q-FIBONACCI POLYNOMIALS

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0. INTRODUCTION

Let MC be the monoid of all Morse code sequences of dots $a(:= \bullet)$ and dashes $b(:= -)$ with respect to concatenation. MC consists of all words in a and b. Let P be the algebra of all polynomials $\sum_{v \in MC} \lambda_v v$ with real coefficients.

We are interested in:

a) polynomials in P which we call *abstract Fibonacci polynomials*. They are defined by the recursion

$$F_n(a, b) = aF_{n-1}(a, b) + bF_{n-2}(a, b)$$

with initial values $F_0(a, b) = 0$, $F_1(a, b) = \varepsilon$.

b) polynomials $F_n(x, s, q)$ in real variables x and s which we call *q-Fibonacci polynomials.* They are defined by the recursion

$$F_n(x, s, q) = xF_{n-1}(x, s, q) + t(q^{n-2}s)F_{n-2}(x, s, q)$$

with initial values $F_0(x, s, q) = 0$, $F_1(x, s, q) = 1$, where $t(s) \neq 0$ is a function of a real variable s and $q \neq 0$ is a real number.

We show how these classes of polynomials are connected, generalize some well-known theorems about the classical Fibonacci polynomials, and study some examples. Related results have been obtained previously by Al-Salam and Ismail [1], Andrews, Knopfmacher, and Paule [4], Carlitz [6], Ismail, Prodinger, and Stanton [11], and Schur [12]. I want to thank Peter Paule for his suggestion to formulate all results in terms of Morse code sequences. My thanks are also due to the referee for drawing my attention to the paper of Ismail, Prodinger, and Stanton [11] and to the polynomials of Al-Salam and Ismail [1]. Most of the cited papers are inspired by the Rogers-Ramanujan identities (e.g., [1], [4], [11], [12]) or by the connections with the general theory of orthogonal polynomials (e.g., [1], [3], [11], [13]), but the aim of this paper is to emphasize the analogy with elementary results on Fibonacci numbers and Fibonacci polynomials and to give as simple and transparent proofs as possible.

1. ABSTRACT FIBONACCI POLYNOMIALS

Let MC be the set of all Morse code sequences of dots $(\bullet)$ and dashes $(-)$. We interpret MC as a monoid with respect to concatenation whose unit element is the empty sequence ε. If we write a for a dot and b for a dash, then MC consists of all words in a and b. Let P be the corresponding monoid algebra over $\mathbb{R}$, i.e., the algebra of all finite sums (polynomials) $\sum_{v \in MC} \lambda_v v$ with real coefficients.

An important element of P is the binomial

$$(a+b)^n = \sum_{k=0}^{n} C_k^n(a, b). \tag{1.1}$$

Here $C_k^n(a, b)$ is the sum of all words with k dashes and $n-k$ dots. It is characterized by the boundary values $C_k^0(a, b) = \delta_{k,0}$ and $C_0^n(a, b) = a^n$ and each of the two recursions

$$C_k^{n+1}(a, b) = bC_{k-1}^n(a, b) + aC_k^n(a, b) \tag{1.2}$$

or

$$C_k^{n+1}(a, b) = C_{k-1}^n(a, b)b + C_k^n(a, b)a. \tag{1.3}$$

We are mainly interested in a class of polynomials which we call *abstract Fibonacci polynomials*. They are defined by the recursion

$$F_n(a, b) = aF_{n-1}(a, b) + bF_{n-2}(a, b) \tag{1.4}$$

and the initial values $F_0(a, b) = 0$ and $F_1(a, b) = \varepsilon$. This sequence begins with 0, ε, a, a^2+b, $a^3+ab+ba$, ...

If we define the length of an element $v \in MC$ as $2k+l$, where k is the number of dashes (elements b) and l is the number of dots (elements a) occurring in v, then it is easily shown by induction that $F_n(a, b)$ is the sum of all words in MC of length $n-1$.

