ON POLYNOMIALS RELATED TO DERIVATIVES OF THE GENERATING FUNCTION OF CATALAN NUMBERS

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1. INTRODUCTION AND SUMMARY

In [3] it has been shown that powers of the generating function $c(x)$ of Catalan numbers $\{C_n\}_{n\in \mathrm{N}_0} = \{1, 1, 2, 5, 14, 42, \ldots\}$, where $\mathrm{N}_0 := \{0, 1, 2, \ldots\}$ (nr. 1459 and A000108 of [8] and references of [3]) can be expressed in terms of a linear combination of 1 and $c(x)$ with coefficients replaced by certain scaled Chebyshev polynomials of the second kind. In this paper, derivatives of $c(x)$ are studied in a similar manner. The starting point is the following expression for the first derivative:

$$\frac{d\,c(x)}{dx} \equiv c'(x) = \frac{1}{x(1-4x)}(1+(-1+2x)\,c(x)). \tag{1}$$

This equation is equivalent to the simple recurrence relation valid for C_n:

$$(n+2)C_{n+1} - 2(2n+1)C_n = 0,\ \ n = -1, 0, 1, \ldots,\ \text{with } C_{-1} = -1/2. \tag{2}$$

Equation (1) can, of course, also be found from the explicit form $c(x) = (1-\sqrt{1-4x})/(2x)$. The result for the n^{th} derivative is of the form

$$\frac{1}{n!}\frac{d^n c(x)}{dx^n} = \frac{1}{(x(1-4x))^n}(a_{n-1}(x) + b_n(x)c(x)), \tag{3}$$

with certain polynomials $a_{n-1}(x)$ of degree $n-1$ and $b_n(x)$ of degree n. These polynomials are found to be

$$b_n(x) = \sum_{m=0}^{n}(-1)^m B(n, m)x^{n-m}$$

with

$$B(n, m) := \binom{2n}{n}\binom{n}{m}\Big/\binom{2m}{m}, \tag{4}$$

which defines a triangle of numbers for $n, m \in \mathrm{N}$, $n \geq m \geq 0$, where $\mathrm{N} := \{1, 2, 3, \ldots\}$. The first terms are depicted in Table 1 with $B(n, m) = 0$ for $n < m$. Another representation for the polynomials $b_n(x)$ is also found, i.e.,

$$b_n(x) = -2\sum_{k=0}^{n} C_{k-1}x^k(4x-1)^{n-k}. \tag{5}$$

Equating both forms of $b_n(x)$ leads to a formula involving convolutions of Catalan numbers with powers of an arbitrary constant $\lambda := (4x-1)/x$. This formula is given in (31). Equation (5) reveals the generating function of the polynomials $b_n(x)$ because it is a convolution of two functional sequences. The result is

$$g_b(x;z):=\sum_{n=0}^{\infty}b_n(x)z^n=\frac{\sqrt{1-4xz}}{1+(1-4x)z}. \tag{6}$$

TABLE 1. *B*(*n*, *m*) Central Binomial Triangle

$n \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0	0
2	6	6	1	0	0	0	0	0	0	0	0
3	20	30	10	1	0	0	0	0	0	0	0
4	70	140	70	14	1	0	0	0	0	0	0
5	252	630	420	126	18	1	0	0	0	0	0
6	924	2772	2310	924	198	22	1	0	0	0	0
7	3432	12012	12012	6006	1716	286	26	1	0	0	0
8	12870	51480	60060	36036	12870	2860	390	30	1	0	0
9	48620	218790	291720	204204	87516	24310	4420	510	34	1	0
10	184756	923780	1385670	1108536	554268	184756	41990	6460	646	38	1

The other family of polynomials is

$$a_n(x)=\sum_{k=0}^{n}(-1)^k A(n+1,k+1)x^{n-k}$$

with the triangular array $A(n,m)$ defined for $m=0$ by $A(n,0)=C_n$, and for $n,m\in\mathbf{N}$ with $n\geq m>0$ by the numbers

$$A(n,m)=\frac{1}{2}\binom{n}{m-1}\left[4^{n-m+1}-\binom{2n}{n}\Big/\binom{2(m-1)}{m-1}\right]. \tag{7}$$

The first terms of this triangular array of numbers are shown in Table 2 with $A(n,m)=0$ for $n<m$. Both results (4) and (7) are solutions to recurrence relations which hold for $b_n(x)$ and $a_n(x)$ and their respective coefficients $B(n,m)$ and $A(n,m)$.

Another representation for the polynomials $a_n(x)$ is found to be

$$a_n(x)=\sum_{k=0}^{n}C_k x^k(4x-1)^{n-k}, \tag{8}$$

which shows that the generating function of these polynomials is

$$g_a(x;z):=\sum_{n=0}^{\infty}a_n(x)z^n=\frac{c(xz)}{1+(1-4x)z}. \tag{9}$$

Comparing (5) with (8) yields the following relation between these two types of polynomials

$$b_n(x)=(4x-1)^n-2xa_{n-1}(x),\ n\in\mathbf{N}_0,\text{ with } a_{-1}(x)\equiv 0, \tag{10}$$

and between the coefficients

$$B(n,m)=\binom{n}{m}4^{n-m}-2A(n,m+1). \tag{11}$$

TABLE 2. ***A*(*n*, *m*) Catalan Triangle**

n \ m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	2	5	1	0	0	0	0	0	0	0	0
3	5	22	9	1	0	0	0	0	0	0	0
4	14	93	58	13	1	0	0	0	0	0	0
5	42	386	325	110	17	1	0	0	0	0	0
6	132	1586	1686	765	178	21	1	0	0	0	0
7	429	6476	8330	4746	1477	262	25	1	0	0	0
8	1430	26333	39796	27314	10654	2525	362	29	1	0	0
9	4862	106762	185517	149052	69930	20754	3973	478	33	1	0
10	16796	431910	848830	781725	428772	152946	36646	5885	610	37	1

