DETERMINING THE DIMENSION OF FRACTALS GENERATED BY PASCAL'S TRIANGLE

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1. INTRODUCTION

Pascal's triangle has long fascinated mathematicians with its intriguing number patterns. The triangle consists of the binomial coefficients of the expansion of $(x+y)^n$, where n is a nonnegative integer. When numbering the rows starting with 0 and the elements of each row starting with 0, the terms are $\binom{\text{row}}{\text{element}}$, where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, assuming $k \le n$. The first eight rows are shown in Figure 1.

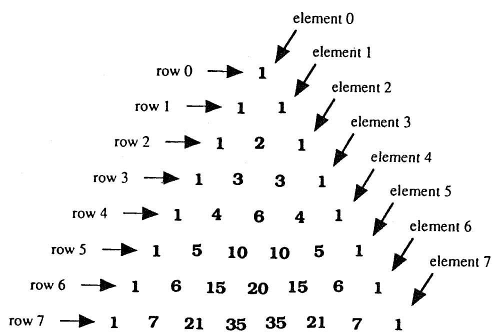


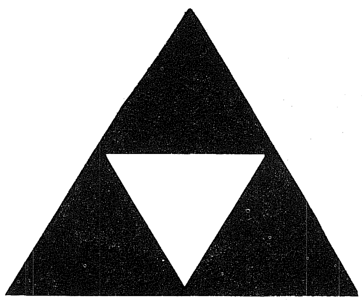
FIGURE 1. Eight rows of Pascal's triangle

A multidimensional pyramid of multinomial coefficients can be generalized from the definition for Pascal's triangle. Each entry is represented as

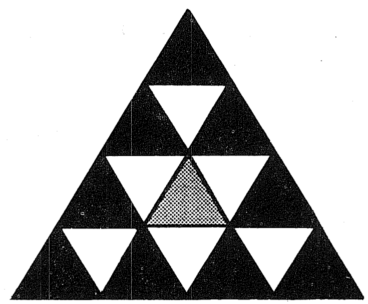
$$\binom{c}{a^1, a^2, a^3, \ldots, a^k} = \frac{c!}{a^1!a^2!a^3!\cdots a^k!},$$

where $c = a^1 + a^2 + a^3 + \cdots + a^k$. (Superscripts are used here to allow subscripts to take on a different meaning later in the paper.) This is the coefficient of the term $x_1^{a^1} x_2^{a^2} x_3^{a^3} \ldots x_k^{a^k}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_k)^c$, where a^i is the exponent of x_i, $i = 1, 2, \ldots, k$. In the case where $k = 3$, a triangular pyramid of integers is formed with each of the lateral faces duplicating Pascal's triangle. The apex of the pyramid is formed by a single 1, and each triangle below corresponds to a particular value of c. The vertices of each such triangle correspond to $a^1 = c$, $a^2 = c$, and $a^3 = c$.

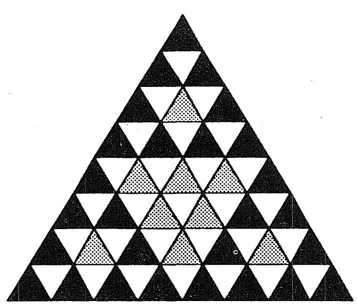
Consider the replacement of each element in Pascal's triangle by its remainder upon division by a prime p. This is called reducing to the least residue modulo p. The set of nonzero entries in this reduced triangle corresponds to a fractal according to the following construction. Consider the first p^n rows of Pascal's triangle, and call this set P_{p^n}. For each p^n, we construct a subset A_{p^n} of the triangle with vertices $(0,0)$, $(1,0)$, and $\left(\frac{1}{2},1\right)$. The fractal generated will lie in this triangle. Let $a_0 = \left(\frac{1}{2}, 1\right)$, and let $a_1, a_2, \ldots, a_{p^n-1}$ be equally spaced points of the segment joining $\left(\frac{1}{2}, 1\right)$ with $(0,0)$ such that a_{p^n} is $(0,0)$. Let $b_1, b_2, \ldots, b_{p^n-1}$ be equally spaced along the segment joining $\left(\frac{1}{2}, 1\right)$ with $(1,0)$. Finally, let $c_1, c_2, \ldots, c_{p^n-1}$ divide the segment $(0,0)$ to $(1,0)$ into p^n equal parts. Connect pairs of points of the form a_i, b_i; b_i, c_i; and a_i, c_{p^n-i}, to form p^{2n} triangular regions, $\frac{(p^n+1)p^n}{2}$ of them "pointing upwards" and $\frac{(p^n+1)p^n}{2}$ "pointing downwards." The first p^n rows of Pascal's triangle have a total of $\frac{(p^n+1)p^n}{2}$ entries. Now every integer in Pascal's triangle can be associated with a triangle which points up. Define the sets A_{p^n} as follows: $A_{p^n} = \{x | x$ belongs to a triangle associated with a nonzero entry in Pascal's triangle$\}$. The fractal associated with Pascal's triangle modulo a prime number is the limiting set A' as n goes to infinity. For $p = 2$, this set is the Sierpinski triangle.



