CONSTRUCTION OF $2*n$ CONSECUTIVE n-NIVEN NUMBERS

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1. INTRODUCTION

Fix a natural number, $n \geq 2$, as our base. For a a natural number, define $s(a)$ to be the sum of the digits of a written in base n. Define $v(a)$ to be the number of digits of a written in base n, i.e., $n^{v(a)-1} \leq a < n^{v(a)}$. For a and b natural numbers, denote the product of a and b by $a*b$. For a and b natural numbers written in base n, let ab denote the concatenation of a and b, i.e., $ab = a*n^{v(b)}+b$. Denote concatenation of k copies of a by a_k, i.e.,

$$a_k = a + a*n^{v(a)} + a*n^{2*v(a)} + \cdots + a*n^{(k-1)*v(a)} = a*\frac{n^{k*v(a)}-1}{n^{v(a)}-1}.$$

Definition: We say a is an *n-Niven number* if a is divisible by its base n digital sum, i.e., $s(a)|a$.

Example: For $n = 11$, we have $15 = 1*11+4*1$, so $s(15) = 1+4 = 5$. Since $5|15$, 15 is an 11-Niven number.

It is known that there can exist at most $2*n$ consecutive n-Niven numbers [3]. It is also known that, for $n = 10$, there exist sequences of twenty consecutive 10-Niven numbers (often just called Niven numbers) [2]. In [1], sequences of six consecutive 3-Niven numbers and four consecutive 2-Niven numbers were constructed. Mimicking a construction of twenty consecutive Niven numbers in [4], we can prove Grundman's conjecture.

Conjecture: For each $n \geq 2$, there exists a sequence of $2*n$ consecutive n-Niven numbers.

Before giving a constructive proof of this conjecture, we give some notation and results that will give us necessary congruence conditions for a number, α, to be the base n digital sum of the first of $2*n$ consecutive n-Niven numbers, β.

For any prime p, let $a(p)$ be such that $p^{a(p)} \leq n$ but $p^{a(p)+1} > n$. For any prime p, let $b(p)$ be such that $p^{b(p)}|(n-1)$ but $p^{b(p)+1}\nmid(n-1)$. Let $\mu = \prod_p p^{a(p)-b(p)}$.

Theorem 1: A sequence of $2*n$ consecutive n-Niven numbers must begin with a number congruent to $n^{\mu*m}-n$ modulo $n^{\mu*m}$ (but not congruent to $n^{\mu*m+1}-n$ modulo $n^{\mu*m+1}$) for some positive integer m.

Proof: It is shown in [3] that the first of $2*n$ consecutive n-Niven numbers, β, must be congruent to 0 modulo n. Suppose $\beta \equiv n^{m'}-n \bmod n^{m'}$ but $\beta \not\equiv n^{m'+1}-n \bmod n^{m'+1}$. We will show that $\mu|m'$. It is enough to show $p^{a(p)-b(p)}|m'$ for all p. Among the n consecutive numbers $s(\beta)$, $s(\beta+1), \ldots, s(\beta+n-1)$, there is a multiple of $p^{a(p)}$. Similarly for $s(\beta+n)$, $s(\beta+n+1), \ldots, s(\beta+2*n-1)$. By the definition of an n-Niven number, this means $p^{a(p)}|s(\beta+i)$, $s(\beta+i)|(\beta+i)$, $p^{a(p)}|s(\beta+n+j)$, and $s(\beta+n+j)|(\beta+n+j)$ for some i, j in $0, 1, \ldots, n-1$. But $s(\beta+i) = S(\beta)+i$

and $s(\beta+n+j)=s(\beta)+n+j-m'*(n-1)$. So, $p^{a(p)}|(n+j-i)$ and $p^{a(p)}|(n+j-i-m'*(n-1))$, and therefore, $p^{a(p)}|m'*(n-1)$. Since $p^{b(p)}$ is the highest power of p dividing $n-1$, we obtain $p^{a(p)-b(p)}|m'$. □

Corollary 1: A sequence of $2*n$ consecutive n-Niven numbers must consist of numbers having at least μ digits written in base n.

