SOME PROPERTIES OF GENERALIZED PASCAL SQUARES AND TRIANGLES*

Richard L. Ollerton
University of Western Sydney, Nepean, 2747, Australia

Anthony G. Shannon
University of Technology, Sydney, 2007, Australia
(Submitted March 1996-Final Revision February 1997)

1. INTRODUCTION

Four properties, related to columns, column sums, diagonal sums, and determinants, will be considered for

(a) the "Pascal square" recurrence relation and its variations,
(b) the "Pascal triangle" recurrence relation and its variations, and
(c) more general recurrence relations which admit these properties.

Associated basic linear recursive sequences are also outlined. Other research may be found in Bollinger [2], Philippou & Georghiou [9], and Carlitz & Riordan [4] who discuss the recurrence relation (2.1) in depth, but with different boundary conditions. In the following, $\left\{ {n \atop p} \right\}$ represents the entry in the n^{th} row, p^{th} column of a square array.

2. GENERALIZED PASCAL SQUARES

Bondarenko [3] presents an extremely useful collation of the myriad results concerning Pascal triangles and their generalizations. We attempt to provide additional insights and unification of some of these by considering properties of square arrays in which the entries are governed by linear partial recurrence relations of a particular form. A number of illustrative cases are given followed by more general results.

2.1 Case 1: The Pascal Square and Variations

The Pascal array in Table 1 is formed by the use of the recurrence relation

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + \left\{ {n \atop p-1} \right\} \quad (n \ge 1, p \ge 0) \tag{2.1}$$

with

$$\left\{ {n \atop -1} \right\} = 0 \;\; (n \ge 0), \quad \left\{ {0 \atop p} \right\} = 1 \;\; (p \ge 1), \quad \text{and} \quad \left\{ {0 \atop 0} \right\} = 1.$$

This is clearly just a rotation of the usual Pascal triangle. We highlight four properties, some well known, which will be generalized.

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* This paper, presented at the Seventh International Research Conference held in Graz, Austria, in July 1996, was scheduled to appear in the Conference Proceedings. However, due to limitations placed by the publisher on the number of pages allowed for the Proceedings, we are publishing the article in this issue of *The Fibonacci Quarterly* to assure its presentation to the widest possible number of readers in the mathematics community.

TABLE 1. Pascal's Square

$n \backslash p$	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2	1	3	6	10	15	21	28	36	45
3	1	4	10	20	35	56	84	120	165
4	1	5	15	35	70	126	210	330	495
5	1	6	21	56	126	252	462	792	1287
6	1	7	28	84	210	462	924	1716	3003
7	1	8	36	120	330	792	1716	3432	6435
8	1	9	45	165	495	1287	3003	6435	12870

Property 1 (Columns): The form of the recurrence relation implies that first differences by column give entries in the previous column. As the 0^{th} column is constant, entries in the p^{th} column are given by the p^{th} order polynomial in n which interpolates the first (or any consecutive) $p+1$ entries in that column

In this case, since $\left\{{n \atop p}\right\} = \binom{n+p}{p}$, the polynomial is $(n+p)(n+p-1)\dots(n+1)/p!$.

Property 2 (Column Sums): For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned}\sum_{i=0}^{n}\left\{{i \atop p}\right\} &= \sum_{i=1}^{n}\left(\left\{{i \atop p+1}\right\}-\left\{{i-1 \atop p+1}\right\}\right)+\left\{{0 \atop p}\right\} \\ &= \left\{{n \atop p+1}\right\}-\left\{{0 \atop p+1}\right\}+\left\{{0 \atop p}\right\}.\end{aligned}$$

