RETROGRADE RENEGADES AND THE PASCAL CONNECTION II: REPEATING DECIMALS REPRESENTED BY SEQUENCES OF DIAGONAL SUMS OF GENERALIZED PASCAL TRIANGLES APPEARING FROM RIGHT TO LEFT

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(Submitted January 1992)

INTRODUCTION

Repeating decimals containing the Fibonacci and Lucas numbers when their repetends are viewed in retrograde fashion, reading from the rightmost digit of the repeating cycle toward the left, have been explored in [1], [2], [3], [4], and [5]. Here, the sequences of generalized Fibonacci numbers $u(n; p, q)$ which can be interpreted as sums along diagonals in Pascal's binomial coefficient triangle [6] and extended to multinomial coefficient arrays [7] are found within repetends, both as read left to right and as read right to left.

1. BINOMIAL DIAGONAL SUMS

Let $u(n; p, q)$ be the sum of terms found along the rising diagonals of Pascal's binomial coefficient array written in left-justified form,

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \tag{1.1}$$

Call the top row the zero$^{\text{th}}$ row and the left-most column the zero$^{\text{th}}$ column. Then $u(n; p, q)$ is the sum of those elements found by beginning in the zero$^{\text{th}}$ column and n^{th} row and taking steps p units up and q units right throughout the left-justified array. Note that $u(n; 1, 1) = F_{n+1}$, the $(n+1)^{\text{st}}$ Fibonacci number. The sequence $u(n; p, 1)$ has the generating function [7]

$$\frac{1}{1-x-x^{p+1}} = \sum_{n=0}^{\infty} u(n; p, 1)x^n \tag{1.2}$$

which converges for $|x| < 1/2$. From the generating function, the recursion for the $u(n; p, 1)$ is

$$u(n; p, 1) = u(n-1; p, 1) + u(n-1-p; p, 1), \quad n \geq p+1,$$

where $u(n; p, 1) = 1$ for $n = 0, 1, \ldots, p$.

Then, taking $x = 1/10$ in (1.2), the decimal representation of the fraction

$$\frac{10^{p+1}}{10^{p+1} - 10^p - 1}$$

has successive terms $u(n; p, 1)$ appearing as successive digits in its repetend until carrying disguises the pattern. When $p = 1$, we display the Fibonacci numbers in the well known

$$\begin{array}{l} 100/89 = 1.12358 \\ \qquad\qquad\qquad\quad 13 \\ \qquad\qquad\qquad\quad 21\ldots \end{array}$$

where the decimal is moved from the usual 1/89 so that the left-most digit is $u(0; 1, 1) = F_1$. We also have

$$\begin{array}{l} 1.00 \\ .11 \\ .0121 \\ .001331\ldots \end{array}$$

or $1+11/10^2+11^2/10^4+\cdots = 10^2/(10^2-11) = 100/89$ by summing the geometric series. Similarly, for $u(n; p, 1)$, since $(10^p+1)^k$ displays the coefficients of the k^{th} row of Pascal's triangle interspersed by $(p-1)$ zeros, we can sum elements that are p units up and 1 unit over by summing the geometric series

$$1+(10^p+1)/10^{p+1}+(10^p+1)^2/10^{2p+2}+\cdots = 10^{p+1}/\left(10^{p+1}-(10^p+1)\right).$$

From [1], since $10(10^p+1)-1 = 10^{p+1}+9$, the repetend of the fraction $1/(10^{p+1}+9)$ ends in powers of (10^p+1), and thus gives $u(n; p, 1)$ reading from right to left in the repetend. Again, we have the symmetric coefficients of Pascal's triangle interspersed with $(p-1)$ zeros, so, for example, for $p = 2$, powers of 101 appear from the right as

$$\begin{array}{r} 1 \\ 101 \\ 10201 \\ 1030301 \\ 104060401 \end{array}$$

making as a sum $\ldots 6432111$ where $u(n; 2, 1)$: $1, 1, 1, 2, 3, 4, 6, \ldots$. Notice that we are summing elements that are up 2 and over 1 in the Pascal array (1.1), applying the Pascal connection of [1].

So far, the sequences $u(n; p, 1)$ mimic the Fibonacci sequence in these applications. However, $u(n; 1, q)$ is more challenging.

