positive integer. Then $n\alpha = [a, \dot{1}, \dot{c}]$ if and only if $n = F_{2m}$, $a = F_{2m+1}$, and $c = L_{2m} - 2$ for some $m \geq 1$.

Theorem 3. If we admit the expansions $\alpha = [2, \dot{1}, -1, 1, \dot{3}]$ and $4\alpha = [6, \dot{1}, 0, 1, \dot{8}]$, then for every integer $r \geq 1$, we have

a) $$L_{2r}\alpha = [L_{2r+1}, \dot{F}_{2r}, 5\dot{F}_{2r}],$$

and

b) $$L_{2r-1}\alpha = [L_{2r} - 1, \dot{1}, F_{2r-1} - 2, 1, 5\dot{F}_{2r-1} - 2] .$$

Unlike Theorems 1 and 2, the converse of Theorem 3 is not true as is easily seen by considering the expansions of 4α, 16α, and 36α. The following theorem, however, provides a partial converse of the first assertion of Theorem 3.

Theorem 4. Let n be a positive integer. Then $n\alpha = [a, \dot{b}, \dot{c}]$ if and only if $nb = F_{2m}$, $ab = F_{2m+1} - 1$, and $bc = L_{2m} - 2$ for some $m \geq 1$.

Before proving these results we derive two lemmas which incidentally provide unusual characterizations of the Fibonacci and Lucas numbers.

Lemma 1. The Pell equation $x^2 - 5y^2 = -4$ is solvable in positive integers if and only if $x = L_{2n-1}$ and $y = F_{2n-1}$ for $n \geq 1$.

Proof. Since $x = y = 1$ is the least positive solution of the given equation, it is well known [2] that every positive solution is given by

$$\begin{aligned} x + y\sqrt{5} &= 2\left(\frac{1 + \sqrt{5}}{2}\right)^{2n-1} \\ &= \frac{1}{2^{2n-2}} \cdot \sum_{k=0}^{n-1} \binom{2n-1}{2k} 5^k + \frac{\sqrt{5}}{2^{2n-2}} \cdot \sum_{j=1}^{n} \binom{2n-1}{2j-1} 5^{j-1} \end{aligned}$$

for $n \geq 1$. On the other hand, by Binet's formula,

$$\begin{aligned} F_{2n-1} &= \frac{1}{\sqrt{5}} \left\{ \left(\frac{1 + \sqrt{5}}{2}\right)^{2n-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2n-1} \right\} \\ &= \frac{1}{2^{2n-2}} \sum_{j=1}^{n} \binom{2n-1}{2j-1} 5^{j-1} , \end{aligned}$$

and

$$L_{2n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1}$$

$$= \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{2k} 5^k \quad .$$

Combining these three results we have that all positive integral solutions to $x^2 - 5y^2 = -4$ are given by $x = L_{2n-1}$ and $y = F_{2n-1}$ for $n \geq 1$ as claimed.

Lemma 2. The Pell equation $x^2 - 5y^2 = 4$ is solvable in positive integers if and only if $x = L_{2n}$ and $y = F_{2n}$ for $n \geq 1$.

Proof. As in the proof of Lemma 1, it is easy to show that

$$\frac{1}{2^{k-1}} (1 + \sqrt{5})^k = L_k + \sqrt{5}\, F_k , \quad k \geq 0$$

where we take $L_0 = 2$. Therefore, since $x = 3$, $y = 1$ is the least positive integral solution of the given equation, every solution in positive integers is given by

$$x + y\sqrt{5} = 2\left(\frac{3+\sqrt{5}}{2}\right)^n$$

$$= 2\left\{\frac{2 + (1 + \sqrt{5})}{2}\right\}^n$$

$$= \sum_{k=0}^{n} \frac{1}{2^{k-1}} (1 + \sqrt{5})^k \binom{n}{k}$$

$$= \sum_{k=0}^{n} L_k \binom{n}{k} + \sqrt{5} \sum_{k=0}^{n} F_k \binom{n}{k}$$

$$= L_{2n} + \sqrt{5}\, F_{2n} \quad .$$

where the last equality is a result of Lucas [3, p. 191]. Thus, all solutions in positive integers are given by $x = L_{2n}$, $y = F_{2n}$ for $n \geq 1$ as claimed.