For the Pascal square,

$$\sum_{i=0}^{n}\left\{{i \atop p}\right\} = \left\{{n \atop p+1}\right\},$$

which is better known as the combinatorial identity

$$\sum_{i=0}^{n}\binom{i+p}{p} = \binom{n+p+1}{p+1}.$$

Property 3 (Diagonal Sums): Let the n^{th} diagonal sum be

$$d_n = \sum_{i=0}^{n}\left\{{n-i \atop i}\right\},$$

then, for $n \geq 1$,

$$\begin{aligned}d_n - d_{n-1} &= \sum_{i=0}^{n-1}\left(\left\{{n-1 \atop i}\right\}-\left\{{n-i-1 \atop i}\right\}\right)+\left\{{0 \atop n}\right\} \\ &= \sum_{i=0}^{n-1}\left\{{n-i \atop i-1}\right\}+\left\{{0 \atop n}\right\}\end{aligned}$$

$$=\sum_{i=1}^{n-1}\begin{Bmatrix}n-i\\i-1\end{Bmatrix}+\begin{Bmatrix}n\\-1\end{Bmatrix}+\begin{Bmatrix}0\\n\end{Bmatrix}$$

$$=\sum_{i=0}^{n-1}\begin{Bmatrix}n-i-1\\i\end{Bmatrix}-\begin{Bmatrix}0\\n-1\end{Bmatrix}+\begin{Bmatrix}0\\n\end{Bmatrix}$$

$$=d_{n-1}-\begin{Bmatrix}0\\n-1\end{Bmatrix}+\begin{Bmatrix}0\\n\end{Bmatrix},$$

and so

$$d_n=2d_{n-1}-\begin{Bmatrix}0\\n-1\end{Bmatrix}+\begin{Bmatrix}0\\n\end{Bmatrix}. \tag{2.2}$$

In this case, $d_n=2d_{n-1}=2^n$ (as $d_0=1$) as expected.

Property 4 (Determinants): Let

$$\left[\begin{Bmatrix}n\\p\end{Bmatrix}\right]_{(a,b)}$$

denote the square array of given entries with $a\le(n,p)\le b$, then, taking determinants and using elementary determinantal row operations:

$$\left|\left[\begin{Bmatrix}n\\p\end{Bmatrix}\right]_{(0,m)}\right|=\begin{vmatrix}\begin{Bmatrix}0\\0\end{Bmatrix} & \begin{Bmatrix}0\\1\end{Bmatrix} & \cdots & \begin{Bmatrix}0\\m\end{Bmatrix}\\ \begin{Bmatrix}1\\0\end{Bmatrix} & \left[\begin{Bmatrix}n\\p\end{Bmatrix}\right]_{(1,m)} & & \\ \cdots & & & \\ \begin{Bmatrix}m\\0\end{Bmatrix} & & & \end{vmatrix}$$

$$=\begin{vmatrix}\begin{Bmatrix}0\\0\end{Bmatrix} & \begin{Bmatrix}0\\1\end{Bmatrix} & \cdots & \begin{Bmatrix}0\\m\end{Bmatrix}\\ \begin{Bmatrix}1\\0\end{Bmatrix}-\begin{Bmatrix}0\\0\end{Bmatrix} & \left[\begin{Bmatrix}n\\p\end{Bmatrix}-\begin{Bmatrix}n-1\\p\end{Bmatrix}\right]_{(1,m)} & & \\ \cdots & & & \\ \begin{Bmatrix}m\\0\end{Bmatrix}-\begin{Bmatrix}m-1\\0\end{Bmatrix} & & & \end{vmatrix}$$

$$=\begin{vmatrix}\begin{Bmatrix}0\\0\end{Bmatrix} & \begin{Bmatrix}0\\1\end{Bmatrix} & \cdots & \begin{Bmatrix}0\\m\end{Bmatrix}\\ 0 & \left[\begin{Bmatrix}n\\p-1\end{Bmatrix}\right]_{(1,m)} & & \\ \cdots & & & \\ 0 & & & \end{vmatrix} \quad \text{(by use of the recurrence relation)}$$

$$=\begin{Bmatrix}0\\0\end{Bmatrix}\left|\left[\begin{Bmatrix}n\\p-1\end{Bmatrix}\right]_{(1,m)}\right|=\begin{Bmatrix}0\\0\end{Bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix}\left|\left[\begin{Bmatrix}n\\p-2\end{Bmatrix}\right]_{(2,m)}\right|$$

$$=\cdots=\prod_{i=0}^{m}\begin{Bmatrix}i\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}^{m+1}.$$

Thus, all square sub-arrays of the Pascal array with top-left corner $\{{0 \atop 0}\} = 1$ are unimodular.