Another result of this theorem is to get restrictions on the digital sum, α, of the first of $2*n$ consecutive n-Niven numbers.

Corollary 2: If $\alpha=s(\beta)$ for β the first of $2*n$ consecutive n-Niven numbers, then for m as in Theorem 1 and for

$$\gamma=\text{lcm}(\alpha,\alpha+1,\ldots,\alpha+n-1) \tag{1}$$

and

$$\gamma'=\text{lcm}(\alpha+n-\mu*m*(n-1),\alpha+n+1-\mu*m*(n-1),\ldots,\alpha+2*n-1-\mu*m*(n-1)), \tag{2}$$

we have $\gcd(\gamma,\gamma')\,|\,\mu*m*(n-1)$.

Proof: For β the first of $2*n$ consecutive n-Niven numbers and for α the base n digital sum of β, since $\beta\equiv 0 \bmod n$, we get

$$(\alpha+i)|(\beta+i) \text{ for } i=0,1,\ldots,n-1$$

and, by Theorem 1, we get

$$(\alpha+n+j-\mu*m*(n-1))\,|\,(\beta+n+j) \text{ for } j=0,1,\ldots,n-1.$$

These imply $\beta\equiv\alpha \bmod \gamma$ and $\beta\equiv\alpha-\mu*m*(n-1) \bmod \gamma'$. These two congruences are compatible if and only if $\gcd(\gamma,\gamma')\,|\,\mu*m*(n-1)$. □

Finally, we will need the following three lemmas in our construction.

Lemma 1: For $\delta=\text{lcm}(\gamma,\gamma')$ there exist positive integer multiples of δ, say $k*\delta$ and $k'*\delta$ so that $\gcd(s(k*\delta),s(k'*\delta))=n-1$. Further, this is the smallest the greatest common divisor of the digital sums of any two integral multiples of δ can be.

Proof: Since $(n-1)|\delta$, we see that $(n-1)|k*\delta$ for any $k\in\mathbb{Z}$. Since $n-1$ is one less than our base, $(n-1)|s(k*\delta)$, so the smallest the greatest common divisor can be is $n-1$.

Now let $a\mathbf{a}b0_\ell$ be the base n expansion of δ with a and b nonzero digits and $\mathbf{a}$ a block of digits of length ℓ'. We can suppose without loss of generality that $\mathbf{a}$ ends in a digit other than $n-1$, for if it does end in $n-1$ we can consider $(n+1)*\delta$ in place of δ. Since $\delta<n^{\ell+\ell'+2}$, there is a multiple of δ between any two multiples of $n^{\ell+\ell'+2}$, so there is some multiple of δ between $(n-1)*n^{\ell+\ell'+2}$ and $n^{\ell+\ell'+3}$, i.e., some κ so that the base n representation of $\kappa*\delta$ is $(n-1)\mathbf{a}'$ with $\nu(\mathbf{a}')=\ell+\ell'+2$. Then, for $k=\kappa*n^{\ell+\ell'+2}+1$ and $k'=(n^{\ell+2*\ell'+4}+1)*k$, we get $(n-1)\mathbf{a}'a\mathbf{a}b0_\ell$ as the base n representation of $k*\delta$ and $(n-1)\mathbf{a}'a(\mathbf{a}+1)(b-1)\mathbf{a}'a\mathbf{a}b0_\ell$ as the base representation of $k'*\delta$. Then we see $s(k*\delta)=n-1+s(\mathbf{a}')+s(a)+s(\mathbf{a})+s(b)$, while $s((k'*\delta))=n-1+2*s(\mathbf{a}')+2*s(a)+2*s(\mathbf{a})+1+2*s(b)-1$; thus, $s(k'*\delta)=2*s(k*\delta)-(n-1)$. This means $\gcd(s(k*\delta), s(k'*\delta))=n-1$. □

