FAMILIES OF IDENTITIES INVOLVING SUMS OF POWERS OF THE FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The well-known identity

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \tag{1.1}$$

has as its Lucas counterpart

$$L_{n+1}^2 + L_n^2 = 5F_{2n+1}. \tag{1.2}$$

Indeed, since $L_{n+1} = F_{n+2} + F_n = F_{n+1} + 2F_n$ and $L_n = F_{n+1} + F_{n-1} = 2F_{n+1} - F_n$, (1.2) follows from (1.1).

As analogs of (1.1) and (1.2) we have

$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n} \quad \text{(see [7], p. 11)}, \tag{1.3}$$

and

$$L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n} \quad \text{(see [4], p. 165)}. \tag{1.4}$$

One aim of this paper is to generalize (1.1)-(1.4). These identities belong to a family of similar identities that involve sums of m^{th} powers ($m \in \mathbf{Z}$, $m \geq 2$) of Fibonacci (Lucas) numbers. As usual, $\mathbf{Z}$ denotes the set of integers. Our second aim is to state a conjecture that proposes a generalization of this family of identities. We state our conjecture in Section 4.

2. PRELIMINARY RESULTS

We require some preliminary results. For $m, n \in \mathbf{Z}$,

$$F_{-n} = (-1)^{n+1} F_n, \tag{2.1}$$

$$L_{-n} = (-1)^n L_n, \tag{2.2}$$

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n, \tag{2.3}$$

$$L_{m+n+1} = F_{m+1}L_{n+1} + F_m L_n, \tag{2.4}$$

$$F_{3(n+2)} = 4F_{3(n+1)} + F_{3n}, \tag{2.5}$$

$$F_{n+4}^3 = 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3. \tag{2.6}$$

Identities (2.1)-(2.4) can be found on pages 28 and 59 in Hoggatt [3]. Identity (2.5) is a special case of (2.3) in Shannon and Horadam [5], and (2.6) occurs as (40) in Long [4]. The recurrences in (2.5) and (2.6) are also satisfied by the Lucas numbers.

We also require the following lemmas.

Lemma 1: $3F_{3(m+3)} + 6F_{3(m+2)} - 3F_{3(m+1)} - F_{3m} = F_{3(m+4)}$, $m \in \mathbf{Z}$.

Proof: By (2.5) we have

$$\begin{aligned}
&3F_{3(m+3)}+6F_{3(m+2)}-3F_{3(m+1)}-F_{3m}\\
&=3F_{3(m+3)}+6F_{3(m+2)}-3F_{3(m+1)}-(F_{3(m+2)}-4F_{3(m+1)})\\
&=3F_{3(m+3)}+5F_{3(m+2)}+F_{3(m+1)}\\
&=3F_{3(m+3)}+5F_{3(m+2)}+F_{3(m+3)}-4F_{3(m+2)}\\
&=4F_{3(m+3)}+F_{3(m+2)}=F_{3(m+4)}. \ \square
\end{aligned}$$

Lemma 2: Let $k,n\in\mathbf{Z}$ with $0\le n\le 3$. Then

$$F_{3k+1}F_{n+k+1}^3+F_{3k+2}F_{n+k}^3-F_{n-2k-1}^3=F_{3k+1}F_{3k+2}F_{3n}. \tag{2.7}$$

Proof: We give the proof only for $n=3$, since the proofs of the remaining cases are similar. For the case $n=3$, identity (2.1) shows that we need to prove

$$F_{3k+1}F_{k+4}^3+F_{3k+2}F_{k+3}^3+F_{2k-2}^3-34F_{3k+1}F_{3k+2}=0. \tag{2.8}$$

This is easily proved by using a powerful technique developed recently by Dresel [1]. Following Dresel, we see that (2.8) is a homogeneous equation of degree 6 in the variable k. Therefore, to prove its validity for all integers k, it suffices to verify its validity for seven different values of k, say $0\le k\le 6$. But (2.8) is easily verified for these values, and so it is true for all integers k. $\square$

3. THE MAIN RESULTS

Our generalizations of (1.1) and (1.2) are contained in the following theorem.

Theorem 1: For $k,n\in\mathbf{Z}$,

$$F_{n+k+1}^2+F_{n-k}^2=F_{2k+1}F_{2n+1} \tag{3.1}$$

and

$$L_{n+k+1}^2+L_{n-k}^2=5F_{2k+1}F_{2n+1}. \tag{3.2}$$

Proof: Using (2.1) and (2.3), we obtain $F_{n+k+1}=F_nF_k+F_{n+1}F_{k+1}$ and $F_{n-k}=F_{n-(k+1)+1}=(-1)^k(F_nF_{k+1}-F_{n+1}F_k)$, so that

$$\begin{aligned}
F_{n+k+1}^2+F_{n-k}^2&=F_k^2F_n^2+F_k^2F_{n+1}^2+F_{k+1}^2F_n^2+F_{k+1}^2F_{n+1}^2\\
&=(F_{k+1}^2+F_k^2)(F_{n+1}^2+F_n^2)=F_{2k+1}F_{2n+1}.
\end{aligned}$$