The triangle of numbers $A(n,m)$ is related to a rectangular array of integers $\hat{A}(n,m)$ with $\hat{A}(0,m)\equiv 1$, $\hat{A}(n,0)=-C_n$ for $n\in\mathbf{N}$, and for $n\geq m\geq 1$ by

$$A(n,m)=-\hat{A}(n-m,m)+2^{2(n-m)+1}\binom{n-1}{m-1}, \tag{12}$$

or with (7) for $m\in\mathbf{N}, n\in\mathbf{N}_0$, by

$$\hat{A}(n,m)=\frac{1}{2}\binom{n+m}{n+1}\left[\binom{2(n+m)}{n+m}\bigg/\binom{2(m-1)}{m-1}-4^{n+1}\frac{m-1}{n+m}\right]. \tag{13}$$

Part of the array $\hat{A}(n,m)$ is shown in Table 3, where it is called $C4(n,m)$.

TABLE 3. ***C*4(*n*, *m*) Catalan Array**

n \ m	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	-1	3	7	11	15	19	23
2	-2	10	38	82	142	218	310
3	-5	35	187	515	1083	1955	3195
4	-14	126	874	2934	7266	15086	27866
5	-42	462	3958	15694	44758	105102	216566
6	-132	1716	17548	80324	259356	679764	1546028
7	-429	6435	76627	397923	1435347	4154403	10338515
8	-1430	24310	330818	1922510	7663898	24281510	65635570
9	-4862	92378	1415650	9105690	39761282	136887322	399429602
10	-16796	352716	6015316	42438076	201483204	749032492	2346750900

It turns out that the m^{th} column of the triangle of numbers $A(n,m)$ for $m=0,1,\dots$ is determined by the generating function

$$c(x)\left(\frac{x}{1-4x}\right)^m.$$

The m^{th} column of the triangle of numbers $B(n,m)$ for $m=0,1,\dots$ is generated by

$$\frac{1}{\sqrt{1-4x}}\left(\frac{x}{1-4x}\right)^m.$$

This fact identifies the infinite dimensional matrices $\mathbf{A}$ and $\mathbf{B}$ as examples of Riordan matrices in the terminology of [7]. The matrix $\hat{\mathbf{A}}$ associated with $\hat{A}(n,m)$ is an example of a Riordan array.

Because differentiation of $c(x)=\sum_{k=0}^{\infty} C_k x^k$ leads to

$$\frac{1}{n!}\frac{d^m c(x)}{dx^n}=\sum_{k=0}^{\infty} C(n,k)x^k,\ \text{with}\ C(n,k):=\frac{1}{n!}\prod_{j=1}^{n}(k+j)C_{n+k}=\frac{(2(n+k))!}{n!\,k!\,(n+k+1)!},\tag{14}$$

where $C(0,k)=C_k$, one finds, together with (3), the following identities for $n\in\mathbf{N}$, $p\in\{0,1,\dots,n-1\}$,

$$(D1)\colon\ \sum_{k=0}^{p}(-1)^k C_k\binom{n}{p-k}\Big/\binom{2(n-p+k)}{n-p+k}=\frac{1}{2}\binom{n}{p+1}\left\{2^{2(p+1)}\Big/\binom{2n}{n}-1\Big/\binom{2(n-p-1)}{n-p-1}\right\}$$
$$=A(n,n-p)\Big/\binom{2n}{n},\tag{15}$$

and for $n\in\mathbf{N}$, $k\in\mathbf{N}_0$,

$$(D2)\colon\ \sum_{j=0}^{n}(-1)^j\left(\binom{n}{j}\Big/\binom{2j}{j}\right)\sum_{l=0}^{k}4^l\binom{n+l-1}{n-1}C_{k+j-l}=C(n,k)\Big/\binom{2n}{n}.\tag{16}$$

The remainder of this paper provides proofs for the above statements.

2. DERIVATIVES

The starting point is equation (1) which can either be verified from the explicit form of the generating function $c(x)$ or by converting the recursion relation (2) for Catalan numbers into an equation for their generating function. A computation of

$$\frac{1}{(n+1)!}\frac{d^{n+1}c(x)}{dx^{n+1}}=\frac{1}{n+1}\frac{d}{dx}\left(\frac{1}{n!}\frac{d^n c(x)}{dx^n}\right),$$

with (3) taken as granted and equation (1), produces the following mixed relations between the quantities $a_n(x)$ and $b_n(x)$ and their first derivatives, valid for $n\in\mathbf{N}_0$,

$$(n+1)a_n(x)=x(1-4x)a'_{n-1}(x)+b_n(x)+n(8x-1)a_{n-1}(x),\tag{17}$$

$$(n+1)b_{n+1}(x)=x(1-4x)b'_n(x)+(-(n+1)+2(1+4n)x)b_n(x),\tag{18}$$

with inputs $a_{-1}(x)\equiv 0$ and $b_0(x)\equiv 1$.