Property 5 (Generalizations): The derivations of Properties 1-4 rely (if at all) only on the left-hand ($p = -1$) zero boundary conditions. They thus apply to the Pascal array generalized by arbitrary top-row entries and hence to left-justified sub-arrays of the Pascal square. In particular, all square sub-arrays of the Pascal array with left side in the $p = 0$ column (or, by symmetry, top row in the $n = 0$ row) are unimodular, as noted by Bicknell & Hoggatt [1] for the simple Pascal array.

Two examples, formed by varying the top row ($n = 0$) boundary conditions, follow.

Case 1, Example 1 (Vieta's Array): Using (2.1) with $\{{0 \atop p}\} = 2$ $(p \geq 1)$ gives the array in Table 2. Applying Properties 1-5, one need only interpolate a p^{th}-order polynomial to $p+1$ (consecutive) entries of the p^{th} column to determine the column; column sums are as indicated for the Pascal array $(p > 1)$; diagonal sums obey $d_n = 2d_{n-1} = 3 \cdot 2^{n-1}$ (as $d_1 = 3$) and all (left-justified) square sub-arrays are unimodular. This is known as Vieta's array [10].

TABLE 2. Vieta's Array

1	2	2	2	2	2	2	2	2
1	3	5	7	9	11	13	15	17
1	4	9	16	25	36	49	64	81
1	5	14	30	55	91	140	204	285
1	6	20	50	105	196	336	540	825
1	7	27	77	182	378	714	1254	2079
1	8	35	112	294	672	1386	2640	4719
1	9	44	156	450	1122	2508	5148	9867
1	10	54	210	660	1782	4290	9438	19305

Case 1, Example 2 (A Fibonacci Array): Similar results hold for the array in Table 3, which is (2.1) with $\{{0 \atop p}\} = F_{p+1}$ $(p \geq 0)$, except now the p^{th} column sum is $\{{n \atop p+1}\} = F_p$, while diagonal sums obey $d_n = 2d_{n-1} + F_{n-1} = 2^n + \sum_{i=1}^{n-1} 2^{i-1} F_{n-1}$.

Again, by Property 5, all (left-justified) square sub-arrays are unimodular.

TABLE 3. Fibonacci Array of Case 1, Example 2

1	1	2	3	5	8	13	21	34
1	2	4	7	12	20	33	54	88
1	3	7	14	26	46	79	133	221
1	4	11	25	51	97	176	309	530
1	5	16	41	92	189	365	674	1204
1	6	22	63	155	344	709	1383	2587
1	7	29	92	247	591	1300	2683	5270
1	8	37	129	376	967	2267	4950	10220
1	9	46	175	551	1518	3785	8735	18955

2.2 Case 2: The Pascal Triangle and Variations

The Pascal triangle array in Table 4 is formed by use of the recurrence relation

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + \left\{ {n-1 \atop p-1} \right\} \quad (n \geq 1, p \geq 0) \tag{2.3}$$

with

$$\left\{ {n \atop -1} \right\} = 0 \quad (n \geq 0), \quad \left\{ {0 \atop p} \right\} = 0 \quad (p \geq 1), \quad \text{and} \quad \left\{ {0 \atop 0} \right\} = 1.$$

Precisely the same methods apply to this case as presented for Case 1. Corresponding results are given.

