THE GIRARD-WARING POWER SUM FORMULAS FOR SYMMETRIC FUNCTIONS AND FIBONACCI SEQUENCES

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Dedicated to Professor Leonard Carlitz on his 90th birthday.

The very widely-known identity

$$\sum_{0\le k\le n/2}(-1)^k \frac{n}{n-k}\binom{n-k}{k}(x+y)^{n-2k}(xy)^k = x^n+y^n, \tag{1}$$

which appears frequently in papers about Fibonacci numbers, and the formula of Carlitz [4], [5],

$$\sum_{i+2j+3k=n}(-1)^j \frac{n}{i+j+k}\frac{(i+j+k)!}{i!\,j!\,k!}(x+y+z)^{n-3k}(xy+yz+zx)^k = x^n+y^n+z^n, \tag{2}$$

summed over all $0\le i,\ j, k\le n,\ n>0$, and where $xyz=1$, as well as the formula

$$\sum_{0\le k\le n/3}\frac{n}{n-2k}\binom{n-2k}{k}(x+y+z)^{n-3k}(xyz)^k = x^n+y^n+z^n, \tag{3}$$

where $xy+yz+zx=0$, are special cases of an older well-known formula for sums of powers of roots of a polynomial which was evidently first found by Girard [12] in 1629, and later given by Waring in 1762 [29], 1770 and 1782 [30]. These may be derived from formulas due to Sir Isaac Newton.

The formulas of Newton, Girard, and Waring do not seem to be as well known to current writers as they should be, and this is the motivation for our remarks: to make the older results more accessible. Our paper was motivated while refereeing a paper [32] that calls formula (1) the "Kummer formula" (who came into the matter very late) and offers formula (3) as a generalization of (1), but with no account of the extensive history of symmetric functions.

The Girard-Waring formula may be derived from what are called Newton's formulas. These appear in classical books on the "Theory of Equations." For example, see Dickson [9, pp. 69-74], Ferrar [11, pp. 153-80], Turnbull [27, pp. 66-80], or Chrystal [6, pp. 436-38]. The most detailed of these is Dickson's account. But the account by Vahlen [28, pp. 449-79] in the old German Encyclopedia is certainly the best historical record.

Let $f(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-2}x^2+a_{n-1}x+a_n=0$ be a polynomial with roots x_1, x_2, ..., x_n, so that $f(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$. Then we define the "elementary" symmetric functions $a_1, a_2, a_3, \ldots$ of the roots as follows:

$$\sum_{1\le i\le n} x_i = x_1+x_2+\cdots+x_n = -a_1,$$

$$\sum_{1\le i<j\le n} x_ix_j = x_1x_2+x_1x_3+x_1x_4+x_2x_3+\cdots = a_2,$$

$$\sum_{1\le i<j<k\le n} x_ix_jx_k = x_1x_2x_3+x_1x_2x_4+x_1x_2x_5+\cdots = -a_3,$$

$$\ldots$$

$$x_1x_2x_3\ldots x_n = (-1)^n a_n.$$

Also, we define the k^{th} power sums of the roots by

$$s_k = \sum_{1 \le i \le n} x_i^k = x_1^k + x_2^k + \cdots + x_n^k. \tag{4}$$

Then the Newton formulas are:

$$\begin{cases} s_1 + a_1 = 0, \\ s_2 + a_1 s_1 + 2a_2 = 0, \\ s_3 + a_1 s_2 + a_2 s_1 + 2a_3 = 0, \\ \ldots \\ s_n + a_1 s_{n-1} + a_2 s_{n-2} + \cdots + n a_n = 0, \\ s_{n+1} + a_1 s_{n-1} + a_2 s_{n-2} + \cdots + s_1 a_n = 0. \end{cases} \tag{5}$$

These equations may be solved by determinants to express s_n in terms of a_n, or conversely. These determinant formulas are as follows:

$$(-1)^n s_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 2a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ na_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}, \tag{6}$$

and

$$(-1)^n n! a_n = \begin{vmatrix} s_1 & 1 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & 2 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}. \tag{7}$$

Turnbull [27, p. 74] gives both these determinants, but not the expanded forms. Father Hagen [16, p. 310] cites Salmon [26] for the determinants. Hagen and Vahlen note that we may write these determinants in expanded form, which is the source of the Girard-Waring formula

$$s_j = j \Sigma (-1)^{k_1 + k_2 + \cdots + k_n} \frac{(k_1 + k_2 + \cdots + k_n - 1)!}{k_1! k_2! \ldots k_n!} a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n} \tag{8}$$

summed over all nonnegative integers k_i satisfying the equation $k_1 + 2k_2 + \cdots + nk_n = j$, and its inverse

$$a_j = \Sigma (-1)^{k_1 + k_2 + \cdots + k_j} \frac{1}{k_1! k_2! \ldots k_j!} \left(\frac{s_1}{1}\right)^{k_1} \left(\frac{s_2}{2}\right)^{k_2} \cdots \left(\frac{s_j}{j}\right)^{k_j} \tag{9}$$

summed over all nonnegative integers k_i satisfying the equation $k_1 + 2k_2 + \cdots + jk_j = j$. Vahlen [28, p. 451] gives (8) and (9) and a very detailed account of the theory of symmetric functions with historical references.