To prove (3.2), we use (2.4) to show that $L_{n+k+1}=L_nF_k+L_{n+1}F_{k+1}$ and $L_{n-k}=L_{-(k+1)+n+1}=(-1)^k(L_nF_{k+1}-L_{n+1}F_k)$, and proceed similarly. $\square$

When $k=0$, (3.1) and (3.2) reduce to (1.1) and (1.2), respectively. For our next theorem, we use a "traditional" approach to prove the first part and, in contrast, the method of Dresel to prove the second part.

Theorem 2: For $k,n\in\mathbf{Z}$,

$$F_{3k+1}F_{n+k+1}^3+F_{3k+2}F_{n+k}^3-F_{n-2k-1}^3=F_{3k+1}F_{3k+2}F_{3n} \tag{3.3}$$

and

$$F_{3k+1}L_{n+k+1}^3+F_{3k+2}L_{n+k}^3-L_{n-2k-1}^3=5F_{3k+1}F_{3k+2}L_{3n}. \tag{3.4}$$

Proof: We proceed by induction on n. Suppose identity (3.3) holds for $n = m$, $m+1$, $m+2$, and $m+3$. Then

$$-(F_{3k+1}F^3_{m+k+1} + F_{3k+2}F^3_{m+k} - F^3_{m-2k-1}) = -F_{3k+1}F_{3k+2}F_{3m},$$
$$-3(F_{3k+1}F^3_{(m+1)+k+1} + F_{3k+2}F^3_{(m+1)+k} - F^3_{(m+1)-2k-1}) = -3F_{3k+1}F_{3k+2}F_{3(m+1)},$$
$$6(F_{3k+1}F^3_{(m+2)+k+1} + F_{3k+2}F^3_{(m+2)+k} - F^3_{(m+2)-2k-1}) = 6F_{3k+1}F_{3k+2}F_{3(m+2)},$$
$$3(F_{3k+1}F^3_{(m+3)+k+1} + F_{3k+2}F^3_{(m+3)+k} - F^3_{(m+3)-2k-1}) = 3F_{3k+1}F_{3k+2}F_{3(m+3)}.$$

Adding, and making use of (2.6), we obtain

$$\begin{aligned} &F_{3k+1}F^3_{(m+4)+k+1} + F_{3k+2}F^3_{(m+4)+k} - F^3_{(m+4)-2k-1} \\ &= F_{3k+1}F_{3k+2}[3F_{3(m+3)} + 6F_{3(m+2)} - 3F_{3(m+1)} - F_{3m}] \\ &= F_{3k+1}F_{3k+2}F_{3(m+4)} \quad \text{(by Lemma 1)}, \end{aligned}$$

and so (3.3) is true for $n = m+4$. But Lemma 2 shows that (3.3) holds for $n = 0, 1, 2$, and 3, and so it holds for $n = 4$ and, by induction, for all integers $n \geq 0$.

To establish (3.3) for all integers $n < 0$, it suffices to replace n by $-n$, and to prove that the resulting identity holds for all integers $n > 0$. That is, it suffices to prove that

$$F^3_{n+2k+1} + (-1)^k F_{3k+2}F^3_{n-k} + (-1)^{k+1}F_{3k+1}F^3_{n-k-1} = F_{3k+1}F_{3k+2}F_{3n} \tag{3.5}$$

holds for all integers $n > 0$. After making use of (2.1) to simplify (3.5) for $0 \leq n \leq 3$, the equivalent of Lemma 2 is established as before, and the proof proceeds as above.

Following Dresel, we see that (3.4) is a homogeneous equation of degree 3 in the variable n. Therefore, to prove its validity for all integers n, it suffices to verify its validity for four different values of n, say $0 \leq n \leq 3$. For $n = 3$, (3.4) becomes

$$F_{3k+1}L^3_{k+4} + F_{3k+2}L^3_{k+3} - L^3_{2k-2} = 380F_{3k+1}F_{3k+2} \quad \text{[by (2.2)]}. \tag{3.6}$$

But (3.6) is a homogeneous equation of degree 6 in the variable k. Therefore, to prove its validity for all integers k, it suffices to verify its validity for seven different values of k, say $0 \leq k \leq 6$. This is easy to verify. We proceed similarly for the other three values of n, and this completes the proof of Theorem 2. □

When $k = 0$, (3.3) and (3.4) reduce to (1.3) and (1.4), respectively.