We note in passing that Lucas [3, p. 199] observes that $L_n^2 - 5F_n^2 = \pm 4$ and that Wasteels [4] proved that if $5x^2 \pm 4$ is the square of an integer then x is a Fibonacci number.

Proof of Theorem 1. By direct calculation we obtain

$$[a,\dot{b}] = \frac{2a - b + \sqrt{b^2 + 4}}{2} \quad .$$

Therefore, $n\alpha = [a,\dot{b}]$ if and only if

$$n = 2a - b \quad \text{and} \quad n\sqrt{5} = \sqrt{b^2 + 4} \quad . \tag{1}$$

The second of these equations is equivalent to

$$b^2 - 5n^2 = -4$$

and, by Lemma 1, this is solvable in positive integers if and only if $n = F_{2m-1}$ and $b = L_{2m-1}$. Finally, since $F_m + L_m = 2F_{m+1}$ for every m, it follows from (1) that

$$a = \frac{n + b}{2} = \frac{F_{2m-1} + L_{2m-1}}{2} = F_{2m}$$

and the proof is complete.

The proofs of Theorems 2 and 4, which depend on Lemma 2, are exactly analogous to the proof of Theorem 1 and will therefore be omitted. Of course, Theorem 2 is the special case of Theorem 4 with $b = 1$.

Proof of Theorem 3. Part (a) follows directly from Theorem 4 with $n = L_{2r}$, $a = L_{2r+1}$, $b = F_{2r}$, $c = 5F_{2r}$, and $m = 2r$ since it is easily shown that $L_{2r}F_{2r} = F_{4r}$, $L_{2r+1}F_{2r} = F_{4r+1} - 1$, and $5F_{2r}^2 = L_{4r} - 2$.

To obtain Part (b) we define the sequence β_i for $c' \geq 1$ by the following series of calculations which depend on Lemma 1: Let

$$\beta_1 = L_{2r-1}\alpha - L_{2r} + 1$$

$$= \frac{\sqrt{5}\,L_{2r-1} - 5F_{2r-1}}{2} + 1$$

$$= \frac{-10}{\sqrt{5}\,L_{2r-1} + 5F_{2r-1}} + 1 .$$

Then

$$\frac{1}{\beta_1} = 1 + \frac{10}{\sqrt{5}L_{2r-1} + 5F_{2r-1} - 10}$$

$$= 1 + \beta_2 ,$$

$$\frac{1}{\beta_2} = \frac{\sqrt{5}\,L_{2r-1} + 5F_{2r-1} - 10}{10}$$

$$= F_{2r-1} - 2 + \frac{\sqrt{5}\,L_{2r-1} - 5F_{2r-1} + 10}{10}$$

$$= F_{2r-1} - 2 + \beta_3 ,$$

$$\frac{1}{\beta_3} = \frac{10}{\sqrt{5}\,L_{2r-1} - 5F_{2r-1} + 10}$$

$$= \frac{10(\sqrt{5}\,L_{2r-1} + 5F_{2r-1})}{-20 + 10(\sqrt{5}\,L_{2r-1} + 5F_{2r-1})}$$

$$= 1 + \frac{2}{\sqrt{5}\,L_{2r-1} + 5F_{2r-1} - 2}$$

$$= 1 + \beta_4 ,$$

and

$$\frac{1}{\beta_4} = \frac{\sqrt{5}\,L_{2r-1} + 5F_{2r-1} - 2}{2}$$

$$= 5F_{2r-1} - 2 + \frac{\sqrt{5}\,L_{2r-1} - 5F_{2r-1}}{2} + 1$$

$$= 5F_{2r-1} - 2 + \beta_5 .$$

Since $\beta_5 = \beta_1$, the sequence now repeats and it follows that

$$
\begin{aligned}
L_{2r-1}\alpha &= L_{2r} - 1 + \beta_1 \\
&= L_{2r} - 1 + \cfrac{1}{1 + \beta_2} \\
&= L_{2r} - 1 + \cfrac{1}{1 + \cfrac{1}{F_{2r-1} - 2 + \beta_3}} \\
&= \cdots \\
&= [L_{2r-1} - 1, \dot{1}, F_{2r-1} - 2, 1, 5\dot{F}_{2r-1} - 2]
\end{aligned}
$$

as claimed.