It is now easy to see that they satisfy also another recursion:

$$F_n(a, b) = F_{n-1}(a, b)a + F_{n-2}(a, b)b. \tag{1.5}$$

To this end, consider all words of length $n-1$ which end with a and those which end with b.

Both recurrences are special cases of the formula

$$F_{m+n}(a, b) = F_{m-1}(a, b)bF_n(a, b) + F_m(a, b)F_{n+1}(a, b), \tag{1.6}$$

which follows from the fact that each word w of length $m+n-1$ can be factored uniquely either as $w = ubv$, where u has length $m-2$ and v has length $n-1$, or as $w = xy$, where x has length $m-1$ and y has length n.

There is a simple formula connecting these Fibonacci polynomials with the $C_k^n(a, b)$:

Theorem 1.1: The abstract Fibonacci polynomials are given by

$$F_n(a, b) = \sum_{k=0}^{n-1} C_k^{n-k-1}(a, b). \tag{1.7}$$

For $F_n(a, b)$ is the sum of all monomials $v \in MC$ with length $n-1$. If such a monomial has exactly k dashes, then it has $n-1-2k$ dots; therefore, $n-k-1$ letters and the sum over all such words is $C_k^{n-k-1}(a, b)$.

Consider now the homomorphism $\phi: P \to \mathbb{R}[x, s]$ defined by $\phi(a) = x$, $\phi(b) = s$, where x and s are commuting variables. Let $F_n(x, s) := \phi(F_n(a, b))$. Then we get the classical Fibonacci polynomials defined by

$$F_n(x, s) = xF_{n-1}(x, s) + sF_{n-2}(x, s)$$

with $F_0(x, s) = 0$, $F_1(x, s) = 1$. Since

$$\phi(C_k^n(a, b)) = \binom{n}{k} s^k x^{n-k},$$

we get from (1.7) the well-known formula

$$F_n(x, s) = \sum_{k=0}^{n-1} \binom{n-k-1}{k} s^k x^{n-2k-1}.$$

2. A CLASS OF q-FIBONACCI POLYNOMIALS

Now we consider Morse code sequences which are defined on some interval $\{m, m+1, \ldots, m+k-1\} \subseteq \mathbb{Z}$. In this case, we say that the sequence starts at place m. We want to associate a weight to such a sequence in the following way: Let $t(s) \neq 0$ be a function of a real variable s and let $q \neq 0$ be a real number. Let v be a Morse code sequence on some interval $\{m, m+1, \ldots, m+k-1\}$. If the place $i \in \{m, m+1, \ldots, m+k-1\}$ is occupied by a dot, we set $w(i) = x$; if it is the endpoint of a dash, we set $w(i) = t(q^i s)$. In the other cases, let $w(i) = 1$. Now the weight of v is defined as the product of the weights of all places of the interval, i.e.,

$$w(v) = \prod_{i=m}^{m+k-1} w(i).$$

If, e.g., the sequence –•–••––• starts at $m = 4$, its weight is $x^4 t(q^5 s) t(q^8 s) t(q^{12} s) t(q^{14} s)$. The weight of all Morse code sequences with length $n-1$ starting at $m = 0$ is denoted by $F_n(x, s, q)$ and is our q-analog of the Fibonacci polynomials.

We can immediately deduce a recursion for $F_n(x, s, q)$.

We show that the recurrence

$$F_n(x, s, q) = xF_{n-1}(x, s, q) + t(q^{n-2} s) F_{n-2}(x, s, q) \tag{2.1}$$

holds with initial values $F_0(x, s, q) = 0$, $F_1(x, s, q) = 1$. This recursion means that we can split a Morse code sequence of length $n-1$ into two parts, those with a dot in the last position and those with a dash there. In the first case, the dot has weight x and the sequence in front has the weight $F_{n-1}(x, s, q)$. Since the weights are multiplicative, the first term is explained. A dash at the end gives the weight $t(q^{n-2} s)$. Since the dash occupies two positions, the sequence in front of the dash now has length $n-2$ and we get the second term in the formula.