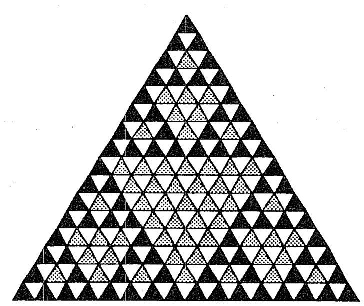
A_{2^1}: In 2^1 rows, there are 3 nonzero entries in $\frac{(2^1+1)2^1}{2} = 3$ upward triangles



A_{2^2}: In 2^2 rows there are 3^2 nonzero entries in $\frac{(2^2+1)2^2}{2} = 10$ upward triangles



A_{2^3}: In 2^3 rows, there are 3^3 nonzero entries in $\frac{(2^3+1)2^3}{2} = 36$ upward triangles.



A_{2^4}: In 2^4 rows, there are 3^4 nonzero entries in $\frac{(2^4+1)2^4}{2} = 136$ upward triangles.

FIGURE 2. Pascal's triangle reduced modulo 2: entries congruent to zero are shaded and nonzero entries are blackened

Figure 3 shows 256 rows of Pascal's triangle reduced modulo 2. Willson [1] showed that a cellular automaton with a linear transformation, that is, one in which each entry is determined by some linear combination of entries in the previous row, may have fractional fractal dimension. Pascal's triangle satisfies this criterion.

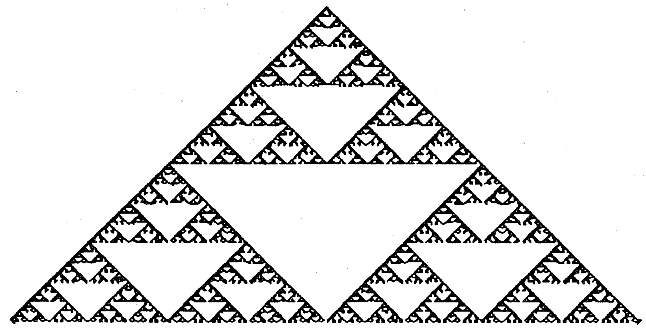


FIGURE 3. Pascal's triangle modulo 2

The fractals described by this construction have fractional fractal dimension. The fractal dimension D is defined as follows (see [2]): Let A be a compact subset of X, where (X, d) is a metric space. For each $\varepsilon > 0$, let $N(A, \varepsilon)$ be the minimum number of closed balls of radius ε which are needed to cover A. Then, the fractal dimension of A is given by

$$D = \lim_{\varepsilon \to 0} \frac{\ln(N(A, \varepsilon)}{\ln(1/\varepsilon)}.$$

By a slight variation of the Box Counting Theorem, the dimension of the fractals can be determined by using $\varepsilon = \frac{1}{p^n}$ and $N(A, \varepsilon) =$ number of triangles in the first p^n rows associated with nonzero entries in the reduced Pascal triangle. Then

$$D = \lim_{\varepsilon \to \infty} \frac{\ln(\text{\# of nonzero entries in the first } p^n \text{ rows})}{\ln(p^n)}.$$

In section 2, a theorem about divisibility of multinomial coefficients by powers of primes is proven. This theorem is used to prove that the fractal dimension of Pascal's triangle modulo a prime p is

$$\frac{\ln[p(p+1)/2]}{\ln p}.$$

This determination is supported with computer results in section 3. Finally, in section 4, a generalization of this formula is proven for the analog of Pascal's triangle which contains multinomial coefficients reduced modulo p.