From (18), it is clear by induction that $b_n(x)$ is a polynomial of degree n. Again by induction, the same statement holds for $a_n(x)$ in (17). Therefore, we write, for $n\in\mathbf{N}_0$,

$$a_n(x) = \sum_{k=0}^{n} (-1)^k a(n,k) x^{n-k}, \tag{19}$$

$$b_n(x) = \sum_{k=0}^{n} (-1)^k B(n,k) x^{n-k}, \tag{20}$$

with the triangular arrays of numbers $a(n, k)$ and $B(n, k)$ with row number n and column number $k \leq n$. The triangular array $a(n, k)$ will later be enlarged to another one which will then be called $A(n, k)$.

We first solve $b_n(x)$ in (18) by inserting (20) and deriving the recursion relation for the coefficients $B(n, m)$ after comparing coefficients of x^{n+1}, x^0, and x^{n-k} for $k = 0, 1, \ldots, n-1$.

$$x^{n+1}\colon\ (n+1)B(n+1,0) = 2(2n+1)B(n,0), \tag{21}$$

$$x^{0}\colon\ B(n+1,n+1) = B(n,n), \tag{22}$$

$$x^{n-k}\colon\ (n+1)B(n+1,k+1) = (k+1)B(n,k) + 2(2(n+k)+3)B(n,k+1). \tag{23}$$

With the input $B(0, 0) = 1$, one deduces from (21) for the leading coefficient of $b_n(x)$

$$B(n,0) = 2^n \frac{(2n-1)!!}{n!} = \frac{(2n)!}{n!n!} = \binom{2n}{n}, \tag{24}$$

and from (22)

$$B(n,n) \equiv 1, \text{ i.e., } b_n(0) = (-1)^n. \tag{25}$$

The double factorial $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)$ appeared in (24).

In order to solve (23), we conjecture from Table 1 that, for $n, m \in \mathbf{N}$,

$$B(n,m) = 4B(n-1,m) + B(n-1,m-1), \tag{26}$$

with input $B(n, 0) = \binom{2n}{n}$ from (24).

If we use this conjecture in (23), written with $n \to n-1$, $k \to m-1$, we are led to consider the simple recursion

$$B(n,m) = \frac{n+1-m}{2(2m-1)} B(n,m-1). \tag{27}$$

The solution of this recursion is, for $n, m \in \mathbf{N}_0$,

$$B(n,m) = \frac{1}{2^m(2m-1)!!} \frac{n!}{(n-m)!} \binom{2n}{n} = \frac{m!n!}{(2m)!(n-m)!} \binom{2n}{n} = \binom{2n}{n} \binom{n}{m} \Big/ \binom{2m}{m}. \tag{28}$$

With the Pochhammer symbol $(a)_n := \Gamma(n+a)/\Gamma(a)$, this result can also be written as

$$B(n,m) = ((2m+1)/2)_{n-m} 4^{m-n} / (n-m)!.$$

This result satisfies (21), i.e., (24), as well as (22), i.e., (25). It is also the solution to (23) provided we prove the conjecture (26) using $B(n, m)$ in (28). This can be done by inserting

$$B(n,m) = \frac{(2n)!m!}{(2m)!n!(n-m)!}$$

in (26). Thus, we have proved the following proposition.

Proposition 1: We have

$$b_n(x) = \sum_{k=0}^{n} (-1)^k B(n,k) x^{n-k}, \text{ where } B(n,k) = \binom{2n}{n}\binom{n}{k}\Big/\binom{2k}{k}.$$

This triangle of numbers as shown in Table 1 appears as A046521 in the database [8].

One can derive another explicit representation for the polynomials $b_n(x)$ by using (27) in (20):

$$(1-4x)b_n'(x) + 2(2n-1)b_n(x) + 2\binom{2n}{n}x^n = 0. \tag{29}$$

This leads, together with (18), to the following inhomogeneous recursion relation for $b_n(x)$:

$$b_{n+1}(x) = (4x-1)b_n(x) - 2C_n x^{n+1}, \; b_0(x) \equiv 1. \tag{30}$$

Equation (29) can also be solved as first-order linear and inhomogeneous differential equation for $b_n(x)$.

Proposition 2: We have

$$b_n(x) = -2\sum_{k=0}^{n} C_{k-1} x^k (4x-1)^{n-k},$$

where the C_k's are the Catalan numbers for $k \in \mathbf{N}_0$ and $C_{-1} = -1/2$.

Proof: Iteration of (30). □

Proposition 3: The generating function $g_b(x; z) := \sum_{n=0}^{\infty} b_n(x) x^n$ for $\{b_n(x)\}$ is given by (6).

Proof: The alternative form of $b_n(x)$ given by equation (5) is a convolution of the functional sequences $\{-2C_{k-1}x^k\}_{n\in\mathbf{N}_0}$ and $\{(4x-1)^n\}_{n\in\mathbf{N}_0}$, with generating functions $1-2xzc(xz) = \sqrt{1-4xz}$ and $1/(1+(1-4x)z)$, respectively. Therefore, $g_b(x; z)$ is the product of these two generating functions. □

Comparing this alternative form (5) for $b_n(x)$ with the one given by (20), together with (28), proves the following identity in n and $\lambda := (4x-1)/x$. The term $k = 0$ in the sum (5) has been written separately.