If we split the Morse code sequences into those with a dot in the first position and those with a dash in the beginning, we get in the same way the recursion

$$F_n(x, s, q) = xF_{n-1}(x, qs, q) + t(qs) F_{n-2}(x, q^2 s, q) \tag{2.2}$$

with the same initial conditions as before.

Let

$$A(x, s) = \begin{pmatrix} 0 & 1 \\ t(s) & x \end{pmatrix} \tag{2.3}$$

and

$$M_n(x, s) = A(x, q^{n-1} s) A(x, q^{n-2} s) \cdots A(x, s). \tag{2.4}$$

Then we get

$$M_n(x, s) = \begin{pmatrix} t(s) F_{n-1}(x, qs, q) & F_n(x, s, q) \\ t(s) F_n(x, qs, q) & F_{n+1}(x, s, q) \end{pmatrix}. \tag{2.5}$$

From (2.4), it follows that the matrices $M_n(x, s)$ satisfy the relation

$$M_{k+n}(x, s) = M_k(x, q^n s) M_n(x, s). \tag{2.6}$$

If we extend this to negative indices—which is uniquely possible—we get

$$M_{-k}(x, s) = (M_k(x, q^{-k} s))^{-1};$$

therefore,

$$M_{-n}(x,s)=\frac{1}{d_n(q^{-n}s)}\begin{pmatrix} F_{n+1}(x,q^{-n}s,q) & -F_n(x,q^{-n}s,q) \\ -t(q^{-n}s)F_n(x,q^{-n+1}s,q) & t(q^{-n}s)F_{n-1}(x,q^{-n+1}s,q)\end{pmatrix},$$

which implies

$$F_{-n}(x,s,q)=(-1)^{n-1}\frac{F_n(x,q^{-n}s,q)}{t\left(\frac{s}{q}\right)t\left(\frac{s}{q^2}\right)\cdots t\left(\frac{s}{q^n}\right)}. \tag{2.7}$$

Taking determinants in (2.5), we obtain the *q-Cassini formula*

$$F_{n-1}(x,qs,q)F_{n+1}(x,s,q)-F_n(x,s,q)F_n(x,qs,q)=(-1)^n t(qs)\cdots t(q^{n-1}s). \tag{2.8}$$

This is a special case of the following theorem.

Theorem 2.1 (q-Euler-Cassini formula): The q-Fibonacci polynomials satisfy the polynomial identity

$$\begin{aligned} F_{n-1}(x,qs,q)F_{n+k}(x,s,q)-F_n(x,s,q)F_{n+k-1}(x,qs,q) \\ =(-1)^n t(qs)\cdots t(q^{n-1}s)F_k(x,q^ns,q). \end{aligned} \tag{2.9}$$

This formula is an immediate consequence of (2.6) if we write it in the form

$$M_{k+n}(x,s)M_n(x,s)^{-1}=M_k(x,q^ns)$$

and compare the upper right entries of the matrices.

A more illuminating proof results from an imitation of the construction given in [14]: Consider all pairs of Morse code sequences of the form (u,v), where u starts at 0 and has length $n+k-1$ for some $k\geq 1$ and v starts at 1 and has length $n-2$. If there is a place i, $0\leq i\leq n-2$, where a dot occurs in one of the sequences, there is also a minimal $i_{\min}$ with this property. Then we exchange the sequences starting at $i_{\min}+1$.

Thus, to each pair (u,v) there is associated a pair $(\hat{u},\hat{v})$, where $\hat{u}$ starts at 0 and has length $n-1$ and $\hat{v}$ starts at 1 and has length $n+k-2$. It is clear that the weights of the pairs are the same, $w(u)w(v)=w(\hat{u})w(\hat{v})$. The only pairs where this bijection fails are, for even n, those where v has only dashes and in u all places up to $n-1$ are occupied by dashes. The weight of these pairs is $t(qs)\cdots t(q^{n-1}s)F_k(x,q^ns,q)$. If n is odd, then this bijection fails at those pairs $(\hat{u},\hat{v})$ where $\hat{u}$ has only dashes and in $\hat{v}$ all places up to $n-1$ are occupied by dashes. Thus, the q-Euler-Cassini formula is proved.