TABLE 4. Pascal Triangle

1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0
1	4	6	4	1	0	0	0	0
1	5	10	10	5	1	0	0	0
1	6	15	20	15	6	1	0	0
1	7	21	35	35	21	7	1	0
1	8	28	56	70	56	28	8	1

Property 6 (Columns): It is interesting to note that the p^{th} column of the Pascal triangle is thus determined by the polynomial in n which interpolates p zeros followed by 1.

Property 7 (Column Sums):

$$\sum_{i=0}^{n} \left\{ {i \atop p} \right\} = \left\{ {n+1 \atop p+1} \right\} - \left\{ {1 \atop p+1} \right\} + \left\{ {0 \atop p} \right\}. \tag{2.4}$$

For the Pascal triangle,

$$\sum_{i=0}^{n} \left\{ {i \atop p} \right\} = \left\{ {n+1 \atop p+1} \right\} \quad (n, p \geq 0)$$

as expected (since here $\{ {n \atop p} \}$ is just the binomial coefficient).

Property 8 (Diagonal Sums):

$$d_n - d_{n-1} = d_{n-2} + \left\{ {0 \atop n} \right\} = d_{n-2},$$

$d_0 = d_1 = 1$ in this case (very well known).

Property 9 (Determinants):

$$\left| \left[\left\{ {n \atop p} \right\} \right]_{(0,m)} \right| = \left\{ {0 \atop 0} \right\} \left| \left[\left\{ {n-1 \atop p-1} \right\} \right]_{(1,m)} \right| = \left\{ {0 \atop 0} \right\}^{m+1}$$

$$= 1 \quad \text{in this case.}$$

Similarly (see Property 5), all (left-justified) square sub-arrays are unimodular (as noted by Bicknell & Hoggatt [1]). Also, as before, Properties 6-9 apply to the array formed with arbitrary initial row. Three examples follow.

Case 2, Example 1 (Division of p-Space by n (p – 1)-Spaces): The array in Table 5 is (2.3) with $\{ {0 \atop 0} \} = 1$, $\{ {0 \atop p} \} = 1$ $(p \geq 1)$. This relation is a generalization of the recurrence relations governing the maximum number of parts into which p-space can be divided by n $(p-1)$-spaces for p = 1, 2, 3. (Shannon [12] discusses these three instances in the context of the pedagogy of problem-solving.)

Properties 6-9 reduce to: entries in the p^{th} column are given by the p^{th}-order polynomial which interpolates $\{1, 2, 4, \ldots, 2^p\}$, the p^{th} column sum is given by

$$\left\{\begin{matrix} n+1 \\ p+1 \end{matrix}\right\} - 1, \quad d_n - d_{n-1} = d_{n-2} + \left\{\begin{matrix} 0 \\ n \end{matrix}\right\} = d_{n-2} + 1 \ (d_0 = 1, d_1 = 2)$$

(thus, the diagonal sums are the partial sums of the Fibonacci sequence), and all left-justified square sub-arrays are unimodular.

TABLE 5. Array in Case 2, Example 1

1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2
1	3	4	4	4	4	4	4	4
1	4	7	8	8	8	8	8	8
1	5	11	15	16	16	16	16	16
1	6	16	26	31	32	32	32	32
1	7	22	42	57	63	64	64	64
1	8	29	64	99	120	127	128	128
1	9	37	93	163	219	247	255	256

Case 2, Example 2 (Another Fibonacci Array): Using (2.3) with $\{{0 \atop p}\} = F_{p+1}$ $(p \geq 0)$ gives the array in Table 6. In this case, the p^{th} column is determined by the interpolating polynomial for the sequence $\{F_{p+1}, F_{p+2}, \ldots, F_{2p+1}\}$, the p^{th} column sum is given by

$$\left\{\begin{matrix} n+1 \\ p+1 \end{matrix}\right\} - F_{p+2},$$

diagonal sums obey $d_n = d_{n-1} + d_{n-2} + F_{n+1}$ $(d_0 = 1, d_1 = 2)$ and all left-justified square sub-arrays are unimodular.