Corollary 1 (convolution of Catalan sequence and the sequence of powers of λ): For $n \in \mathbf{N}$ and $\lambda \neq \infty$,

$$s_{n-1}(\lambda) := \lambda^{n-1}\sum_{k=0}^{n-1}\frac{C_k}{\lambda^k} = \frac{1}{2}\left(\lambda^n - \binom{2n}{n}\sum_{k=0}^{n}(-1)^k(4-\lambda)^k\binom{n}{k}\Big/\binom{2k}{k}\right). \tag{31}$$

Therefore, the generating function for the sequence $s_n(\lambda)$ is

$$g(\lambda; x) := \sum_{n=0}^{\infty} s_n(\lambda)x^n = c(x)/(1-\lambda x).$$

From the generating function, the recurrence relation is found to be $s_n(\lambda) = \lambda s_{n-1}(\lambda) + C_n$, $s_{-1}(\lambda) \equiv 0$. The connection with the polynomial $b_n(x)$ is

$$s_n(\lambda) = \frac{1}{2}(\lambda^{n+1} - (4-\lambda)^{n+1} b_{n+1}(1/(4-\lambda))).$$

The case $\lambda = 0$ $(x = 1/4)$ is also covered by this formula. It produces from $s_n(0) = C_n$ the following identity.

Example 1: Case $\lambda = 0$ $(x = 1/4)$,

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 4^k \Big/ \binom{2k}{k} = \frac{1}{2n-1}. \tag{32}$$

This identity occurs in one of the exercises 2.7, 2, page 32 of [4].

We note that from (5) one has $-2b_{n+1}(1/4) = C_n / 4^n$. The large n behavior of this sequence is known to be (see [2], Exercise 9.60):

$$C_n / 4^n \sim \frac{1}{\sqrt{\pi}} \frac{1}{n^{3/2}}.$$

If one puts $4x - 1 = x$, i.e., $x = 1/3$, in (5), one can identify the partial sum $s_n(1)$ of Catalan numbers:

$$s_n(1) := \sum_{k=0}^{n} C_k = \frac{1}{2}(1 - 3^{n+1} b_{n+1}(1/3)). \tag{33}$$

This sequence $\{1, 2, 4, 9, 23, 65, 197, 626, 2056, \ldots\}$ appears as A014137 in the web encyclopedia [8]. If one puts $\lambda - 1$ in Corollary 1, one also finds the following example.

Example 2:

$$2s_{n-1}(1) = 1 + \binom{2n}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 3^k \Big/ \binom{2k}{k}. \tag{34}$$

Another interesting example is the case $\lambda = 4$ $(x = \infty)$. Here one finds a simple result for the convolution of Catalan's sequence with powers of 4.

Example 3: $\lambda = 4$ $(x = \infty)$,

$$2s_{n-1}(4) = 4^n - \binom{2n}{n}. \tag{35}$$

This sequence $\{1, 5, 22, 93, 386, 1586, 6476, \ldots\}$ appears in the book [8] as Nr. 3920 and as A000346 in the web encyclopedia [8]. It will show up again in this work as $A(n+1, 1)$, the second column in the $A(n, m)$ triangle (see Table 2).

The sequence for $\lambda = -1$ $(x = 1/5)$ is also nonnegative, as can be seen by writing

$$s_{2k}(-1) = C_2 + \sum_{l=2}^{k} (C_{2l} - C_{2l-1}) \quad \text{for } k \in \mathbf{N}$$

and

$$s_{2k+1}(-1) = \sum_{l=1}^{k} (C_{2l+1} - C_{2l}),$$

and using

$$\Delta C_n := C_n - C_{n-1} = 3\frac{n-1}{n+1} C_{n-1} \geq 0.$$

This is the sequence $\{1, 0, 2, 3, 11, 31, 101, 328, 1102, 3760, \ldots\}$ which appears now as A032357 in the web encyclopedia [8].

Recursion (26) for $B(n,m)$ can be transformed into an equation for the generating function for the sequence appearing in the m^{th} column of the $B(n,m)$ triangle

$$G_B(m;x) := \sum_{n=m}^{\infty} B(n,m)x^n, \tag{36}$$

with input

$$G_B(0;x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

the generating function for the central binomial numbers. So (26) implies, for $m \in \mathbf{N}_0$,

$$G_B(m;x) = \left(\frac{x}{1-4x}\right)^m \frac{1}{\sqrt{1-4x}}. \tag{37}$$

For $x\frac{d}{dx}G_B(m;x)$, see (53). Therefore, we have proved the following proposition.

Proposition 4 (column sequences of the B(n, m) triangle): The sequence $\{B(n,m)\}_{n=m}^{\infty}$, defined for fixed $m \in \mathbf{N}_0$ and $n \in \mathbf{N}_0$ by (28), is the convolution of the central binomial sequence

$$\left\{\binom{2k}{k}\right\}_{k \in \mathbf{N}_0}$$

and the m^{th} convolution of the (shifted) power sequence $\{0, 1, 4^1, 4^2, \ldots\}$.