Start with $u(n; 1, 2)$: $1, 1, 1, 2, 4, 7, 12, 21, \ldots$, which has zero column elements in its definition. Then $u(n; 1, 2)$ has the associated sequence $v(n; 1, 2)$: $0, 1, 2, 3, 5, 9, \ldots$, $n = 0, 1, 2, \ldots$, formed by summing up 1 and over q throughout array (1.1) but starting with column one instead of column zero. Consider

$$\begin{array}{l} 10.00 \\ \;\;\, .11 \\ \;\;\, .00121 \\ \;\;\, .00001331 \\ \;\;\, .00000014641\ldots \end{array}$$

which is $1/10^{-1}+11/10^2+11^2/10^5+\cdots = 10^4/(10^3-11) = 10000/989$. The coefficients of successive powers of ten appearing are $1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, \ldots$, where the odd terms give $u(n; 1, 2)$ and the even terms give $v(n; 1, 2)$, and we see powers of 11, and $989 = 10^3 - 11$.

Now, $10^2 \cdot 11 - 1 = 1099$ gives powers of 11 shifted in groups of 2 to make the same sum from the right in 1/1099, which ends in

$$\begin{array}{r} 1211101 \\ 1331 \\ 14641 \\ \dots \end{array}$$

which sums to ...975432211101, where the $u(n; 1, 2)$ and the $v(n; 1, 2)$ are interleaved.

Now, $u(n; 1, 3)$ begins on column zero, and has two related sequences $v(n; 1, 3)$ and $w(n; 1, 3)$ that begin with columns one and two in array (1.1):

$$\begin{array}{l} u(n; 1, 3){:}\ 1, 1, 1, 1, 2, 5, 11, \dots \\ v(n; 1, 3){:}\ 0, 1, 2, 3, 4, 6, 11, \dots \\ w(n; 1, 3){:}\ 0, 0, 1, 3, 6, 10, \dots. \end{array}$$

Then $1/10^{-2} + 11/10^2 + 11^2/10^6 + \cdots = 10^6/(10^4 - 11) = 1000000/9989 = 100.11012\dots$, where the coefficients of 10^k are the three sequences interleaved with $u(n; 1, 3)$ appearing as every third term. That is,

$$\begin{array}{l} 100.00 \\ .11 \\ .000121 \\ .0000001331 \\ .00000000014641\dots \end{array}$$

which sums with coefficients

$$\underline{1}, 0, 0, \underline{1}, 1, 0, \underline{1}, 2, 1, \underline{1}, 3, 3, \underline{2}, 4, 6, \underline{5}, 6, 10, \underline{11}, 11, \dots.$$

Now, $9989 = 10^4 - 11$, and $10^3 \cdot 11 - 1 = 10999$. The three sequences $u(n; 1, 3)$, $v(n; 1, 3)$, and $w(n; 1, 3)$ are interleaved from right to left in the decimal repetend of 1/10999.

In general, $u(n; 1, q)$ appears as one of q sequences that interleave from left to right in $10^{2q}/(10^{q+1} - 11)$ and from right to left in the repetend of $1/(10^q \cdot 11 - 1)$. The q sequences are formed by summing up 1 and over q throughout array (1.1), beginning with column k, $k = 0, 1, \dots, q - 1$.

Things get more peculiar if we take $q \neq 1$. $p \neq 1$. Take $p = 2$, $q = 2$, and let $v(n; 2, 2)$ be the related sèquence beginning at column one:

$$\begin{array}{l} u(n; 2, 2){:}\ 1, 1, 1, 1, 2, 4, 7, 11, 17, 27, 44, 72, 117, 189, \dots \\ v(n; 2, 2){:}\ 0, 1, 2, 3, 4, 6, 10, 17, 28, 45, 72, 116, \dots. \end{array}$$

We have to split Pascal's triangle into even and odd rows:

$$\begin{array}{l} 10.000 \\ .121 \\ .0014641 \\ .000016(15)(20)(15)61 \\ .000000\ \ 1\ \ \ \ 8\ \ \ \dots \end{array}$$

which is $1/10^{-1}+11^2/10^3+11^4/10^7+\cdots=100000/9879=10^5/(10^4-11^2)$, and which has for coefficients of successive powers of 10 from left to right

$$\underline{1}, 0, \underline{1}, 2, \underline{2}, 4, \underline{7}, 10, \underline{17}, 28, \ldots$$

while

.11
.001331
.000015(10)(10)51
.000000 1 7 ...

has sum $11/10^2+11^3/10^6+11^5/10^{10}+\cdots=1100/9879$ with coefficients of successive powers of 10 given by

$$\underline{1}, 1, \underline{1}, 3, \underline{4}, 6, \underline{11}, 17, \underline{27}, 45, \ldots,$$

where we see in both sequences that every second term of $u(n; 2, 2)$ is interleaved with every second term of $v(n; 2, 2)$. Now, $10^4-11^2=9879$ and $11^2\cdot 10^2-1=12099$ so 1/12099 has powers of 11^2 in groups of 2 digits to give the same interleaved sequence from right to left as in the even split above.