2. THEORETICAL DETERMINATION OF DIMENSION OF PASCAL'S TRIANGLE

The symbol $p^r | x$ means that p^r divides x with remainder zero. The symbol $p^r \| x$ means r is the largest integer for which $p^r | x$. Throughout the paper, p will refer to a prime.

First, we will work toward the dimension of Pascal's triangle reduced modulo p. The following lemma, proven by C. T. Long [3], will allow us to determine the requirements for divisibility of multinomial coefficients by a prime.

Long's Lemma: If p is prime, $n = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots + a_r p^r$ with $a_r \neq 0$ and $0 \le a_i < p$ for each $i \le r$, and $p^e \| n!$, then

$$e = \frac{n - (a_0 + a_1 + a_2 + a_3 + \cdots + a_r)}{p-1}.$$

Now we will apply this to determine the divisibility of multinomial coefficients. Let a^1, a^2, $a^3, \ldots, a^k$ be positive integers and let a_j^i denote the coefficient of p^j in the base p representation of a^i, so that if a^i has m_i digits in its base p representation, then

$$a^i = \sum_{j=0}^{m_i} a_j^i p^j.$$

The sum of these a^i is denoted by c, so that

$$c = \sum_{i=1}^{k} a^i = \sum_{j=0}^{m+s} c_j p^j,$$

where m is the maximum value of m_i. The c_j are the base p digits of c, and the additional s digits in c allow for large carries in the sum of the a^i.

Theorem 1—Multinomial Divisibility Theorem: For prime p, $p^r \| \binom{c}{a^1, a^2, a^3, \ldots, a^k}$, iff r is the sum of the carries made when adding the a^i in base p.

Proof: Let $d_0, d_1\, d_2, d_3, \ldots, d_{m+s-1}$ be the carries when adding the a^i, so that the sum of the digits in each position equals the digit for c in that position plus p times the carry to the next digit in c:

$$\sum_{i=1}^{k} a_0^i = c_0 + pd_0, \quad d_0 + \left(\sum_{i=1}^{k} a_1^i\right) = c_1 + pd_1, \quad d_i + \left(\sum_{i=1}^{k} a_2^i\right) = c_2 + pd_2, \ldots,$$

$$d_{m-1} + \left(\sum_{i=1}^{k} a_m^i\right) = c_m + pd_m, \qquad d_m = c_{m+1} + pd_{m+1},$$

$$d_{m+1} = c_{m+2} + pd_{m+2}, \ldots, \quad d_{m+s-2} = c_{m+s-1} + pd_{m+s-1}, \quad d_{m+s-1} = c_{m+s}.$$

Notice that extra digits beyond c_{m+1} in c occur if the carry from the m^{th} digit of c is greater than p.

Solving for the d_i, we have:

$$\frac{\left(\sum_{i=1}^{k} a_0^i\right) - c_0}{p} = d_0, \quad \frac{\left(\sum_{i=1}^{k} a_1^i\right) - c_1 + d_0}{p} = d_1, \ldots, \quad \frac{\left(\sum_{i=1}^{k} a_m^i\right) - c_m + d_{m-1}}{p} = d_m,$$

$$\frac{-c_{m+1} + d_m}{p} = d_{m+1}, \ldots, \quad \frac{-c_{m+s-1} + d_{m+s-2}}{p} = d_{m+s-1}, \quad d_{m+s-1} - c_{m+s} = 0.$$

The sum of the carries is $\sum_{i=0}^{m+s-1} d_i$, which equals

$$\frac{\left(\sum_{i=1}^{k} a_0^i\right) - c_0 + \left(\sum_{i=1}^{k} a_1^i\right) - c_1 + d_0 + \cdots + \left(\left(\sum_{i=1}^{k} a_m^i\right) - c_m + d_{m-1}\right) + (-c_{m+1} + d_{m-1}) + \cdots + (-c_{m+s-1} + d_{m+s-2}) + (-c_{m+s} + d_{m+s-1})}{p}$$

$$= \frac{\left(\sum_{i=1}^{k} a_0^i\right) + \left(\sum_{i=1}^{k} a_1^i\right) + \left(\sum_{i=1}^{k} a_2^i\right) + \cdots + \left(\sum_{i=1}^{k} a_m^i\right) - (c_0 + c_1 + \cdots + c_{m+s}) + (d_0 + d_1 + \cdots + d_{m+s-1})}{p}$$

$$= \frac{\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right] - \left(\sum_{i=0}^{m+s} c_i\right) + \left(\sum_{i=0}^{m+s-1} d_i\right)}{p} = \sum_{i=0}^{m+s-1} d_i.$$