Note 1: The infinite dimensional matrix $\mathbf{B}$ with elements $B(n,m)$ given for $n \ge m \ge 0$ by (28) and $B(n,m) \equiv 0$ for $n < m$ is an example of a Riordan matrix [7]. With the notation of this reference,

$$\mathbf{B} = \left(\frac{1}{\sqrt{1-4x}}, \frac{x}{1-4x}\right).$$

Note 2:(Sheffer-type identities from Riordan matrices): Triangular Riordan matrices

$$\mathbf{M} = (M_{i,j})_{i \ge j \ge 0} = (g(x), f(x)),$$

$M_{i,j} = 0$ for $j > i$, in the notation of [7], lead to polynomials that satisfy Sheffer-type identities (see [5] and its references, and also [1]),

$$S_n(x+y) = \sum_{k=0}^{n} S_k(y)P_{n-k}(x) = \sum_{k=0}^{n} P_k(y)S_{n-k}(x), \tag{38}$$

$$P_n(x+y) = \sum_{k=0}^{n} P_k(y)P_{n-k}(x) = \sum_{k=0}^{n} P_k(x)P_{n-k}(y), \tag{39}$$

where the polynomials $S_n(x)$ and $P_n(x)$ are defined by

$$S_n(x) = \sum_{m=0}^{n} M_{n,m}\frac{x^m}{m!},\ n \in \mathbf{N}_0,\ \ P_n(x) = \sum_{m=1}^{n} P_{n,m}\frac{x^m}{m!},\ n \in \mathbf{N},\ P_0(x) \equiv 1, \tag{40}$$

with $P_{n,m} := [z^n](f^m(z))$, $n \ge m \ge 1$. Here $g(x)$ defines the first column of $\mathbf{M}$: $M_{n,0} = [x^n]g(x)$.

If one uses $s_n(x) := n!\,S_n(x)$ and $p_n(x) := n!\,P_n(x)$, one obtains the Sheffer identities (also called binomial identities) treated in [5]. Then $s_n(x)$ is Sheffer for $(1/g(\bar{f}(t)), \bar{f}(t))$, and $p_n(x)$ is associated to $\bar{f}(t)$—or Sheffer for $(1, \bar{f}(t))$—in the terminology of [5]. Here $\bar{f}(t)$ stands for the inverse of $f(t)$ as a function.

Let us give the relation between $g_b(x; z)$ and $G_B(m; x)$.

Proposition 5: We have

$$g_b(x; z) = \sum_{m=0}^{\infty} (-1)^m G_B(m; xz)\left(\frac{1}{x}\right)^m. \tag{41}$$

Proof: One inserts the value of $b_n(x)$ given in (20) into the definition (6) of $g_b(x; z)$ and rewrites the Cauchy sum as two infinite sums which are then interchanged. Finally, the definition of $G_B(m; x)$ in (36) is used. □

One can check (41) by using the explicit form of $G_B(m; xz)$ given in (36) and comparing with (6).

In a similar vein, we can solve $a_n(x)$ in (17) with $b_n(x)$ given by (20) and (28). The coefficients $a(n, k)$, defined by (19), have to satisfy, after comparing coefficients of x^n, x^0, and x^{n-k} for $k = 1, 2, \ldots, n-1$ and $n \in \mathbf{N}_0$:

$$x^n\!: \; a(n, 0) = 4a(n-1, 0) + C_n, \tag{42}$$

$$x^0\!: \; (n+1)a(n, n) = 1 + na(n-1, n-1), \tag{43}$$

$$x^{n-k}\!: \; (n+1)a(n, k) = ka(n-1, k-1) + 4(n+1+k)a(n-1, k) + B(n, k). \tag{44}$$

In (42) we have used (24), i.e., $B(n, 0) = (n+1)C_n$; in (43) we have used (25), i.e., $B(n, n) \equiv 1$. From (42) one finds, with input $a(0, 0) = 1$,

$$a(n, 0) = \sum_{k=0}^{n} C_k 4^{n-k}, \tag{45}$$

and from (43),

$$a(n, n) \equiv 1 \;\text{ or }\; a_n(0) = (-1)^n. \tag{46}$$

Note that $a(n, 0) = s_n(4)$ of (31) with solution (35). It is convenient to define $a(n-1, -1) := C_n$, $n \in \mathbf{N}_0$. Then the sequence $\{a(n, 0)\}_{-1}^{\infty}$ is, with $a(-1, 0) := 0$, the convolution of the sequence $\{a(k, -1)\}_{-1}^{\infty}$ and the shifted power sequence $\{0, 1, 4^1, 4^2, \ldots\}$. Before solving (44), with $B(n, k)$ from (28) inserted, we add to the triangular array of numbers $a(n, m)$ the $m = -1$ column and an extra row for $n = -1$, and define a new enlarged triangular array for $n, m \in \mathbf{N}_0$ as

$$A(n, m) := a(n-1, m-1) \tag{47}$$

with $A(n, 0) = a(n-1, -1) = C_n$ and $A(0, m) = a(-1, m-1) = \delta_{0,m}$. An inspection of the $A(n, m)$ triangular array, partly depicted in Table 2, leads to the conjecture