Remark 1: It follows from the proof that we can choose k, k' in the lemma with $\nu(k*\delta)\le 5+2*\ell+2*\ell'$ and $\nu(k'*\delta)\le 9+3*\ell+4*\ell'$ when δ [or $(n+1)*\delta$ if $\mathbf{a}$ ends in $n-1$] has

$\ell+\ell'+2$ digits in base n. Since ℓ is the number of terminal zeros in δ and ℓ' is the number of digits strictly between the first and last nonzero digit of δ, we have

$$\nu(k*\delta) \le 5+2*(\ell+\ell') \le 5+2*(\nu(\delta)-1).$$

Since $\delta < (\alpha+n-1)^{2*n}$, we have $\nu(\delta) \le 2*n*(\log_n(\alpha+n-1)+1)$.. This inequality leads to

$$\nu(k*\delta) \le 5+2*(2*n*(\log_n(\alpha+n-1)+1)-1) \le 5+4*n*(\log_n(\alpha+n-1)+1).$$

Similarly,

$$\nu(k*\delta) \le 2*(5+4*(n*(\log_n(\alpha+n-1)+1))).$$

This comes into play in constructing a "growth condition" in the next section.

Lemma 2: For any positive integer z, if $\alpha \equiv z \bmod \gamma$, then $(n-1)|(\alpha - s(z))$.

Proof: This is equivalent to showing $\alpha \equiv s(z) \bmod (n-1)$. We know $z \equiv s(z) \bmod (n-1)$ as $n-1$ is one less than our base. Since $(n-1)|\gamma$, we get $z \equiv \alpha \bmod (n-1)$ which, taken with the previous congruence, gives the result. □

Lemma 3: For positive integers x, y, z, if $\gcd(x, y)|z$ and $z \ge x*y$, then we can express z as a nonnegative linear combination of x and y.

Proof: That we can write z as a linear combination of x and y follows from the extended Euclidean algorithm. To see that we can obtain a nonnegative linear combination, suppose $z = r*x+t*y$. Since $x, y, z > 0$, at least one of r and t is positive. If they are both nonnegative, we are done, so suppose without loss of generality that $r<0$. Then $z = z+(y*x-x*y) = (r+y)*x+(t-x)*y$. We can repeat this until we have a nonnegative coefficient on x, so assume without loss of generality that $r+y \ge 0$. If $t-x \ge 0$, then we have a nonnegative linear combination and so are done. This means we are left to consider $r<0, t>0, r+y \ge 0$, and $t-x<0$. However, if $z = r*x+t*y$ with $r<0, x>0$, then $t*y>z$ so that $(t-x)*y > z-x*y \ge 0$ by hypothesis. But $y>0$ and $(t-x)*y \ge 0$ means $t-x \ge 0$, a contradiction. □

2. CONSTRUCTION

In this section we shall construct an α that can serve as the digital sum of the first of $2*n$ consecutive n-Niven numbers. We then use this α to actually construct the first of $2*n$ consecutive n-Niven numbers, β, with $\alpha = s(\beta)$. We present the construction using the results of the previous section. In that section, we derived congruence restrictions on the digital sum of the first of $2*n$ consecutive n-Niven numbers (if such a sequence exists). We now use these restrictions to construct such a sequence.

Let $a(p), b(p)$, and μ be as in the previous section. For our construction, we specifically fix $m = \prod_{p|n} p$. For p a prime, define $c(p)$ by

$$p^{c(p)}|(\mu*m*(n-1)-i) \text{ for some } i = 1, 2, \ldots, 2*n-1$$

and

$$p^{c(p)+1}\nmid(\mu*m*(n-1)-i) \text{ for any } i = 1, 2, \ldots, 2*n-1.$$

To produce an α satisfying $\gcd(\gamma, \gamma')|\mu*m*(n-1)$, we impose the following condition.