Corollary 2.2: In the special case $t(s)=s$, the Euler-Cassini formula reduces to

$$F_{n-1}(x,qs,q)F_{n+k}(x,s,q)-F_n(x,s,q)F_{n+k-1}(x,qs,q)=(-1)^n q^{\binom{n}{2}}s^{n-1}F_k(x,q^ns,q).$$

This corollary was first proved by Andrews, Knopfmacher, and Paule [4] with other methods. Another proof for the more general polynomials of Al-Salam and Ismail was given in [11], and yet another combinatorial proof was recently obtained by Berkovich and Paule [5].

Remark: If we choose a function $x(s)$ instead of the constant x and define the weight of the place i as $x(q^is)$ if the place is occupied by a dot, then we get a polynomial $K_n(s)$ as the weight of the set of all Morse code sequences of length n starting at position 0. We call it the *q-continuant* corresponding to the set of all Morse code sequences of length n, since for $t(s) \equiv 1$ and $x(q^k s) = x_{k+1}$ we obtain the continuants considered in [10].

The continuant is intimately connected with continued fractions. If we set $x(q^i s) = x_i$ and $t(q^i s) = y_i$ and write

$$x_0 + \cfrac{y_1}{x_1 + \cfrac{y_2}{x_2 + \cdots}} = x_0 + \frac{y_1}{x_1 +}\frac{y_2}{x_2 +}\frac{y_3}{x_3 +}\cdots,$$

then it is easy to see that

$$x_0 + \frac{y_1}{x_1 +}\frac{y_2}{x_2 +}\cdots\frac{y_n}{x_n} = \frac{K_{n+1}(s)}{K_n(qs)}.$$

As a special case, we obtain

$$\frac{F_{n+2}(1, s, q)}{F_{n+1}(1, qs, q)} = 1 + \frac{t(qs)}{1+}\frac{t(q^2 s)}{1+}\cdots\frac{t(q^n s)}{1}.$$

If we let $n \to \infty$, it is easy to see that, at least in the case where $t(s)$ is a formal power series with $t(0) = 0$, we have $\lim_{n\to\infty} F_n(x, s, q) = F_\infty(x, s, q)$ in the sense that the coefficients of each power q^k remain constant beginning with some index $n(k)$. Therefore, we obtain the infinite continued fraction

$$\frac{F_\infty(1, s, q)}{F_\infty(1, qs, q)} = 1 + \frac{t(qs)}{1+}\frac{t(q^2 s)}{1+}\cdots$$

and the functional equation $F_\infty(x, s, q) = F_\infty(x, qs, q) + t(qs)F_\infty(x, q^2 s, q)$.

3. q-FIBONACCI OPERATORS

Now we want to establish a connection between the abstract Fibonacci polynomials and the q-Fibonacci polynomials. To this end, we consider the ring R of linear operators on the vector space of polynomials $\mathbb{R}[x, s]$. We are interested only in multiplication operators with polynomials and the operator η in R defined by $\eta f(x, s) = f(x, qs)$.

We define a homomorphism $\Phi : P \to R$ by

$$\Phi(a) = x\eta, \quad \Phi(b) = t(qs)\eta^2.$$

Then we have

$$\Phi(F_n(a, b)) = F_n(x, s, q)\eta^{n-1}. \tag{3.1}$$

This is easily verified by induction from

$$\begin{aligned}\Phi(F_n(a, b)) &= \Phi(a)\Phi(F_{n-1}(a, b)) + \Phi(b)\Phi(F_{n-2}(a, b)) \\ &= x\eta F_{n-1}(x, s, q)\eta^{n-2} + t(qs)\eta^2 F_{n-2}(x, s, q)\eta^{n-3} \\ &= xF_{n-1}(x, qs, q)\eta^{n-1} + t(qs)F_{n-2}(x, q^2 s, q)\eta^{n-1} \\ &= (xF_{n-1}(x, qs, q) + t(qs)F_{n-2}(x, q^2 s, q))\eta^{n-1} = F_n(x, s, q)\eta^{n-1}.\end{aligned}$$