3. <u>More general results.</u> Since F_{n+1}/F_n is a convergent in the simple continued fraction expansion of $(1 + \sqrt{5})/2$, the results of the preceding section suggest that one ask if there is any interesting connection between the simple continued fraction expansion of a quadratic surd ξ and the simple continued fraction expansion of $q_n\xi$ where p_n/q_n is the n^{th} convergent to ξ. The following theorems, which generalize those of Section 2, answer this question in the affirmative for surds of the form $\xi = [a, \dot{b}]$ or $\xi = [a, \dot{b}, \dot{c}]$.

<u>Theorem 5.</u> Let $\xi = [a, \dot{b}]$, let n be a positive integer, let p_k/q_k denote the k^{th} convergent to ξ and let $t_k = q_{k-1} + q_{k+1}$. Then $n\xi = [r, \dot{s}]$ if and only if $n = q_{2m-2}$, $r = p_{2m-2}$, and $s = t_{2m-2}$ for some integer $m \geq 1$.

<u>Theorem 6.</u> Let $\xi, n, p_k/q_k$ and t_k be as in Theorem 5. Then $n\xi = [u, \dot{v}, \dot{w}]$ if and only if $vn = q_{2m-1}$, $vu = p_{2m-1} - 1$, and $vw = t_{2m-1} - 2$ for some integer $m \geq 1$.

<u>Theorem 7.</u> Let $\xi = [a, \dot{b}, \dot{c}]$, let p_k/q_k be the k^{th} convergent to ξ, let $t_k = q_{k-1} + q_{k+1}$, and let $s_k = p_{k-1} + p_{k+1}$. Then, for every integer $r \geq 1$, we have

a) $$q_{2r}\xi = \left[p_{2r}, \dot{t}_{2r}, \frac{c}{b}\dot{t}_{2r}\right],$$

b) $$q_{2r-1}\xi = [p_{2r-1} - 1, \dot{1}, \dot{t}_{2r} - 2],$$

c) $$t_{2r-1}\xi = \left[s_{2r-1}, \dot{q}_{2r-1}, \left(c^2 + \frac{4c}{b}\right)\dot{q}_{2r}\right],$$

and

d) $$t_{2r}\xi = [s_{2r} - 1, \dot{1}, q_{2r} - 2, 1, (bc + 4)\dot{q}_{2r} - 2]$$

Proof of Theorem 5. The convergents to $\xi = [a, \dot{b}]$ are given by the difference equations

$$q_n = bq_{n-1} + q_{n-2}$$

$$p_n = bp_{n-1} + p_{n-2}$$

with the initial conditions $q_0 = 1$, $q_1 = b$, $p_0 = a$, and $p_1 = ab + 1$. These are easily solved to obtain

$$q_n = \frac{1}{\sqrt{b^2 + 4}}\left\{\left(\frac{b + \sqrt{b^2 + 4}}{2}\right)^{n+1} - \left(\frac{b - \sqrt{b^2 + 4}}{2}\right)^{n+1}\right\},$$

(2) $$p_n = \frac{a}{2} \cdot t_{n-1} + \frac{ab + 2}{2} \cdot q_{n-1},$$

where

$$t_{n-1} = \left(\frac{b + \sqrt{b^2 + 4}}{2}\right)^n + \left(\frac{b - \sqrt{b^2 + 4}}{2}\right)^n$$

and it is easily shown by induction that $t_n = q_{n-1} + q_{n+1}$ for $n \geq 0$.