If we take $p = 3$ and $q = 2$,

$$u(n; 3, 2)\text{: } 1, 1, 1, 1, 1, 2, 4, 7, 11, 16, 23, 34, 52, \ldots$$
$$v(n; 3, 2)\text{: } 0, 1, 2, 3, 4, 5, 7, 11, 18, 29, 45, 68, \ldots$$

then $1/10^{-1}+11^3/10^4+11^6/10^9+11^9/10^{14}+\cdots=10^6/(10^5-11^3)$ has as coefficients of 10^k from left to right

$$\underline{1}, 0, \underline{1}, 3, \underline{4}, 7, \underline{16}, 29, \underline{52}, \ldots,$$

where every second term comes from every third term in $u(n; 3, 2)$ and $v(n; 3, 2)$. There are three similar cases, where the other two come from $11^2/10^3+11^5/10^8+11^8/10^{13}+\cdots =11^2\cdot 10^2/(10^5-11^3)$ and $11/10^2+11^4/10^7+11^7/10^{12}+\cdots=11\cdot 10^3/(10^5-11^3)$. Now, $10^3\cdot 11^3-1=1330999$, and the repetend of 1/1330999 has powers of 11^3 appearing in groups of 3 from right to left, and has the primary interleaved sequence appearing from right to left.

In general, for $u(n; p, q)$, $q\ge 1$, $p\ge 1$, the primary case of q sequences interleaved where every q^{th} term is every p^{th} one in the q sequences, appears from left to right in the coefficients of 10^k in the decimal expansion of the fraction $10^{p+2q-1}/(10^{p+q}-11^p)$ while the repetend has the primary case appearing right to left in the repetend of $1/(10^q\cdot 11^p-1)$, where powers of 11^p appear in groups of q from right to left. If we take $q = 1$ in the formula for $u(n; p, q)$, we get $10^{p+1}/(10^{p+1}-11^p)$, which makes every p^{th} term of $u(n; p, 1)$ appear, in contrast to $10^{p+1}/(10^{p+1}-10^p-1)$, which makes all terms of $u(n; p, 1)$ appear.

These representations of $u(n; p, q)$ come from summing the geometric series

$$\frac{1}{10^{1-q}}+\frac{11^p}{10^{p+1}}+\frac{11^{2p}}{10^{2p+q+1}}+\cdots=\frac{10^{p+2q-1}}{10^{p+q}-11^p},$$

where 11^p gives coefficients of every p^{th} row of Pascal's triangle, 10^{p+1} gives a separate place value for each coefficient, the ratio $11^p / 10^{p+q}$ moves p rows up and q columns over in the array (1.1), $1/10^{1-q}$ puts all zero terms of the q sequences to the left of the decimal point, and $u(1; p, q)$ is the coefficient of 1/10 in the decimal expansion. Summing all columns down catches all summands in the infinite sum, and makes q sequences interleaved. The repetend of $1/(10^q \cdot 11^p - 1)$, read from right to left, ends in p^{th} powers of 11 moved over q columns, again giving q interleaved sequences.

It is possible to make decimals for $u(n; p, q)$ that list every term of the q interleaved sequences if $(p, q) = 1$. If we sum

$$\frac{1}{10^{1-q}} + \frac{10^p+1}{10^{p+1}} + \frac{10^p+1}{10^{p+q+1}} + \cdots = \frac{10^{p+2q-1}}{10^{p+q} - 10^p - 1},$$