As a special case, we see that applying the Fibonacci operators to the polynomial 1 we get

$$F_n(x, s, q) = F_n(x, s, q)\eta^{n-1}1 = \Phi(F_n(a, b))1. \tag{3.2}$$

4. THE q-FIBONACCI POLYNOMIALS OF L. CARLITZ

If we choose $t(s)=\frac{s}{q}$, we get the q-Fibonacci polynomials of Carlitz [6]. They are a special case of the Al-Salam and Ismail polynomials $U_n(x;a,b)$ introduced in [1], which are defined by $U_{n+1}(x;a,b)=x(1+aq^n)U_n(x;a,b)-bq^{n-1}U_{n-1}(x;a,b)$ for $n\geq 1$ with initial values $U_0(x;a,b)=1$, $U_1(x;a,b)=x(1+a)$. It is clear that $F_n(x,s,q)=U_{n-1}(x;0,-sq^{-1})$.

In this case, the recurrences are

$$F_n(x,s,q)=xF_{n-1}(x,s,q)+q^{n-3}sF_{n-2}(x,s,q),$$
$$F_0(x,s,q)=0,\ F_1(x,s,q)=1,$$

and

$$F_n(x,s,q)=xF_{n-1}(x,qs,q)+sF_{n-2}(x,q^2s,q),$$
$$F_0(x,s,q)=0,\ F_1(x,s,q)=1.$$

The matrix form reduces to

$$M_n(x,s)=\begin{pmatrix} sF_{n-1}(x,qs,q) & F_n(x,s,q) \\ sF_n(x,qs,q) & F_{n+1}(x,s,q)\end{pmatrix}$$

and the Cassini formula is

$$F_{n+1}(x,s,q)F_{n-1}(x,qs,q)-F_n(x,s,q)F_n(x,qs,q)=(-1)^n q^{\binom{n-1}{2}}s^{n-1}.$$

Remark: A special case of these polynomials has already been obtained by Schur [12]. In [4], these Schur polynomials are called e_n and d_n. In our terminology, these are $e_n=F_n(1,q,q)$ and $d_n=F_{n-1}(1,q^2,q)$.

For the following, we need the Gaussian q-binomial coefficients $\left[{n \atop k}\right]$ (cf., e.g., [3] or [7]). We define them by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\prod_{i=1}^{k}\frac{q^{n-i+1}-1}{q^i-1} \text{ for } n\in\mathbb{Z} \text{ and } k\in\mathbb{N}.$$

They satisfy the following recursions:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}=q^k\begin{bmatrix} n \\ k \end{bmatrix}+\begin{bmatrix} n \\ k-1 \end{bmatrix}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}=\begin{bmatrix} n \\ k \end{bmatrix}+q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

The q-binomial theorem (see, e.g., [3] or [7]) states that, for $n\in\mathbb{N}$,

$$(A+B)^n=\sum_{k\geq 0}\begin{bmatrix} n \\ k \end{bmatrix}B^kA^{n-k} \text{ if } AB=qBA.$$

Now we have $x\eta\cdot s\eta^2=qs\eta^2\cdot x\eta$ or, in other words, $\Phi(a)\Phi(b)=q\Phi(b)\Phi(a)$. This may be stated in the following way

$$\Phi(C_k^n(a,b))=\begin{bmatrix} n \\ k \end{bmatrix}(s\eta^2)^k(x\eta)^{n-k}. \tag{4.1}$$