Moreover, since $[a, \dot{b}] = (2a - b + \sqrt{b^2 + 4})/2$, it follows that $n\xi = [r, \dot{s}]$ if and only if the equations

(3) $$n(2a - b) = 2r - s,$$ $$n\sqrt{b^2 + 4} = \sqrt{s^2 + 4}$$

simultaneously hold. The second of these equations is equivalent to

$$s^2 - n^2(b^2 + 4) = -4$$

and $s = b$, $n = 1$ is clearly the minimal positive solution. Therefore, every solution (s, n) in positive integers is given by the equation

$$s + n\sqrt{b^2 + 4} = 2\left(\frac{s + \sqrt{b^2 + 4}}{2}\right)^{2m-1}, \quad m = 1, 2, \cdots,$$

and it is easily shown by expanding the powers here and in (2) that this reduces to

$$s + n\sqrt{b^2 + 4} = t_{2m-2} + q_{2m-2}\sqrt{b^2 + 4} .$$

Also, from the second equation in (2), we have

$$\begin{aligned} r &= \frac{n(2a - b) + s}{2} \\ &= \frac{(2a - b)q_{2m-2} + t_{2m-2}}{2} \\ &= aq_{2m-2} + \frac{q_{2m-3} + q_{2m-1} - bq_{2m-2}}{2} \\ &= aq_{2m-2} + q_{2m-3} \\ &= p_{2m-2} \end{aligned}$$

since it is easily proved by induction that $aq_n + q_{n-1} = p_n$ for all n. This completes the proof.

Proof of Theorem 6. Note in particular that the preceding argument essentially shows that

$$\text{(4)} \qquad \frac{(b + \sqrt{b^2 + 4})^k}{2^{k-1}} = t_{k-1} + q_{k-1}\sqrt{b^2 + 4}, \quad k \geq 1 .$$

Also, it is easily shown by induction that

$$\sum_{k=0}^{m} \binom{m}{k} b^k q_{k-1} = q_{2m-1} \quad ,$$

$$\sum_{k=0}^{m+1} \binom{m}{k-1} b^k q_{k-1} = bq_{2m} \quad ,$$

$$\sum_{k=0}^{m} \binom{m}{k} b^k t_{k-1} = q_{2m-1} \quad ,$$

and

$$\sum_{k=0}^{m} \binom{m}{k} b^k q_{k-2} = q_{2m-2} \quad .$$

Now, as in the preceding proof, one can show that $n\xi = [u, \dot{v}, \dot{w}]$ if and only if $vw + 2$ and vn are simultaneously positive integral solutions of the Pell equation

$$(vw + 2)^2 - (vn)^2(b^2 + 4) = 4 \tag{6}$$

and of $n(2a - b) = 2u - w$. Also, the general solution of (6) is given by

$$(vw + 2) + vn\sqrt{b^2 + 4} = 2\left\{\frac{b^2 + 2 + b\sqrt{b^2 + 4}}{2}\right\}^m, \quad m = 1, 2, \cdots .$$

Using the equalities in (4) and (5) this may be simplified to give

$$2\left\{\frac{(b^2 + 2) + b\sqrt{b^2 + 4}}{2}\right\}^m = 2\left\{\frac{2 + b(b + \sqrt{b^2 + 4})}{2}\right\}^m$$
$$= \sum_{k=0}^{m} \binom{m}{k} \frac{(b + \sqrt{b^2 + 4})^k b^k}{2^{k-1}}$$

$$= \sum_{k=0}^{m} \binom{m}{k} (t_{k-1} + q_{k-1}\sqrt{b^2 + 4})b^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} t_{k-1} + \sqrt{b^2 + 4}\sum_{k=0}^{m} \binom{m}{k} b^k q_{k-1}$$

$$= t_{2m-1} + \sqrt{b^2 + 4} \cdot q_{2m-1} \quad .$$

Thus, $vw + 2 = t_{2m-1}$, $vn = q_{2m-1}$, and

$$vu = \frac{vn(2a - b) + vw}{2}$$

$$= \frac{(2a - b)q_{2m-1} + t_{2m-1} - 2}{2}$$

$$= p_{2m-1} - 1$$

as in the preceding proof.