Congruence Condition I: For all $p \nmid n$ with $c(p) > a(p)$, we require

$$\alpha \equiv 1, 2, \ldots, p^{a(p)+1} - n \bmod p^{a(p)+1}$$

or

$$\alpha + n - \mu * m * (n-1) \equiv 1, 2, \ldots, p^{a(p)+1} - n \bmod p^{a(p)+1}. \tag{3}$$

This assures that the "prime to n" part of $\gcd(\gamma, \gamma')$ will divide $\mu * (n-1)$. But, for $p|n$, we require stronger conditions in order to have an α for which $\gcd(\gamma, \gamma') | \mu * m * (n-1)$.

Congruence Condition II: For all $p|n$, we require both of the following:

$$\alpha + n - \mu * m * (n-1) \equiv 1, 2, \ldots, p^{a(p)+2} - n \bmod p^{a(p)+2}; \tag{4}$$

$$\alpha \equiv p^{a(p)+1} - n \bmod p^{a(p)+1}. \tag{5}$$

Remark 2: There exist α simultaneously satisfying these conditions. It is clear we can find an α satisfying Condition I for every p. For Condition II, (5) is equivalent to

$$\alpha \equiv p^{a(p)+1} - n, 2 * p^{a(p)+1} - n, \ldots, p * p^{a(p)+1} - n \bmod p^{a(p)+2}. \tag{6}$$

Then (4) restricts α to one of $p^{a(p)+2} - n$ consecutive residue classes modulo $p^{a(p)+2}$, but at least one of these must also be a solution to (6) since those solutions are spaced every $p^{a(p)+1}$. and $p^{a(p)+1} > n$ implies $p^{a(p)+2} - n > p^{a(p)+1}$.

Finally, as there are infinitely many α satisfying Congruence Conditions I and II, we are free to choose one as large as we like. We choose α large enough to satisfy the following

Growth Condition:

$$\begin{aligned} \alpha \geq\ & (n-1) * (\mu * m + 2 * n * (\log_n(\alpha + n - 1) + 1)) \\ & + (n-1)^2 * 2 * (5 + 4 * n * (\log_n(\alpha + n - 1) + 1))^2. \end{aligned} \tag{7}$$

Again, it is possible to find such an α because the left-hand side grows linearly while the right-hand side grows logarithmically in α.

Theorem 2: Any α satisfying Congruence Conditions I and II and the Growth Condition is the digital sum of the first of $2*n$ consecutive n-Niven numbers. In particular, for each $n \geq 2$, there exists a sequence of $2*n$ consecutive n-Niven numbers.

Proof: We start with an α satisfying Congruence Conditions I and II and the Growth Condition. For $\gamma = \operatorname{lcm}(\alpha, \alpha+1, \ldots, \alpha+n-1)$ and $\gamma' = \operatorname{lcm}(\alpha + n - \mu * m * (n-1), \ldots, \alpha + 2 * n - 1 - \mu * m * (n-1))$, we can solve

$$b \equiv \alpha \bmod \gamma \text{ and } b \equiv \alpha - \mu * m * (n-1) \bmod \gamma'. \tag{8}$$

To see this, note that, for $p \nmid n$, we have $v_p(\mu * m * (n-1)) = a(p)$ and Congruence Condition I assures that $v_p(\gcd(\gamma, \gamma')) \leq a(p)$. For $p|n$, we have $v_p(\mu * m * (n-1)) = a(p) + 1$ and, by (5), $v_p(\gcd(\gamma, \gamma')) \leq a(p)$.

Let b be the least positive solution to (8). Any other solution to (8) differs from the minimal positive one by a multiple of $\delta = \operatorname{lcm}(\gamma, \gamma')$. We can modify b by adding multiples of δ to create a number, b', so that

$$b' \equiv n^{\mu*m} - n \mod n^{\mu*m}$$

but (9)

$$b' \not\equiv n^{\mu*m+1} - n \mod n^{\mu*m+1}.$$

This is possible by Congruence Condition II: For $p|n$, Condition II assures that $\alpha \equiv p^{a(p)+1} - n$ mod $p^{a(p)+1}$. Since $\mu*m*(n-1) \equiv 0$ mod $p^{a(p)+1}$, we have $\alpha + n - \mu*m*(n-1) \equiv 0$ mod $p^{a(p)+1}$.Now (8) assures $b+n \equiv 0$ mod $p^{a(p)+1}$. By Condition II, $v_p(\delta) \le a(p)+1 = v_p(\mu*m)$, so

$$b \equiv n^{\mu*m} - n \mod \prod_{p|n} p^{v_p(\delta)}.$$

This means we can add multiples of δ to b to get b' as above.