$$A(n, m) = 4A(n-1, m) + A(n-1, m-1), \tag{48}$$

with $A(n, 0) = C_n$ and $A(n, m) \equiv 0$ for $n < m$. This recursion relation can be used to extend the array $A(n, m)$ to negative integer values of m. This conjecture is correct for $A(n+1, 1) = a(n, 0)$ found in (45), as well as for $A(n+1, n+1) = a(n, n) \equiv 1$ known from (46). The generating function for the sequence appearing in the m^{th} column,

$$G_A(m;x) := \sum_{n=m}^{\infty} A(n,m)x^n, \tag{49}$$

satisfies, due to (48), $G_A(m;x) = \frac{x}{1-4x}G_A(m-1;x)$, remembering that $A(m-1,m)\equiv 0$ and that $G_A(0;x) = c(x)$. Therefore,

$$G_A(m;x) = \left(\frac{x}{1-4x}\right)^m c(x). \tag{50}$$

Note 3: The infinite dimensional matrix **A** with elements $A(n,m)$ given for $n \ge m \ge 0$ by (48) and $A(n,m)\equiv 0$ for $n<m$ is another example of a Riordan matrix, written in the notation of [7] as $(c(x), x/(1-4x))$.

Because of (37) and $\sqrt{1-4x}\,c(x) = 2-c(x)$, these generating functions of the conjectured $A(n,m)$ column sequences obey

$$G_A(m;x) = (2-c(x))G_B(m;x). \tag{51}$$

If we use the conjecture (48) in (44), which is written with (47) in the form

$$(n+1)A(n+1,m+1) = mA(n,m) + 4(n+m+1)A(n,m+1) + B(n,m)$$

for $n \in \mathbf{N}_0$, $m \in \{1, 2, \dots, n-1\}$, we have

$$mA(n+1,m+1) - (n+1)A(n,m) + B(n,m) = 0. \tag{52}$$

This recursion relation can be written with the help of the generating functions (36) and (49) as

$$\left(x\frac{d}{dx}+1\right)G_A(m;x) - \frac{m}{x}G_A(m+1;x) = G_B(m;x), \tag{53}$$

or with (50) (i.e., the conjecture) as

$$\left(x\frac{d}{dx}+1-\frac{m}{1-4x}\right)G_A(m;x) = G_B(m;x). \tag{54}$$

Together with (51), this means

$$x\frac{d}{dx}((2-c(x))G_B(m;x)) = \left[\left(\frac{m}{1-4x}-1\right)(2-c(x))+1\right]G_B(m;x). \tag{55}$$

If we can prove this equation with $G_B(x)$ given by (37), we have shown that (44) is equivalent to the conjecture (48). In order to prove (55), we first compute from (37) for $m \in \mathbf{N}_0$,

$$x\frac{d}{dx}G_B(m;x) = \left(2+\frac{m}{x}\right)G_B(m+1;x) = \frac{2x+m}{1-4x}G_B(m;x). \tag{56}$$

With this result, (55) reduces to

$$\left(-xc'(x) + (2-c(x))\frac{1-2x}{1-4x} - 1\right)G_B(m;x) = 0, \tag{57}$$

and with (1), the factor in front of $G_B(m;x)$ vanishes identically for $x \ne 1/4$. Therefore, we have proved the following two propositions concerning the column sequences of the $A(n,m)$ triangular array and the triangular $A(n,m)$ array, respectively.

Proposition 6: The triangular array of numbers $A(n,m)$, defined for $n, m \in \mathbf{N}_0$ by equation (48), $A(n,0) = C_n$, $A(n,m) \equiv 0$ for $n < m$ has as its m^{th} column sequence $\{A(n,m)\}_{n=m}^{\infty}$ the convolution of the Catalan sequence and the m^{th} convolution of the shifted power sequence $\{0, 1, 4^1, 4^2, \ldots\}$.

Proof: Use (50) with (49). □

Proposition 7: The triangular array $A(n,m)$ of Proposition 6 coincides with the one defined by (47) and (42), (43) and (44) with $B(n,m)$ given by (28).

Proof: On one hand, $a(n,0) = A(n+1,1)$ and $a(n,n) = A(n+1,n+1) \equiv 1$ of (42) and (43), i.e., (45) and (46), respectively, satisfy (48). On the other hand, (44) is rewritten with the aid of (47) as (52), and (52) has been proved by (53)-(57). □

Alternatively, one can use the now proven conjecture (48), together with (47), in (44) and derive for $n \in \mathbf{N}_0$, $m \in \mathbf{N}_0$,

$$4ma(n-1,m) = (n+1-m)a(n-1,m-1) - B(n,m). \tag{58}$$

This is written in terms of the polynomials $a_{n-1}(x)$ of (19) and $b_n(x)$ of (20) as

$$x(1-4x)a'_{n-1}(x) + (1-4x+4nx)a_{n-1}(x) - \binom{2n}{n}x^n + b_n(x) = 0. \tag{59}$$

With this result, (17) becomes an inhomogeneous recursion relation for $a_n(x)$:

$$a_n(x) = (4x-1)a_{n-1}(x) + C_n x^n, \; a_0(x) \equiv 1. \tag{60}$$

Moreover, (59) can also be considered as an inhomogeneous linear differential equation for $a_{n-1}(x)$ with given $b_n(x)$. To find the solution this way is, however, a bit tedious. Let us give an alternative form for $a_n(x)$ in the following proposition.