Multiplying by p on both sides,

$$p\left(\sum_{i=0}^{m+s-1} d_i\right) = \left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right] - \left(\sum_{i=0}^{m+s} c_i\right) + \left(\sum_{i=0}^{m+s-1} d_i\right).$$

Hence,

$$(p-1)\left(\sum_{i=0}^{m+s-1} d_i\right) = \left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right] - \left(\sum_{i=0}^{m+s} c_i\right).$$

Dividing by $p-1$, we get

$$\sum_{i=0}^{m+s-1} d_i = \frac{\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right] - \left(\sum_{i=0}^{m+s} c_i\right)}{p-1}.$$

Since $c = \sum_{i=1}^{k} a^i$, we can add $c - \sum_{i=1}^{k} a^i$, so that

$$\sum_{i=0}^{m+s-1} d_i = \frac{\left[\sum_{j=0}^{m}\sum_{i=1}^{k} a_j^i\right] - \left(\sum_{i=0}^{m+s} c_i\right) + c - \left(\sum_{i=1}^{k} a^i\right)}{p-1}$$

$$= \frac{c - \left(\sum_{i=0}^{m+s} c_i\right)}{p-1} - \frac{\sum_{i=1}^{k}\left(a^i - \sum_{j=0}^{m} a_j^i\right)}{p-1}.$$

By Long's Lemma, $\left(c - \sum_{i=0}^{m+s} c_i\right)/(p-1)$ is the highest power of p which divides $c!$ Likewise, each $\left(a^i - \sum_{j=0}^{m} a_j^i\right)/(p-1)$ is the highest power of p which divides a^i. Thus, the previous expression simplifies to

$$\sum_{i=0}^{m+s-1} d_i = (\text{highest power of } p \text{ which divides } c!) - \sum_{i=1}^{k} (\text{highest power of } p \text{ which divides } a^i!)$$

The highest power of p which divides the multinomial coefficient $\frac{c!}{a^1!a^2!a^3!\ldots a^k!}$ is the highest power which will divide $c!$ minus the highest powers which divide each of the a^i. Therefore,

$$\sum_{i=0}^{m+s-1} d_i = \text{ highest power of } p \text{ which divides } \binom{c}{a^1, a^2, \ldots, a^k}.$$

This theorem can now be used to develop a more efficient method for determining entries in Pascal's triangle which are not divisible by p, in order to determine the dimension of Pascal's triangle. When computing the self-similarity dimension using p^m rows, each entry corresponds to a triangle of length $1/p^m$. If we consider covering the fractal with triangular boxes, the number of boxes needed to cover the fractal is equal to the number of entries which are not congruent to zero. The dimension is then

$$\lim_{\#\text{ rows}\to\infty} \frac{\ln(\#\text{ nonzero entries})}{\ln(\#\text{ rows considered})}.$$

Theorem 2—Dimension of Pascal's Triangle Modulo p: The fractal generated by Pascal's triangle modulo p has fractal dimension

$$\frac{\ln[p(p+1)/2]}{\ln p}.$$

Proof: Consider entries $\binom{A}{B}$ in Pascal's triangle, such that all $a_i \ge b_i$ in the base p representations :

$$A = a_0p^0 + a_1p^1 + a_2p^2 + a_3p^3 + \cdots + a_mp^m;$$
$$B = b_0p^0 + b_1p^1 + b_2p^2 + b_3p^3 + \cdots + b_mp^m.$$

We require that $a_m \neq 0$ so that m cannot be reduced, but it is not necessary that $b_m \neq 0$. Using the binomial case of Theorem 1, the highest power of p which divides the term $\binom{A}{B}$ is equal to the number of carries when $(A-B)$ is added to B in base p.

$$A - B = (a_0 - b_0)p^0 + (a_1 - b_1)p^1 + (a_2 - b_2)p^2 + \cdots + (a_m - b_m)p^m.$$

$$B + (A-B) = (b_0 + (a_0 - b_0))p^0 + (b_1 + (a_1 - b_1))p^1 + \cdots + (b_m + (a_m - b_m))p^m.$$

Since each $a_i \ge b_i$, and $a_i < p$, no carries will occur when adding $a_i - b_i$ and b_i. Conversely, if, for any i, $b_i > a_i$, then the sum $(a_i - b_i) + b_i$ will cause a carry so that $p \mid \binom{A}{B}$. Thus, in order to determine the entries which are not divisible by p, we need only that the $a_i \ge b_i$ for each digit in the base p representations.