Therefore, from (1.7), we get

$$F_n(x,s,q)=\Phi(F_n(a,b))1=\sum_{k=0}^{n-1}\begin{bmatrix}n-k-1\\k\end{bmatrix}(s\eta^2)^k(x\eta)^{n-2k-1}1$$

or, equivalently,

$$F_n(x,s,q)=\sum_{k=0}^{n-1}\begin{bmatrix}n-k-1\\k\end{bmatrix}q^{2\binom{k}{2}}s^kx^{n-2k-1}. \tag{4.2}$$

5. ANOTHER INTERESTING CASE

Another interesting special case is given by

$$t(s)=\frac{4qs}{(1+s)(1+qs)}.$$

We denote the corresponding Fibonacci polynomials by $f_n(x,s,q)$. Here we get

$$(x\eta+t(qs)\eta^2)^n1=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}\frac{4^kq^{2\binom{k+1}{2}}s^k}{\prod_{j=1}^{k}(1+q^js)(1+q^{n+j}s)}. \tag{5.1}$$

If we set

$$d_{n,k}(s)=\frac{4^kq^{2\binom{k+1}{2}}s^k}{\prod_{j=1}^{k}(1+q^js)(1+q^{n+j}s)},$$

we can write this in the form

$$(x\eta+t(qs)\eta^2)^n1=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}d_{n,k}(s).$$

It is easy to see that

$$d_{n,k}(qs)=q^k\frac{1+qs}{1+q^{k+1}s}d_{n+1,k}(s)$$

and

$$t(qs)d_{n,k-1}(q^2s)=\frac{(1+q^{n+2}s)}{1+q^{k+1}s}d_{n+1,k}(s).$$

We have to show that

$$(x\eta+t(qs)\eta^2)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}d_{n,k}(s)=\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}x^{n+1-k}d_{n+1,k}(s).$$

The left-hand side is

$$\sum_{k=0}^{n}x\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}d_{n,k}(qs)+\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}t(qs)d_{n,k}(q^2s)$$

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n+1-k}d_{n,k}(qs)+\sum_{k=0}^{n}\begin{bmatrix}n\\k-1\end{bmatrix}x^{n+1-k}t(qs)d_{n,k-1}(q^2s)$$

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n+1-k}q^k\frac{1+qs}{1+q^{k+1}s}d_{n+1,k}(s)+\sum_{k=0}^{n}\begin{bmatrix}n\\k-1\end{bmatrix}x^{n+1-k}\frac{1+q^{n+2}s}{1+q^{k+1}s}d_{n+1,k}(s).$$

The recurrences of the q-binomial coefficients imply

$$\begin{bmatrix} n \\ k \end{bmatrix} q^k \frac{1+qs}{1+q^{k+1}s} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \frac{1+q^{n+2}s}{1+q^{k+1}s} = \begin{bmatrix} n+1 \\ k \end{bmatrix},$$

from which the right-hand side follows.

From (5.1), it follows that

$$\Phi(C_k^n(a,b)) = \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \frac{4^k q^{2\binom{k+1}{2}} s^k}{\prod_{j=1}^{k}(1+q^j s)(1+q^{n+j}s)}.$$

In this case, (1.7) implies the following theorem.

Theorem 5.1: The q-Fibonacci polynomials $f_n(x,s,q)$ are given by

$$\begin{aligned} f_n(x,s,q) &:= \Phi(F_n(a,b))1 \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} x^{n-2k-1} \frac{4^k q^{2\binom{k+1}{2}} s^k}{\prod_{j=1}^{k}(1+q^j s)(1+q^{n-k+j-1}s)}. \end{aligned} \tag{5.2}$$

6. A CONNECTION WITH THE CATALAN NUMBERS

The classical Fibonacci polynomials are intimately related to the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Fibonacci polynomials $F_n(x,1)$, $n>0$, are a basis of the vector space of polynomials. If we define the linear functional L by $L(F_{n+1}) = \delta_{n,0}$, then we get $L(x^{2n+1}) = 0$ and $L(x^{2n}) = (-1)^n C_n$.