we have lined up the array to give successive terms of $u(n; p, q)$, $n = 0, 1, 2, \ldots$, interleaved with the successive terms of the other $q-1$ related sequences. Note that $10^p + 1 = 10\ldots01, (p-1)$ zeros, will give coefficients of rows of Pascal's triangle interspersed with $(p-1)$ zeros, when raised to powers. The ratio $(10^p+1)/10^{p+q}$ gives successive rows shifted p units over and q units up to line up coefficients for summing. Then, $u(1; p, q)$ is the coefficient of 1/10 and $u(0; p, q)$ appears to the left of the decimal point, as do the zero terms of the other $(q-1)$ sequences. The terms of the sequence $u(n; p, q)$ are interspersed with the terms of the q related sequences as before. However, if $(p, q) \neq 1$, coefficients will not line up for proper summing to make $u(n; p, q)$. If $p = q$, we get $u(n; p, q)$ as given by the fraction

$$\frac{10^{3p-1}}{10^{2p} - 10^p - 1} = \frac{10^{p-1} \cdot (10^p)^2}{(10^p)^2 - (10^p)^1 - 1}$$

where $u(n; 1, 1)$ is given by $10^2(10^2 - 10^1 - 1)$ from our earlier fraction for $u(n; p, 1)$. If we replace 10 by 10^p, we write a fraction where $u(n; 1, 1)$ appears as every p^{th} term, interspersed by $(p-1)$ zeros, and we get the fraction for $u(n; p, p)$ except for a shift of $(p-1)$ places in the decimal point. We also line up previously derived sequences whenever $(p, q) \neq 1$. Let $(p, q) = d$. Then the fraction for $u(n; p, q)$ gives the sequences $u(n; p/d, q/d)$ as every d^{th} coefficient, interspersed so that the q/d sequences are interleaved, but the decimal point is moved $(d-1)$ places to the right. When $(p, q) = 1$, $u(n; p, q)$ is given from the right in the repetend of the fraction $1/[10^q \cdot (10^p + 1) - 1]$ appearing as part of the q interleaved sequences.

Of course, [7] gives the generating function for $u(n; p, q)$ as

$$\frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n; p, q)x^n$$

which converges for $|x| < 1/2$. Taking x = 1/10 and simplifying, the decimal expansion of $9^{q-1} \cdot 10^{p+2q-1} / (9^q \cdot 10^p - 1)$ had $u(n; p, q)$ appearing as coefficients of 10^k from left to right but carrying makes the pattern disappear quickly. The pattern continues longer if we use $x = 1/10^k$, $k > 1$, and look at groups of k digits. This representation, however, does not lead to the same sequences being found in patterns from right to left except when $q = 1$.

2. TRINOMIAL DIAGONAL SUMS

The coefficients appearing in expansions of the trinomial $(1+x+x^2)^n, n = 0, 1, 2, \ldots$, written in left-justified form, are

$$
\begin{array}{ccccccccccc}
1 \\
1 & 1 & 1 \\
1 & 2 & 3 & 2 & 1 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\tag{2.1}
$$

Call the top row the "zero$^{\text{th}}$ row" and the left column the "zero$^{\text{th}}$ column." Let $u(n; p, q)$ be the sum of the term in the left column and the n^{th} row and the terms obtained by taking steps p units up and q units right throughout the array. Then, from [7],

$$\frac{1}{1-x-x^{p+1}-x^{2p+1}} = \sum_{n=1}^{\infty} u(n; p, 1)x^n. \tag{2.2}$$

As in §1, the decimal expansion of $10^{2p+1}/(10^{2p+1}-(10^{2p}+10^p+1))$ has $u(n; p, 1)$ as the coefficients of successive powers of 1/10 where $u(0; p, 1)$ appears left of the decimal point, and powers of $(10^{2p}+10^p+1)$ appear as

$$1+\frac{(10^{2p}+10^p+1)}{10^{2p+1}}+\frac{(10^{2p}+10^p+1)^2}{10^{4p+2}}+\cdots = \frac{10^{2p+1}}{10^{2p+1}-(10^{2p}+10^p+1)}.$$

Since $10(10^{2p}+10^p+1)-1 = 10^{2p+1}+10^{p+1}+9$, the repetend of $1/(10^{2p+1}+10^{p+1}+9)$ ends in $u(n; p, 1)$ as in the binomial case. Note that $p = 1$ gives the Tribonacci case reported in [1]. As before, the case for $u(n; p, 1)$ is simple because the known generating function almost takes care of it, but we are on our own when $q > 1$.

Now, suppose that $p = 1$, and consider $u(n; 1, q)$, $q \geq 1$. Then the decimal expansion of $10^{2q+1}/(10^{q+2}-111)$ gives q interleaved sequences as in the binomial case from left to right, while the repetend of $1/(10^q \cdot 111-1)$ gives q interleaved sequences from right to left.