TABLE 6. Array in Case 2, Example 2

1	1	2	3	5	8	13	21	34
1	2	3	5	8	13	21	34	55
1	3	5	8	13	21	34	55	89
1	4	8	13	21	34	55	89	144
1	5	12	21	34	55	89	144	233
1	6	17	33	55	89	144	233	377
1	7	23	50	88	144	233	377	610
1	8	30	73	138	232	377	610	987
1	9	38	103	211	370	609	987	1597

It is of interest to note that Lavers [8] found the corresponding "Fibonacci triangle" in his investigation of certain idempotent transformations.

Case 2, Example 3 (The Lucas Triangle): Setting $\{{0 \atop 0}\} = 1$, $\{{0 \atop 1}\} = 2$, and $\{{0 \atop p}\} = 0$ $(p \geq 2)$ in (2.3) gives the array in Table 7. This is the so-called Lucas triangle ([3], p. 26). Here the p^{th} column is determined by the interpolating polynomial for the sequence $\{0, \ldots, 0, 2, 2p+1\}$, the p^{th} column sum is given by

$$\left\{\begin{matrix} n+1 \\ p+1 \end{matrix}\right\} \ (p \geq 1),$$

diagonal sums obey $d_n = d_{n-1} + d_{n-2}$ $(d_0 = 1, d_1 = 3)$ (the Lucas numbers) and, once again, all left-justified square sub-arrays are unimodular.

TABLE 7. Array in Case 2, Example 3

1	2	0	0	0	0	0	0	0
1	3	2	0	0	0	0	0	0
1	4	5	2	0	0	0	0	0
1	5	9	7	2	0	0	0	0
1	6	14	16	9	2	0	0	0
1	7	20	30	25	11	2	0	0
1	8	27	50	55	36	13	2	0
1	9	35	77	105	91	49	15	2
1	10	44	112	182	196	140	64	17

2.3 Generalizations

We now generalize the foregoing to recursive relations of the form

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = b\begin{Bmatrix} n-1 \\ p \end{Bmatrix} + \sum_{i=r}^{s} a_i \begin{Bmatrix} n+i \\ p-1 \end{Bmatrix}, \tag{2.5}$$

with $\{{n \atop -1}\} = 0 \ \forall n$ and other boundary conditions for $\{{n \atop p}\}$ $(n \le 0,\ p \ge 0)$ given as necessary.

The above covers each of the earlier cases and others including:

(a) the complementary binomial coefficients of Puritz ([3], p. 33) (Table 8) where

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = \begin{Bmatrix} n-1 \\ p \end{Bmatrix} - \begin{Bmatrix} n \\ p-1 \end{Bmatrix} \ (n \ge 1,\ p \ge 0),$$

with

$$\begin{Bmatrix} n \\ -1 \end{Bmatrix} = 0 \ \forall n, \quad \begin{Bmatrix} 0 \\ p \end{Bmatrix} = 0 \ (p \ge 1), \quad \text{and} \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1;$$

TABLE 8: Array in 2.3(a)

1	0	0	0	0	0	0	0	0
1	−1	1	−1	1	−1	1	−1	1
1	−2	3	−4	5	−6	7	−8	9
1	−3	6	−10	15	−21	28	−36	45
1	−4	10	−20	35	−56	84	−120	165
1	−5	15	−35	70	−126	210	−330	495
1	−6	21	−56	126	−252	462	−792	1287
1	−7	28	−84	210	−462	924	−1716	3003
1	−8	36	−120	330	−792	1716	−3432	6435

(b) an analog of the Pascal triangle (Table 9) studied by Wong & Maddocks ([3], p. 36), where

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = \begin{Bmatrix} n-1 \\ p \end{Bmatrix} + \begin{Bmatrix} n-1 \\ p-1 \end{Bmatrix} + \begin{Bmatrix} n-2 \\ p-1 \end{Bmatrix} \ (n \ge 1,\ p \ge 0),$$

with

$$\begin{Bmatrix} n \\ -1 \end{Bmatrix} = 0 \ \forall n, \quad \begin{Bmatrix} 0 \\ p \end{Bmatrix} = 0 \ (p \ge 1),$$

$$\begin{Bmatrix} -1 \\ p \end{Bmatrix} = 0 \ (p \ge 0), \quad \text{and} \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1,$$

for which row sums are Pell numbers and diagonal sums are "Tribonacci" numbers;