Our next task is to modify b' by concatenating copies of multiples of δ so that we obtain a number, β, with $s(\beta) = \alpha$. Since δ is less than the product of the $2*n$ numbers $\alpha, \alpha+1, \ldots,$ $\alpha + 2*n - 1 - \mu*m*(n-1)$, the largest of which has $v(\alpha+n-1) \le \log_n(\alpha+n-1)+1$, we get

$$v(\delta) \le 2*n*(\log_n(\alpha+n-1)+1).$$

Since b was the minimal solution to (8), we have $v(b) \le v(\delta)$. We created b' by adding multiples of δ to b. Keeping track of the digits, we see that

$$v(b') \le \mu*m + v(\delta) + 1$$

as we modify b to get a terminal 0 with $\mu*m-1$ penultimate $(n-1)$'s. To do this by adding multiples of δ, we will be left with not more than $v(\delta)+1$ digits in front of the penultimate $(n-1)$'s, since we can first choose a multiple of δ less than $n*\delta$ to change the second base n digit (from right) of b to $n-1$ and then choose a multiple of $n*\delta$ less than $n^2*\delta$ to change the third base n digit (from right) to $n-1$, and so on. We continue until we add a multiple of $n^{\mu*m-2}*\delta$ less than $n^{\mu*m-1}*\delta$ to change the $\mu*m$ base n digit to $n-1$. A final multiple of $n^{\mu*m-1}*\delta$ may need to be added to assure that the $\mu*m+1$ digit is not $n-1$.

Since each digit can contribute at most $n-1$ to the digital sum, we get

$$s(\delta) \le 2*n*(\log_n(\alpha+n-1)+1)*(n-1)$$

and

$$s(b') \le (\mu*m + 2*n*(\log_n(\alpha+n-1)+1))*(n-1).$$

Since $b' \equiv b \equiv \alpha \mod \gamma$, Lemma 2 gives $(n-1)|(\alpha - s(b'))$. By Lemma 1, there exist k and k' so that $\gcd(s(k*\delta), s(k'*\delta)) = n-1$; thus,

$$\gcd(s(k*\delta), s(k'*\delta))|(\alpha - s(b')).$$

Remark 1 says that our k and k' may be chosen so that

$$s(k*\delta) \le (n-1)*(5+2*(2*n*(\log_{(n)}(\alpha+n-1)+1)))$$

and

$$s(k'*\delta) \le (n-1)*2*(5+2*(2*n*(\log_{(n)}(\alpha+n-1)+1))).$$

These two inequalities and the Growth Condition assure $\alpha - s(b') \ge s(k*\delta)*s(k'*\delta)$, so we can use Lemma 3 with $z = \alpha - s(b')$, $x = s(k*\delta)$, and $y = s(k'*\delta)$. We conclude that there are nonnegative integers r and t such that $\alpha - s(b') = r*s(k*\delta) + t*s(k'*\delta)$. But then

$$\alpha = r*s(k*\delta)+t*s(k'*\delta)+s(b'), \text{ so}$$
$$\alpha = s((k*\delta)_r(k'*\delta)_t b').$$