Proof of Theorem 7. For

$$\xi = [a, \dot{b}, \dot{c}] = a + \frac{-bc + \sqrt{b^2c^2 + 4bc}}{2b} \quad ,$$

define

$$A = \xi - a, \quad B = A + c, \quad C = bA/c,$$

and

$$D = bB/c = C + b \, .$$

The following identities are useful:

a) $$q_{2k}^2 = 1 + cq_{2k+1}q_{2k-1}/b$$

b) $$q_{2k+1}^2 = b(q_{2k+2}q_{2k} + 1)/c$$

c) $$p_{2k} = aq_{2k} + cq_{2k-1}/b$$

d) $$p_{2k+1} = aq_{2k+1} + q_{2k}$$

e) $$(q_{2k}B - cq_{2k+1}/b)(q_{2k}A + cq_{2k+1}/b) = c/b$$

f) $$(q_{2k}D - q_{2k+1})(q_{2k}C + q_{2k+1}) = b/c$$

g) $$(q_{2k+1}B - q_{2k+2} + 1)(q_{2k+1}A + q_{2k+2} - 1) = q_{2k+2} + q_{2k} - 2 = t_{2k+1} - 2$$

h) $$t_{2r} = bt_{2r-1} + t_{2r-2}$$

i) $$t_{2r+1} = ct_{2r} + t_{2r-1}$$

j) $$s_{2k} = at_{2k} + t_{2k-1}$$

k) $$s_{2k+1} = at_{2k+1} + ct_{2k}/b$$

m) $$ct_{2k}^2 - bt_{2k+1}t_{2k-1} = ct_{2k}t_{2k+2} - bt_{2k+1}^2 = -b(bc + 4)$$

n) $$(t_{2r-1}B - ct_{2r}/b)(t_{2r-1}A + ct_{2r}/b) = c(bc + 4)/b$$

o) $$(t_{2r-1}D - t_{2r})(t_{2r-1}C + t_{2r}) = b(bc + 4)/c \quad .$$

These are proved in a straightforward manner.

To prove 7a, we have by identity c) and the definition of B that

$$\begin{aligned} q_{2k}\xi &= q_{2k}a + q_{2k}A \\ &= q_{2k}a + cq_{2k-1}/b - cq_{2k-1}/b + q_{2k}A \\ &= p_{2k} + Bq_{2k} - cq_{2k+1}/b \,, \qquad \text{definition of } B, \\ &= p_{2k} + 1/\beta_1 \quad . \end{aligned}$$

Using identity e) and the definitions of C, D and t_{2k}, we have

$$\beta_1 = b(q_{2k}A + cq_{2k+1}/b)/c$$

$$= q_{2k}C + q_{2k+1}, \qquad \text{definition of } C,$$

$$= q_{2k+1} + q_{2k-1} + q_{2k}C - q_{2k-1}$$

$$= t_{2k} + q_{2k}D - q_{2k+1}$$

$$= t_{2k} + 1/\beta_2 \quad .$$

Using identity f) and the definitions of C and t_{2k}, we have

$$\beta_2 = c(q_{2k}C + q_{2k+1})/b$$

$$= q_{2k}A + cq_{2k+1}/b$$

$$= c(q_{2k+1} + q_{2k-1})/b + q_{2k}A - cq_{2k-1}/b$$

$$= ct_{2k}/b + 1/\beta_1 \quad .$$

We therefore have

$$q_{2k}\xi = [p_{2k},\ \dot{t}_{2k},\ c\dot{t}_{2k}/b] \quad ,$$

proving 7a.

The proof of 7b is similar and uses identity g) at a key point in the argument. The argument will not be presented here.

To prove 7c, we note that

$$t_{2r-1}\xi = t_{2r-1}a + t_{2r-1}A$$
$$= s_{2r-1} + t_{2r-1}A - ct_{2r-2}/b$$
$$= s_{2r-1} + (t_{2r-1}B - ct_{2r}/b)$$
$$= s_{2r-1} + \frac{c(bc+4)/b}{t_{2r-1}A + ct_{2r}/b}$$