4. A CONJECTURE FOR HIGHER POWERS

The identity

$$\sum_{j=0}^{m} (-1)^{\frac{j(j+3)}{2}} \begin{bmatrix} m \\ j \end{bmatrix} F^{m+1}_{n+m-j} = F_1 \ldots F_m F_{(m+1)(n+\frac{m}{2})} \tag{4.1}$$

is a special case of identity (5) of Torretto and Fuchs [6]. Here $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]$ is the Fibonomial coefficient defined for integers $m \geq 0$ by

$$\begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} 0 & j < 0 \text{ or } j > m, \\ 1 & j = 0, m, \\ \frac{F_m F_{m-1} \ldots F_{m-j+1}}{F_1 F_2 \ldots F_j} & 0 < j < m. \end{cases}$$

For an excellent discussion on generalized binomial coefficients, and a comprehensive list of references, see Gould [2].

In identity (4.1), $m=1$ yields (1.1) and $m=2$ yields an identity equivalent to (1.3). For $m=3$ and $m=4$, identity (4.1) becomes, respectively,

$$F_{n+3}^4+2F_{n+2}^4-2F_{n+1}^4-F_n^4=2F_{4n+6}, \tag{4.2}$$

and

$$F_{n+4}^5+3F_{n+3}^5-6F_{n+2}^5-3F_{n+1}^5+F_n^5=6F_{5n+10}. \tag{4.3}$$

Our generalizations in Theorems 1 and 2 prompted us to search for similar generalizations of (4.2) and (4.3) and their Lucas counterparts. We accomplished this by introducing the parameter k, assuming the existence of an identity of the required shape, and solving systems of simultaneous equations to find the coefficients. Indeed, after employing this constructive approach on several more instances of (4.1), we were led to a conjecture on a generalization of (4.1). First, we need some notation. Write, for example, $(F_5)_{(4)}=F_5F_4F_3F_2$ and $(F_4)_{(6)}=F_4F_3F_2F_1F_{-1}F_{-2}$. In general, we take $(F_n)_{(m)}$ to be the "falling" factorial, which begins at F_n for $n\neq 0$, and is the product of m Fibonacci numbers *excluding* F_0. Define $(F_0)_{(0)}=1$ and, for $m\geq 1$, $(F_0)_{(m)}=F_{-1}\dots F_{-m}$. We now state our conjecture in two parts.

Conjecture: Let $k,m,n\in\mathbf{Z}$ with $m\geq 1$. Then:

(a)
$$\sum_{j=0}^{m-1}\frac{F_{n+k+m-j}^{m+1}}{(F_{m-1-j})_{(m-1)}F_{(m+1)k+m-j}}+(-1)^{\frac{m(m+3)}{2}}\frac{F_{n-mk}^{m+1}}{\prod_{j=1}^{m}F_{(m+1)k+j}}=F_{(m+1)\left(n+\frac{m}{2}\right)}.$$

(b) To obtain the Lucas counterpart of (a), we first replace each occurrence of F in the numerators on the left by L. Then, if m is even, we replace the right side by

$$5^{\frac{m}{2}}L_{(m+1)\left(n+\frac{m}{2}\right)}.$$

If m is odd, we replace the right side by

$$5^{\frac{m+1}{2}}F_{(m+1)\left(n+\frac{m}{2}\right)}$$

When $m=1$ our conjecture yields (3.1) and (3.2), and when $m=2$ it yields (3.3) and (3.4). For $k=0$ we claim that part (a) of our conjecture reduces to (4.1), but this is not obvious. It is useful to consider an example. If we take $m=4$, part (a) becomes

$$\frac{F_{n+k+4}^5}{2F_{5k+4}}+\frac{F_{n+k+3}^5}{F_{5k+3}}-\frac{F_{n+k+2}^5}{F_{5k+2}}-\frac{F_{n+k+1}^5}{2F_{5k+1}}+\frac{F_{n-4k}^5}{F_{5k+1}\dots F_{5k+4}}=F_{5n+10}. \tag{4.4}$$

Now putting $k=0$ we see that (4.4) reduces to (4.3). Indeed, we have performed similar verifications for $1\leq m\leq 9$. With these values of m we have verified that our conjecture is true for a wide selection of the parameters k and n.

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REFERENCES

1. L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5**:169-84. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1993.
2. H. W. Gould. "The Bracket Function and Fontené-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients." *The Fibonacci Quarterly* **7.1** (1969):23-40, 55.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. The Fibonacci Association, Santa Clara, CA, 1979.
4. C. T. Long. "Discovering Fibonacci Identities." *The Fibonacci Quarterly* **24.2** (1986):160-67.
5. A. G. Shannon & A. F. Horadam. "Special Recurrence Relations Associated with the Sequence $\{w_n(a, b; p, q)\}$." *The Fibonacci Quarterly* **17.4** (1979):294-99.
6. R. F. Torretto & J. A. Fuchs. "Generalized Binomial Coefficients." *The Fibonacci Quarterly* **2.4** (1964):296-302.
7. N. N. Vorob'ev. *Fibonacci Numbers*. New York: Pergamon Press, 1961.

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