Proposition 8: The solution of the recursion relation (60) is given by (8).

Proof: Iteration of (60). □

Next, we give a corollary.

Corollary 2: The generating function $g_a(x;z) := \sum_{n=0}^{\infty} a_n(x) z^n$ is given by (9).

Proof: Equation (8) above shows that $a_n(x)$ is a convolution of the functional sequences $\{C_k x^k\}_{n \in \mathbf{N}_0}$ and $\{(4x-1)^k\}_{k \in \mathbf{N}_0}$ with generating functions $c(xz)$ and $1/(1+(1-4x)z)$. Therefore, $g_a(x;z)$ is the product of these generating functions. □

We now have a relation between $g_a(x;z)$ and $G_A(m;x)$.

Proposition 9:

$$g_a(x;z) = \frac{1}{1-4xz} \sum_{m=0}^{\infty} (-1)^m G_A(m;xz) \left(\frac{1}{x}\right)^m. \tag{61}$$

Proof: Analogous to the proof of Proposition 5. □

One can check (61) by putting in the explicit form (50) of $G_A(m;x)$ and compare with (9). Let us state the relation between $b_n(x)$ and $a_{n-1}(x)$ as Proposition 10.

Proposition 10: For $n \in \mathbf{N}_0$ and $a_{-1}(x) \equiv 0$, the relation between $b_n(x)$ and $a_{n-1}(x)$ is given by (10).

Proof: The alternative expressions (5) and (8) for these two families of polynomials are used. One splits off the $k = 0$ term in (5) with $C_{-1} = -1/2$ from the sum and shifts the summation variable. □

Corollary 3: The coefficients of the triangular arrays $A(n, m)$ and $B(n, m)$ are related as given by (11).

Proof: The relation (10) between the polynomials is, with the help of (19) and (20), written for the coefficients $a(n-1, m)$, or by (47) for $A(n, m+1)$ and $B(n, m)$. □

It remains to compute the explicit expression for the coefficients $a(n, k)$ of $a_n(x)$ defined by (19). Because of (47), it suffices to determine $A(n, m)$.

Corollary 4: The triangular array numbers $A(n, m)$ are given explicitly by formula (7).

Proof: The formula (4) written for $B(n, m-1)$ is used in relation (11). □

Note 4: This formula for $A(n, m)$ satisfies indeed the recursion relation (48) with the given input. The first term,

$$\frac{1}{2}4^{n-m+1}\binom{n}{m-1}$$

satisfies it because of the binomial identity

$$\binom{n}{m-1} = \binom{n-1}{m-1} + \binom{n-1}{m-2}.$$

For the second term of $A(n, m)$ in (7) one has to prove

$$\binom{n}{m-1}\binom{2n}{n} = 4\binom{n-1}{m-1}\binom{2(n-1)}{n-1} + \binom{n-1}{m-2}\binom{2(n-1)}{n-1}\frac{2(2m-3)}{m-1},$$

or after division by $\binom{2(n-1)}{n-1}$,

$$\frac{2n-1}{n}\binom{n}{m-1} = 2\binom{n-1}{m-1} + \binom{n-1}{m-2}\frac{2m-3}{m-1},$$

which reduces to the trivial identity $2n-1 = 2(n-m+1)+2m-3$. Both terms together, i.e., (7), satisfy the input $A(n, n) \equiv 1$.

Note 5: $A(n, m)$ was found originally after iteration in the form (with $n \geq m > 0$ and $(-1)!! := 1$)

$$A(n, m) = 2 \cdot 4^{n-m}\binom{n}{m-1} - \frac{\prod_{k=1}^{m}(2(n-m)+2k-1)}{(2m-3)!!}C_{n-m}. \tag{62}$$

$A(n, 0) = C_n$. It is easy to establish the equivalence with (7).

In the original derivation of the formula (7) for $A(n, m)$, it turned out to be convenient to introduce a rectangular array of integers $\hat{A}(n, m)$ for $n, m \in \mathbf{N}_0$ as follows: $\hat{A}(0, m) \equiv 1$, $\hat{A}(n, 0) := -C_n$ for $n \in \mathbf{N}$, and for $m \in \mathbf{N}$ and $n \in \mathbf{N}_0$, $\hat{A}(n, m)$ is defined by (12) or, equivalently, by (13). The $A(n, m)$ recursion (48) translates (with the help of the Pascal-triangle identity) into

$$\hat{A}(n,m) = 4\hat{A}(n-1,m) + \hat{A}(n,m-1). \tag{63}$$

This leads, after iteration and use of $\hat{A}(0,m) \equiv 1$ from (12) with $A(n,n) \equiv 1$, to

$$\hat{A}(n,m) = 4^n \sum_{k=0}^{n} \hat{A}(k,m-1)/4^k. \tag{64}$$

Thus, the following proposition describes column sequences of the $\hat{A}(n,m) \equiv C4(n,m)$ array.

Proposition 11: The m^{th} column sequence of the $\hat{A}(n,m)$ array, $\{\hat{A}(n,m)\}_{n\in\mathbb{N}_0}$, is the convolution of the sequence $\{\hat{A}(n,0)\}_{n\in\mathbb{N}_0} = \{1,-1,-2,-5,\dots\}$, generated by $2-c(x)$, and the m^{th} convolution of the power sequence $\{4^k\}_{k\in\mathbb{N}_0}$.