Using $(k*\delta)_r(k'*\delta)_t b' \equiv n^{\mu*m}-n \mod n^{\mu*m}$, we get

$$\alpha+i = s((k*\delta)_r(k'*\delta)_t b'+i)$$
and
$$\alpha+n+i-\mu*m*(n-1) = s((k*\delta)_r(k'*\delta)_t b'+n+i) \tag{10}$$

for $i=0,1,\ldots,n-1$. Since $(k*\delta)_r(k'*\delta)_t b' \equiv b \mod \delta$, (8) assures

$$(\alpha+i)|((k*\delta)_r(k'*\delta)_t b'+i)$$
and
$$(\alpha+n+i)|((k*\delta)_r(k'*\delta)_t b'+n+i) \tag{11}$$

for all $i=0,1,\ldots,n-1$. By (10), (11), and the definition of an n-Niven number, $(k*\delta)_r(k'*\delta)_t b'$ is the first of $2*n$ consecutive n-Niven numbers. □

Remark 3: We note that we have proved something stronger than the theorem, namely, that there exist infinitely many sequences of $2*n$ consecutive n-Niven numbers, since there exist infinitely many α satisfying Condition I, Condition II, and the Growth Condition.

3. EXAMPLES

Example 1: For $n=2$ we get $\mu=2, m=2$, and the conditions (3)-(5), (7),

$$\alpha \equiv 0,1 \mod 3,$$
$$\alpha \equiv 6 \mod 8,$$
$$\alpha \geq 36033.$$

Taking, for example, $\alpha = 36046$, we get the base 2 representations:

$$b = 1_5001_40101_3001001010 1_70_31011010_51_40_41010_{(2)};$$

$$\delta = 101_301101101011010110110_4100110101010 10_31_601001_400_{(2)}.$$

Then, letting $b' = b+7*\delta$, we get the right number of penultimate 1's:

$$b' = 1011001_3001_300101010_31101010_31101101_50_31_501_301 01_30_{(2)}.$$

We easily see that $s(b')=37$. Now we want to follow Lemma 1 to get multiples of δ with relatively prime base 2 digital sums. First, we want $\delta' = (n+1)*\delta$ as **a** has a terminal $n-1$. Using δ' in place of δ, we get $k = 2*2^{62}+1$ and $k' = (2^{122}+1)k = 2^{185}+2^{122}+2^{63}+1$. Then we see that

$$\begin{aligned} k*\delta' = {} & 10_31100100 10_410_410_310_31_3001_90101_401_4011010_310_3 \\ & 110010010_410_410_310_31_3001_90101_401_40110100_{(2)} \end{aligned}$$

with $s(k*\delta')=64$ and

$$\begin{aligned} k'*\delta' = {} & 10_3110010010_410_410_310_31_3001_90101_401_4011010_3 \\ & 10_3110010010_410_410_310_31_3001_90101_401_401_30_4 \\ & 110010010_410_410_310_31_3001_90101_401_4011010_310_3 \\ & 110010010_410_410_310_31_3001_90101_401_40110100_{(2)} \end{aligned}$$

with $s(k'*\delta')=127$. It is easy to see that

$$\alpha - s(b') = 36009 = 517*64+23*127 = 517*s(k*\delta')+23*s(k'*\delta'),$$

so

$$s((k*\delta')_{517}(k'*\delta')_{23}b') = 36046 = \alpha.$$

Thus, $(k*\delta')_{517}(k'*\delta')_{23}b'$ is the base 2 representation of the first number in a sequence of 4 consecutive 2-Niven numbers.

We note that the Growth Condition, while assuring we can get α as a digital sum, results in large numbers. In practice, much smaller α satisfying Congruence Conditions I and II can be digital sums of $2*n$ consecutive n-Niven numbers.

Example 2: For $n=2$, we get $\mu=2, m=2$ and the congruence conditions $\alpha\equiv 0,1$ mod 3 and $\alpha\equiv 6$ mod 8. $\alpha=6$ is such an α (although it clearly does not satisfy the Growth Condition). This leads to $b=342=101010110_{(2)}$ and $\delta=420=110100100_{(2)}$. It is easy to see that $\beta=b+14*\delta=6222$ has base 2 expansion $1000010011110_{(2)}$, so $s(\beta)=\alpha$. This means β is the first of a sequence of four consecutive 2-Niven numbers.

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