$$= s_{2r-1} + \frac{1}{\beta_1} \quad ,$$

$$\beta_1 = (bt_{2r-1}A + ct_{2r})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A + ct_{2r} - c(bc + 4)q_{2r-1})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A + cq_{2r+1} - c(bc + 3)q_{2r-1})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r}A + cbq_{2r} - c(bc + 2)q_{2r-1})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A + cbq_{2r} - (bc + 2)(q_{2r} - q_{2r-2}))/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A - 2q_{2r} + (bc + 2)q_{2r-2})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A - 2(cq_{2r-1} + q_{2r-2}) + (bc + 2)q_{2r-2})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A - 2cq_{2r-1} + bcq_{2r-2})/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A - c(q_{2r-1} + q_{2r-3}))/c(bc + 4)$$

$$= q_{2r-1} + (bt_{2r-1}A - c(t_{2r-2}))/c(bc + 4)$$

$$= q_{2r-1} + (t_{2r-1}C - t_{2r-2})/(bc + 4)$$

$$= q_{2r-1} + (t_{2r-1}D - t_{2r})/(bc + 4)$$

$$= q_{2r-1} + \frac{b(bc + 4)/c}{(bc + 4)(t_{2r-1}C + t_{2r})}$$

$$= q_{2r-1} + \frac{1}{t_{2r-1}A + ct_{2r}/b}$$

$$= q_{2r-1} + \frac{1}{\beta_2} = q_{2r-1} + \frac{1}{\beta_1((bc^2 + 4c)/b)}$$

and

$$\beta_2 = (bc^2 + 4c)\beta_1/b = ((bc^2 + 4c)/b)(q_{2r-1} + \frac{1}{\beta_2})$$

$$= ((bc^2 + 4c)/b)q_{2r-1} + \frac{1}{\beta_1} .$$

Hence we obtain

$$t_{2k-1}\xi = [s_{2k-1}, \dot{q}_{2k-1}, \dot{((bc^2 + 4c)/b)q_{2k-1}}] .$$

The proof of part d) is similar and the argument is omitted.

The following theorem, which is stated without proof, is a partial converse of Theorem 7.

Theorem 8. Let ξ, p_k, q_k, s_k, and t_k be as in Theorem 7 and let n, u, and v be positive integers.

a) If v is such that b divides cv and $n\xi = [u, \dot{v}, c\dot{v}/b]$, then $n = q_{2r}$, $u = t_{2r}$, and $v = p_{2r}$ for some positive integer r.

b) If $n\xi = [u, \dot{1}, \dot{v}]$, then $n = q_{2r-1}$, $u = p_{2r-1} - 1$, and $v = t_{2r} - 2$ for some positive integer r.

Remark. When a simple continued fraction has a partial quotient 1 the corresponding approximation of the convergent to the number in question is not as good as when other integers are partial quotients. The 1's can be eliminated as all but the first partial quotient if it is permitted to have -1's as numerators. The corresponding convergents would then be better approximations than the original ones.

Setting about to purge the 1's from the expressions obtained in Theorems 2, 3b, 7b and 7d we ran across an interesting pattern that allowed us to simplify the notation. Let us define the symbol $-[a_0, a_1, a_2, \cdots]$ to be the expression

$$a_0 + \cfrac{-1}{a_1 + \cfrac{-1}{a_2 + \cdots}}$$

Although this expression might not always be meaningful, it is in the cases we consider here.

With the new notation we are able to restate a few of the theorems as Theorem 9:

a) $$F_k\alpha = (-1)^{k+1}\left[F_{k+1}, \dot{L}_k\right].$$

b) $$L_k\alpha = (-1)^k\left[L_{k+1}, \dot{F}_k, 5\dot{F}_k\right].$$

c) $$\text{If } \xi = [a, \dot{b}] \text{ then } q_k\xi = (-1)^k[p_k, \dot{t}_k].$$

d) $$\text{If } \xi = [a, \dot{b}, \dot{c}] \text{ and } k \text{ odd, then } q_k\xi = -[p_k, \dot{t}_k].$$

e) If $\xi = [a, \dot{b}, \dot{c}]$ and k even, then $t_k\xi = -[s_k, \dot{q}_k, (bc + \dot{4})q_k]$.

The proofs are quite similar to the original proofs and are omitted.

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