TABLE 9. Array in 2.3(b)

1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	3	1	0	0	0	0	0	0
1	5	5	1	0	0	0	0	0
1	7	13	7	1	0	0	0	0
1	9	25	25	9	1	0	0	0
1	11	41	63	41	11	1	0	0
1	13	61	129	129	61	13	1	0
1	15	85	231	321	231	85	15	1

and

(c) the recurrence relation

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = b \begin{Bmatrix} n-1 \\ p \end{Bmatrix} + a_{-1} \begin{Bmatrix} n-1 \\ p-1 \end{Bmatrix}$$

studied by Cadogan ([3], p. 30).

Property 10 (Column Sums): The most straightforward generalization is for the special case of (2.5),

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = \begin{Bmatrix} n-1 \\ p \end{Bmatrix} + a_{-k} \begin{Bmatrix} n-k \\ p-1 \end{Bmatrix} \quad (k \geq 0), \tag{2.6}$$

with $\{{n \atop -1}\} = 0 \ \forall n$ and other boundary conditions for $\{{n \atop p}\}$ $(n \leq 0,\, p \geq 0)$ given as necessary.

For $n \geq 1$ and $p \geq 0$,

$$\begin{aligned} a_{-k} \sum_{i=0}^{n} \begin{Bmatrix} i \\ p \end{Bmatrix} &= \sum_{i=1}^{n} \left(\begin{Bmatrix} i+k \\ p+1 \end{Bmatrix} - \begin{Bmatrix} i+k-1 \\ p+1 \end{Bmatrix} \right) + \begin{Bmatrix} 0 \\ p \end{Bmatrix} \\ &= \begin{Bmatrix} n+k \\ p+1 \end{Bmatrix} - \begin{Bmatrix} k \\ p+1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ p \end{Bmatrix}. \end{aligned}$$

Property 11 (Diagonal Sums): For $n \geq 1$,

$$\begin{aligned} d_n - bd_{n-1} &= \sum_{i=0}^{n-1} \left(\begin{Bmatrix} n-i \\ i \end{Bmatrix} - b \begin{Bmatrix} n-i-1 \\ i \end{Bmatrix} \right) + \begin{Bmatrix} 0 \\ n \end{Bmatrix} \\ &= \sum_{i=0}^{n-1} \sum_{j=r}^{s} a_j \begin{Bmatrix} n-i+j \\ i-1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ n \end{Bmatrix} \\ &= \sum_{j=r}^{s} a_j d_{n+j-1} - \sum_{j=r}^{s} a_j \sum_{i=0}^{j} \begin{Bmatrix} j-i \\ n+i-1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ n \end{Bmatrix} \quad (n+r-1 \geq 0). \end{aligned}$$

Property 12 (Determinants): For $b \neq 0$, and noting that $\{{n \atop 0}\} = ab^n$ $(a = \{{0 \atop 0}\},\, n \geq 0)$,

$$\left| \left[\begin{Bmatrix} n \\ p \end{Bmatrix} \right]_{(0,m)} \right| = \begin{vmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} & \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} & \cdots & \begin{Bmatrix} 0 \\ m \end{Bmatrix} \\ \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - b \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} & \left[\begin{Bmatrix} n \\ p \end{Bmatrix} - b \begin{Bmatrix} n-1 \\ p \end{Bmatrix} \right]_{(1,m)} & & \\ \cdots & & & \\ \begin{Bmatrix} m \\ 0 \end{Bmatrix} - b \begin{Bmatrix} m-1 \\ 0 \end{Bmatrix} & & & \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \left| \left[\sum_{i=r}^{s} a_i \begin{Bmatrix} n+i \\ p-1 \end{Bmatrix} \right]_{(1,m)} \right|$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \sum_i a_i \begin{Bmatrix} 1+i \\ 0 \end{Bmatrix} \sum_{i,j} a_i a_j \begin{Bmatrix} 2+i+j \\ 0 \end{Bmatrix} \cdots \sum_{i,j,\dots k} a_i a_j \dots a_k \begin{Bmatrix} m+i+j+\cdots+k \\ 0 \end{Bmatrix}$$