Proof: Iteration of (64) with the $\hat{A}(n,0)$ input. □

Corollary 5: The ordinary generating function of the m^{th} column sequence of the $\hat{A}(n,m)$ array (13) is given by

$$G_{\hat{A}}(m;x) := \sum_{n=0}^{\infty} \hat{A}(n,m)x^n = (2-c(x))\left(\frac{1}{1-4x}\right)^m \tag{65}$$

for $m \in \mathbb{N}_0$.

Proof: Use Proposition 11 written for generating functions. □

Because of the convolution of the (negative) Catalan sequence with powers of 4, we shall call this $\hat{A}(n,m)$ array also $C4(n,m)$. A part of it is shown in Table 3 above. The second column sequence is given by

$$\hat{A}(n,1) \equiv C4(n,1) = \binom{2n+1}{n}$$

and appears as nr. 2848 in the book [8], or as A001700 in the web encyclopedia [8]. The sequence of the third column $\{\hat{A}(n,2) \equiv C4(n,2)\}_{n\in N_0} = \{1,7,38,187,\dots\}$ is, from (64) and (62) with (12), determined by

$$4^n \sum_{k=0}^{n} \binom{2k+1}{k} \Big/ 4^k = (2n+3)(2n+1)C_n - 2^{2n+1},$$

and is listed as A000531 in the web encyclopedia [8]. There the fourth column sequence is now listed as A029887.

Note 6: The infinite dimensional lower triangular matrix $\tilde{\mathbf{A}}$ related to the array $\hat{A}(n,m) \equiv C4(n,m)$ by $\tilde{A}(n,m) := \hat{A}(n-m,m+1)$ for $n \ge m \ge 0$ and $\tilde{A}(n,m) := 0$ for $n < m$ is again an example of a Riordan matrix [7]. In the notation of [7], $\tilde{\mathbf{A}} = (c(x)/\sqrt{1-4x}, x/\sqrt{1-4x})$.

Finally, we derive identities by using, for $n \in \mathbb{N}_0$, equation (14) for the left-hand side of (3) and the results for $a_{n-1}(x)$ and $b_n(x)$ for the right-hand side. Because there are no negative powers of x on the left-hand side of (3), such powers have to vanish on the right-hand side. This leads to the first family of identities. Because

$$(1-4x)^{-n} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} 4^k x^k,$$

with Pochhammer's symbol defined after (28), this means that $x^p](a_{n-1}(x)+b_n(x)c(x))$, the coefficient proportional to x^p, has to vanish for $p=0,1,\ldots,n-1$, $n\in\mathbf{N}$. This requirement reads

$$(-1)^{n-1-p}a(n-1,n-1-p)+\sum_{k=0}^{p}(-1)^{n-k}B(n,n-k)\,C_{p-k}\equiv 0. \tag{66}$$

The sum is restricted to $k\le p$ $(<n)$ because no number C_l with negative index is found in $c(x)$. Inserting the known coefficients produces (15).

Proposition 12: For $n\in\mathbf{N}$ and $p\in\{0,1,\ldots,n-1\}$ identity ($D1$), given by (15), holds.

Proof: With (47), (66) becomes

$$\sum_{k=0}^{p}(-1)^{p-k}C_{p-k}B(n,n-k)=A(n,n-p), \tag{67}$$

which is ($D1$) of (15) if the summation index k is changed into $p-k$, and the symmetry of the binomial coefficients is used. □

Example 4: Take $p=n-1\in\mathbf{N}_0$:

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{1}{2k+1}=4^n\Big/\binom{2n}{n}-1=2A(n,1)\Big/\binom{2n}{n}. \tag{68}$$

With this identity we have found a sum representation for the convolution of the Catalan sequence and powers of 4:

$$s_{n-1}(4):=4^{n-1}\sum_{k=0}^{n-1}C_k/4^k=\frac{1}{2}\binom{2n}{n}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{1}{2k+1}$$

[cf. (35) with (31)].

The second family of identities, ($D2$) of (16), results from comparing powers x^k with $k\in\mathbf{N}_0$ on both sides of (3) after expansion of $(1-4x)^{-n}$ as given above in the text before (66). Only the second term $b_n(x)c(x)$ contributes because $a_{n-1}(x)/x^n$ has only negative powers of x. Thus, with definition (14), one finds, for $k\in\mathbf{N}_0$ and $n\in\mathbf{N}$,

$$C(n,k)=\sum_{l=0}^{k}\frac{(n)_l4^l}{l!}\sum_{j=0}^{n}(-1)^{n-j}B(n,n-j)C_{n-j+k-l}, \tag{69}$$

which is, after interchange of the summations and insertion of $B(n,n-j)$ from (4), the desired identity ($D2$) if also the summation index j is changed to $n-q$.

Thus, we have shown

Proposition 13: For $k\in\mathbf{N}_0$ and $n\in\mathbf{N}$, identity ($D2$) of (16) with $C(n,k)$ defined by (14) holds true.

Example 5: Take $k=0$, $n\in\mathbf{N}$. Then we have

$$\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\equiv 1, \tag{70}$$

which is elementary.

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