We will now sketch how this fact can be generalized. The polynomials $F_n(x,s,q)$, $n>0$, are a basis of the vector space $\mathbb{P}$ of all polynomials in x whose coefficients are rational functions in s and q. We can therefore define a linear functional L on $\mathbb{P}$ by

$$L(F_n) = \delta_{n,1}. \tag{6.1}$$

Let

$$\hat{F}_n(x,s,q) = \frac{F_n(x,s,q)}{t(s)t(qs)\cdots t(q^{n-1}s)}.$$

Then we have $x\hat{F}_n = t(q^n s)\hat{F}_{n+1} - \hat{F}_{n-1}$.

Now, define the numbers

$$a_{n,k} = (-1)^{\lceil \frac{n+k}{2} \rceil} L(x^n \hat{F}_{k+1}),$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x. They satisfy

$$\begin{aligned} a_{0,k} &= \delta_{0,k} \\ a_{n,k} &= a_{n-1,k-1} + t(q^{k+1}s)a_{n-1,k+1}, \end{aligned} \tag{6.2}$$

where $a_{n,k} = 0$ if $k<0$.

These numbers have an obvious combinatorial interpretation (consider, e.g., [8], [9]). Consider all nonnegative lattice paths in $\mathbb{R}^2$ that start in $(0, 0)$ with upward steps $(1, 1)$ and downward steps $(1, -1)$. We associate to each upward step ending on the height k the weight 1 and to each downward step ending on the height k the weight $t(q^{k+1}s)$. The weight of the path is the product of the weights of all steps of the path.

Then $a_{n,k}$ is the weight of all lattice paths from $(0, 0)$ to (n, k). It is clear that $a_{2n+1,0} = 0$. If we set $a_{2n,0} = C_n(s, q)$, then $C_n(s, q)$ is a q-analog of the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

because it is well known that the number of such paths equals C_n (cf., e.g., [10]). It is easy to give a recurrence for these q-Catalan numbers. To this end, decompose each lattice path from $(0, 0)$ to $(2n, 0)$ into the first path which returns to the x-axis and the rest path. The first path goes from $(0, 0)$ to $(2k+2, 0)$, $0 \le k \le n-1$, and consists of a rising segment followed by a path from $(0, 0)$ to $(2k, 0)$ (but one level higher) and a falling segment. Thus,

$$C_n(s, q) = \sum_{k=0}^{n-1} C_k(qs, q)t(qs)C_{n-k-1}(s, q).$$

This is a q-analog of the recursion

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \; C_0 = 1,$$

for the classical Catalan numbers.

For

$$t(s) = \frac{4qs}{(1+s)(1+qs)},$$

the corresponding q-Catalan numbers $C_n(1, q)$ have been found by Andrews [2] and are given by the explicit formula

$$C_n(1, q) = \frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\frac{(2q)^{2n}}{(1+q)(1+q^{n+1})\prod_{j=2}^{n}(1+q^j)^2}.$$

The q-Catalan numbers appear also as coefficients of the following power series associated with the Fibonacci polynomials. Consider the q-Fibonacci polynomials corresponding to $-zt(s)$ in place of $t(s)$. Then we have $F_n(1, s, q, z) = F_{n-1}(1, qs, q, z) - zt(qs)F_{n-2}(1, q^2s, q, z)$ If we define

$$g_n(s, z) = \frac{F_{n-1}(1, qs, q, z)}{F_n(1, s, q, z)},$$

then we have $g_n(s, z) = 1 + zt(qs)g_{n-1}(qs, z)g_n(s, z)$.

For $n \to \infty$, these formal power series in z converge coordinatewise toward a formal power series $g(s, z)$ which satisfies $g(s, z) = 1 + zt(qs)g(s, z)g(qs, z)$. Comparing coefficients we see that $g(s, z) = \sum C_n(s, q)z^n$, where the $C_n(s, q)$ are the q-Catalan numbers defined by

$$C_n(s, q) = \sum_{k=0}^{n-1} C_k(qs, q)t(qs)C_{n-k-1}(s, q).$$

This result is also an easy consequence of Theorem 5.8.2 in [3].

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