$$= a^{m+1} \left(b \sum_{i=r}^{s} a_i b^i \right)^{m(m+1)/2}.$$

Sufficient conditions for this derivation are that each element below and including $\{^{m+(m-1)r}_{p}\}$ $(m \geq 2, 0 \leq p < m-1)$ has been formed by the given recurrence relation. (Thus, if $m+(m-1)r \leq 0$, the result will only apply to sub-arrays beginning at row $1-m-(m-1)r$. This is not a restriction if $r \geq -1$.)

When $b = 0$, we need only restrict the previous formula to $r = s = -1$ (though it will apply to $s \geq -1$), hence,

$$\left| \left[\begin{Bmatrix} n \\ p \end{Bmatrix} \right]_{(0,m)} \right| = a^{m+1} a_{-1}^{m(m+1)/2}.$$

Thus, other unimodular arrays can be formed by setting, for example, $a = b = \sum_i a_i = 1$.

3. GENERALIZED PASCAL TRIANGLES

Consider the square with the rule of formation,

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = \begin{Bmatrix} n \\ p-2 \end{Bmatrix} + \begin{Bmatrix} n-1 \\ p-1 \end{Bmatrix} + \begin{Bmatrix} n-2 \\ p-1 \end{Bmatrix}, \tag{3.1}$$

with

$$\begin{Bmatrix} n \\ p \end{Bmatrix} = 0 \ (p < 0), \quad \begin{Bmatrix} -1 \\ p \end{Bmatrix} = 0 \ (p > 0), \quad \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p \end{Bmatrix} = 1 \ (p > 0).$$

TABLE 10. Generalized Pascal Triangle

1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	2	3	3	4	4	5	5	6	6	7	7
1	2	3	5	6	9	10	14	15	20	21	27		
1	2	5	7	13	16	26	30	45	50	71			
1	3	6	13	19	35	45	75	90	140				
1	3	9	16	35	51	96	126	216					
1	4	10	26	45	96	141	267						
1	4	14	30	75	126	267							
1	5	15	45	90	216								
1	5	20	50	140	266								
1	6	21	71	161									
1	6	27	77										
1	7	28	105										
1	7	35											

If we add along the diagonals in Table 10, we get the sequence $\{1, 2, 3, 6, 18, 27, 54, 81, 162, \dots\}$, which is generated by the recurrence relation $W_n = W_{n-1} + W_{n-2} + \delta(2,n) W_{n-3}$, $n > 3$, where

$$\delta(m,n) = \begin{cases} 1 & \text{if } m \mid n, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(Shannon [11])}$$

and the initial terms are $W_i = i$, $i = 1, 2, 3$. We can bifurcate this sequence into $W_{1n} = \{1, 3, 9, 27, 81, \ldots\}$ and $W_{2n} = (2, 6, 18, 54, 162, \ldots\}$, which are generated by the recurrence relation

$$W_{jn} = 2W_{j,n-1} + 3W_{j,n-2}, \quad n > 2. \tag{3.3}$$

This bifurcation enables us to distinguish two triangles within the square array, as in Tables 11 and 12.