The determinant (7) and expansion (9), without mention of their use in connection with Girard-Waring power sums, were noted by Thomas Muir [21, pp. 216-17]. Muir cites G. Mola [10] for his second paper on questions of E. Fergola [10] for the expanded form of (7), i.e., (9). Note Problem 67-13 [31] where W. B. Gragg has an incorrect citation, misunderstanding Muir's citation.

Of course, (9) was already known to Waring. A short proof of (9) was given by Richter [23] using generating functions and the multinomial theorem.

With $n = 2$, and suitable change in notation, relation (8) yields

$$x^n + y^n = n \sum_{i+2j=n} (-1)^{i+j} \frac{(i+j+1)!}{i!\,j!} a^i b^j \text{ summed for } 0 \le i,\ j \le n, \tag{10}$$

where $-a = x+y$ and $b = xy$. Clearly, this is the same as formula (1) when we make the dummy variable substitution, $i = n-2j$.

With $n = 3$, and suitable change in notation, relation (8) yields

$$x^n + y^n + z^n = n \sum_{i+2j+3k=n} (-1)^{i+j+k} \frac{(i+j+k-1)!}{i!\,j!\,k!} a^i b^j c^k, \tag{11}$$

where $-a = x+y+z$, $b = xy+yz+zx$, and $-c = xyz$. When we set $c = 1$ in this, we obtain formula (2) above of Carlitz.

Or, if we specialize (11) by setting $b = 0$, then the only value assumed by j in the summation is $j = 0$, so that we obtain

$$x^n + y^n + z^n = \sum_{i+3k=n} \frac{n}{i+k} \frac{(i+k)!}{i!\,k!} (x+y+z)^i (xyz)^k \tag{12}$$

summed over all $0 \le i,\ k \le n$. Replace i by $n-3k$ to get

$$x^n + y^n + z^n = \sum_{0 \le k \le n/3} \frac{n}{n-2k} \frac{(n-2k)!}{k!\,(n-3k)!} (x+y+z)^{n-2k} (xyz)^k, \tag{13}$$

which is formula (3).

For the sum of four n^{th} powers, we obtain

$$x^n + y^n + z^n + w^n = n \sum (-1)^{i+j+k+\ell} \frac{(i+j+k+\ell-1)!}{i!\,j!\,k!\,\ell!} a^i b^j c^k d^\ell \tag{14}$$

summed over all $0 \le i, j, k, \ell \le n$ with $i + 2j + 3k + 4\ell = n$, and where

$$\begin{aligned} -a &= x+y+z+w, & b &= xy+xz+xw+yz+yw+zw, \\ -c &= xyz+xyw+xzw+yzw, & d &= xyzw. \end{aligned}$$

By making special choices for a, b, c, and d, we could obtain simpler formulas somewhat analogous to (1), (2), or (3). For example, let $b = c = 0$, and we get

$$x^n + y^n + z^n + w^n = \sum_{0 \le k \le n/4} (-1)^{n-3k} \frac{n}{n-3k} \binom{n-3k}{k} a^{n-3k} b^k, \tag{15}$$

subject to $xy+xz+xw+yz+yw+zw = xyz+xyw+xzw+yzw = 0$. With similar conditions on the symmetric functions of the roots, we can obtain power sum formulas of the type

$$s_n = \sum_{1 \le i \le r} x_i^n = \sum_{0 \le k \le n/r+1} (-1)^{n-rk} \frac{n}{n-rk} \binom{n-rk}{k} a^{n-rk} b^k. \tag{16}$$

H. A. Rothe [25], a student of Carl F. Hindenburg of the old German school of combinatorial analysis, found the formula

$$\sum_{k=0}^{n} A_k(a,b) A_{n-k}(c,b) = A_n(a+c,b) \tag{17}$$

where

$$A_k(a,b) = \frac{a}{a+bk} \binom{a+bk}{k}, \tag{18}$$

where a, b, and c may be any complex numbers. Formula (17) generalizes the Vandermonde convolution that occurs when $b = 0$.