The case for $u(n; p, q)$ is given from left to right by $10^{2p+2q-1}/(10^{2p+q}-111^p)$, which generates every p^{th} term of q interleaved sequences. Each of the q sequences is generated by starting in the k^{th} column, n^{th} row, and summing elements found by taking steps of up p, right q, throughout array (2.1), for $k = 0, 1, \ldots, q-1$. Similarly to the binomial case, we sum

$1/10^{1-q} + 111^p/10^{2p+1} + 111^{2p}/10^{4p+q+1} \ldots$, where the geometric ratio is $111^p/10^{2p+q}$ to select every p^{th} row and move q units right in the array, 111^p contains $2p+1$ terms, and the zero term for each sequence appears to the left of the decimal point. The repetend of the fraction $1/(10^q \cdot 111^p - 1)$ will have the same interleaved sequences appearing from right to left and will show powers of 111^p diagonalized from the right.

Similarly to the binomial, we can write every term of the q interleaved sequences for $u(n; p, q)$, $(p, q) = 1$ from left to right by summing

$$\frac{1}{10^{1-q}} + \frac{10^{2p} + 10^p + 1}{10^{2p+1}} + \frac{(10^{2p} + 10^p + 1)^2}{10^{4p+q+1}} + \cdots = \frac{10^{2p+2q-1}}{10^{2p+q} - 10^{2p} - 10^p - 1}$$

and the same q sequences appear from right to left in the repetend of $1/(10^q \cdot (10^{2p} + 10^p + 1) - 1)$.

3. MULTINOMIAL DIAGONAL SUMS

Write the coefficients appearing in expansions of the multinomial $(1 + x + x^2 + \cdots + x^m)^n$, $n =$ 0, 1, 2, ..., in left-justified form. Call the top row the zero$^{\text{th}}$ row and the left column the zero$^{\text{th}}$ column. Let $u(n; p, q)$ be the sum of the term in the zero$^{\text{th}}$ column and n^{th} row and the terms obtained by taking steps p units up and q units right throughout the array. Then, from [7],

$$\frac{1}{1 - \mathrm{x} - \mathrm{x}^{\mathrm{p}+1} - x^{2p+1} - \cdots - x^{mp+1}} = \sum_{n=0}^{\infty} u(n; p, 1) x^n.$$

Thus, the decimal expansion of $10^{mp+1}/(10^{mp+1} - (10^{mp} + 10^{(m-1)p} + \cdots + 10 + 1))$ has $u(n; p, 1)$ appearing as coefficients of successive powers of 1/10, where $u(0; p, 1)$ appears left of the decimal. The repetend of the fraction $1/(10^{mp+1} + 10^{(m-1)p+1} + 10^{(m-2)p+1} + \cdots + 10^{p+1} + 9)$ has $u(n; p, 1)$ appearing from right to left as before. We expect that the repetend $1/(10^q \cdot 11\ldots1^p - 1)$, where $(m + 1)$ 1's appear in the multiplier of 10^q, would generate the p^{th} terms of q interleaved sequences related to $u(n; p, q)$ from right to left as before, and that the repetend of the fraction $10^{mp+2q-1}/(10^{mp+q} - (11\ldots1)^p)$ would generate those same interleaved sequences from left to right because we still have a "Pascal connection" available. The $(mp + 1)$ coefficients of the p^{th} row are generated by $(11\ldots1)^P$ (there are $m + 1$ 1's), and the geometric ratio is $(11\ldots1)^p/10^{mp+q}$ to select every p^{th} row and move q units right, so we sum $1/10^{1-q} + (11\ldots1)^p/10^{mp+1} +$ $(11\ldots1)^{2p}/10^{2mp+q+1} + \cdots$ to form $10^{mp+2q-1}/(10^{mp+q} - (11\ldots1)^p)$. As before, we can write all the terms of $u(n; p, q)$, $(p, q) = 1$, interleaved as part of the q sequences, left to right by

$$\frac{10^{mp+2q-1}}{10^{mp+q} - (10^{mp} + 10^{(m-1)p} + \cdots 10^p + 1}$$

and from right to left in the repetend of

$$\frac{1}{10^q \cdot (10^{mp} + 10^{(m-1)p} + \cdots + 10^p + 1) - 1}.$$

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AMS numbers: 11A63; 05A10; 11B39

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