TABLE 11. Triangle Corresponding to $\{W_{1n}\}$ [1]

1												
1	1	1										
1	2	3	2	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
1	6	21	50	90	126	141	126	90	50	21	6	1

TABLE 12. Triangle Corresponding to $\{W_{2n}\}$

1	1										
1	2	2	1								
1	3	5	5	3	1						
1	4	9	13	13	9	4	1				
1	5	14	26	35	35	26	14	5	1		
1	6	20	45	75	96	96	75	45	20	6	1

Notice that the triangle in Table 12 has the feature that

$$\sum_{p=0}^{2n+1} \left\{ \begin{matrix} n \\ p \end{matrix} \right\} = 2 \cdot 3^n, \quad n = 0, 1, \ldots.$$

Obviously we get the ordinary Pascal triangle if we take the diagonals of the Pascal square (see Table 1). Similarly, if we consider

$$\left\{ \begin{matrix} n \\ p \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ p \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ p-1 \end{matrix} \right\} + \left\{ \begin{matrix} n \\ p-1 \end{matrix} \right\} \tag{3.4}$$

with $\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \} = 1$, $\{ \begin{smallmatrix} 0 \\ p \end{smallmatrix} \} = 1$, which is also of the form (2.5), we get the square in Table 13(a) and the triangle in Table 13(b). In addition to the properties of Section 2, the numbers in Table 11(a), $D(n, m)$, are Delanoy numbers [14] and are linked to minimal paths in lattices. We observe here that the row sums yield the Pell sequence $\{P_n\} = \{1, 2, 5, 12, 29, 70, \ldots\}$ defined by the initial terms $P_1 = 1$, $P_2 = 2$, and the second-order recurrence relation (Horadam [7]) $P_n = 2P_{n-1} + P_{n-2}$, $n > 2$.

TABLE 13. Arrays Corresponding to (3.4)

(a)

1	1	1	1	1
1	3	5	7	9
1	5	13	25	41
1	7	25	63	129
1	9	41	129	321

(b)

1					
1	1				
1	3	1			
1	5	5	1		
1	7	13	7	1	
1	9	25	25	9	1

This is a particularly rich triangle because, when we add along the diagonals, we obtain the third-order sequence $\{0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots\}$.

Again, if we take the triangle from the diagonals of the array formed from equation (2.3),

$$\left\{ {n \atop p} \right\} = \left\{ {n-1 \atop p} \right\} + \left\{ {n-1 \atop p-1} \right\} \tag{3.5}$$

with $\left\{ {n \atop 0} \right\} = 1$, $\left\{ {0 \atop p} \right\} = 1$, we get the square and triangle of Table 14(a) and (b).

TABLE 14. Arrays Associated with (3.5)

(a)

1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2
1	3	4	4	4	4	4	4	4
1	4	7	8	8	8	8	8	8
1	5	11	15	16	16	16	16	16
1	6	16	26	31	32	32	32	32
1	7	22	42	57	63	64	64	64
1	8	29	64	99	120	127	128	128
1	9	37	93	163	219	247	255	256

(b)

1							
1	1						
1	2	1					
1	3	2	1				
1	4	4	2	1			
1	5	7	4	2	1		
1	6	11	8	4	2	1	
1	7	16	15	8	4	2	1

Here, too, we find two sequences. The row sums yield $\{x_n\} = \{1, 2, 4, 7, 12, 20, 33, \ldots\}$, with the nonhomogenous second-order recurrence relation $x_n = x_{n-1} + x_{n-2} + 1$, while the diagonal sums yield $\{y_n\} = \{1, 1, 2, 3, 5, 7, 11, 16, 24, 35, 52, \ldots\}$, which is formed from the fifth-order recurrence relation $y_n = y_{n-1} + y_{n-2} - y_{n-5}$, $n > 5$, which is a particular case of equation (1) found in Dubeau & Shannon [5].

4. CONCLUSION

We have demonstrated that a number of well-known properties of Pascal-type arrays are consequences of a more general partial recurrence relation. Further investigations could include relating the various sequences to standard sequences identified by Sloane [13]. Algebraic structural properties can be studied along the lines of Korec [6] who has, in effect, generalized some of the work of Wells [15].

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AMS Classification Numbers: 11B39, 11B65

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