It is perhaps of interest to note that J. L. Lagrange evidently believed it would be possible to solve the polynomial equation $f(x) = 0$ with a finite closed formula by manipulating the symmetric functions of the roots. Closed formulas using root extractions are, of course, possible for the quadratic, cubic, and biquadratic, but the work of Galois and Abel showed that this is impossible for equations of fifth degree or higher.

Lagrange did succeed in getting infinite series expansions of the roots in his memoir of 1770. For the simple trinomial equation $x^n + px + q = 0$, the series involve binomial coefficient functions of the form (18). More generally, Lagrange developed what is now called the Lagrange-Bürmann inversion formula for series. See any advanced complex analysis text.

Reference [15] contains a list of twelve of my papers since 1956 dealing with the generalized Vandermonde convolution. The first two [13] and [14] deal with Rothe's work. I first became interested in (17) when I found it in Hagen's *Synopsis* [16].

Closely associated with (17)-(18) are the coefficients

$$C_k(a, b) = \binom{a+bk}{k}, \tag{19}$$

and, in fact, we easily obtain

$$\sum_{k=0}^{n} C_k(a, b) A_{n-k}(c, b) = C_n(a+c, b). \tag{20}$$

Relating to Fibonacci research is the fact that $C_k(n, -1)$ occurs in a popular formula for the Fibonacci numbers that is easily noted by looking at diagonal sums. This formula is

$$F_{n+1} = \sum_{0 \le k \le n/2} C_k(n, -1) = \sum_{0 \le k \le n/2} \binom{n-k}{k}. \tag{21}$$

This arises because the dual to relation (1) is

$$\sum_{0 \le k \le n/2} (-1)^k \binom{n-k}{k} (x+y)^{n-2k} (xy)^k = \frac{x^{n+1} - y^{n+1}}{x - y}. \tag{22}$$

The classical Fibonacci numbers arise with x and y as the roots of the characteristic equation $z^2 - z - 1 = 0$ that is associated with the Fibonacci recurrence $F_{n+2} - F_{n+1} - F_n = 0$. With the same choices, relation (1) yields the classical Lucas numbers L_n for which, of course, the recurrence relation is $L_{n+2} - L_{n+1} - L_n = 0$.

The A coefficients (18) and the C coefficients (19) may be thought of as relating to generalized Lucas and Fibonacci numbers, respectively, the difference being in the presence of the fraction $a/(a+bk)$. Charles A. Church [7] noted precisely this, defining

$$F_{n+1} = \sum_{0 \le k \le n/b+1} \binom{n-bk}{k} \quad \text{and} \quad L_{n+1} = \sum_{0 \le k \le n/b+1} \frac{n}{n-bk} \binom{n-bk}{k}.$$

More general Fibonacci numbers U_n with

$$U_n = \sum_{0 \le k \le n/r+1} \binom{n-rk}{k} a^{n-rk} b^k \tag{23}$$

were studied in the first volume of *The Fibonacci Quarterly* by J. A. Raab [22]. V. C. Harris and Carolyn Stiles [17] then introduced a generalized Fibonacci sequence that satisfies

$$A_n = \sum_{0\le k\le n/p+q} \binom{n-pk}{qk} a^{n-rk} b^k \tag{24}$$

with $a = b = 1$. Verner E. Hoggatt, Jr. [18] wrote about this. Papers too numerous to list (before and since) have studied such sequences. David Dickinson [8] and others have noted that expressions of the form

$$S_m(t) = \sum_k \binom{n+ak}{b+ck} t^{b+ck}, \quad \text{where } m = nc - ab, \ |t| \le 1, \tag{25}$$

summed over meaningful ranges of k essentially satisfy recursive relations and therefore may be seen as generalized Fibonacci numbers. He finds the associated trinomial equation $x^c - tx^a - 1 = 0$, which has distinct roots α_r, $r = 0, 1, \ldots, c-1$, and obtains the formula

$$S_m(t) = \sum_{0\le r\le c-1} \frac{\alpha_r^m}{c - at\alpha_r^{a-c}}. \tag{26}$$

The late Leon Bernstein [1], [2], [3] examined the zeros of the function

$$f(n) = \sum_{0\le i\le n/2} (-1)^i \binom{n-2i}{i}, \tag{27}$$

finding that it has only the zeros 3 and 12. He also found several new combinatorial identities. Carlitz [4], [5] then studied Bernstein's work and in these papers he developed relation (2) above. His techniques are the usual generating function and multinomial theorem approach. Neither Bernstein nor Carlitz mentions or uses the work of Girard and Waring, but (11) with $c = 1$ is found by Carlitz, so that the Girard-Waring formula is implicit there.

We may note that the late John Riordan [24, p. 47] related power sum symmetric functions to the general Bell polynomials he loved to study.

The Girard-Waring formulas offer many extensions of the ordinary Fibonacci-Lucas formulas. These formulas should be more well known.

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