The next step will be to determine the fractal dimension of Pascal's triangle modulo p. As discussed above, to find the dimension of this fractal, we find the number N of triangles of side length ε which correspond to nonzero entries. If we consider Pascal's triangle down to row p^m, scaled to have side length 1, then the triangles have side length $\varepsilon = 1/p^m$, such that each triangle corresponds to exactly one entry.

We are interested in determining how many entries $\binom{A}{B}$ in Pascal's triangle, through the first p^m rows, are *not* divisible by p. By the above argument, this is equal to the number of ways to choose A and B such that $0 \le B \le A < p^m$ where $0 \le b_i \le a_i < p$ for $i = 0, 1, \ldots, m$. The number of ways to choose the first such pair of base p digits a_0, b_0 is $p(p+1)/2$ by a simple counting argument. Therefore, the number of ways to choose $m+1$ such pairs independently is $[p(p+1)/2]^m$. The number of boxes of size $(1/p^m)$ needed to cover the p^m rows of the triangle is $[p(p+1)/2]^m$. Using the self-similarity definition of dimension, the fractal has dimension

$$\lim_{m\to\infty} \frac{\ln(\#\text{ nonzero entries})}{\ln(p^m \text{rows considered})} = \lim_{m\to\infty} \frac{\ln[p(p+1)/2]^m}{\ln(p^m)},$$

which simplifies to $\frac{\ln[p(p+1)/2]}{\ln(p)}$.

3. COMPUTER VERIFICATION OF THEORETICAL RESULTS

In 1989, N. S. Holter et al. [4] proposed without proof a dimension for Pascal's triangle modulo p. Their formula agrees with the one determined here. Their determination was based on a computer program which considers all elements whose distance form the top of the triangle is less than n and counts the number of elements x which are not divisible by the modulus. The values $D_n = \frac{\ln(x)}{\ln(n)}$ are approximations to the dimension, and $\lim_{n\to\infty} D_n$ is the fractal dimension. In their paper, they reported values of D_n for $n = 198$, 500, and 1000.

This experimental determination of dimension has two shortfalls. First, these cutoff values fall at different places in the approximations to the fractal, so that the figures cannot be rescaled to produce similar images. Second, since the determination is based on distance from the top rather than row numbers, the method sweeps out sectors rather than the triangular fractals studied here. These two problems make it difficult to determine the true limit, which is obscured by changes in marking places. Figure 4 illustrates these differences in the two determinations. (See, also, Table 1 on page 119.)

For this paper, a different experimental determination was performed using values of n which were powers of the modulus used. Also, triangles were used rather than sectors. Using this method and larger values of n, the values did approach the theoretically determined limit of

$$\frac{\ln 3}{\ln 2} = 1.58496\ldots.$$

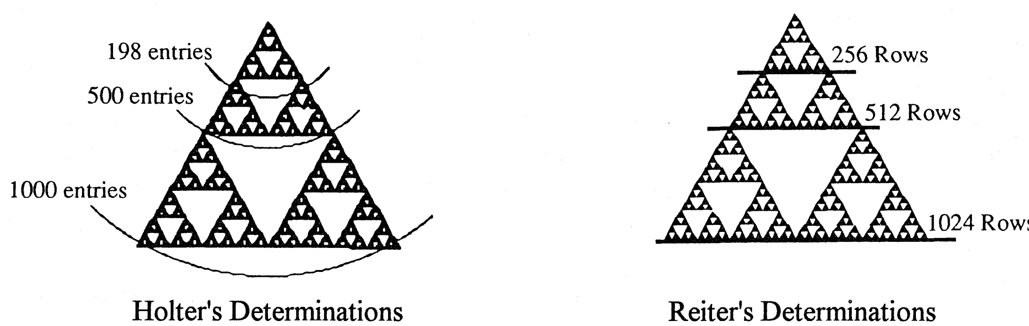


FIGURE 4. Diagrams of cutoff values in computer determinations of the dimension of Pascal's triangle modulo 2

TABLE 1. Data from computer determinations of the dimension of Pascal's triangle modulo 2 and 4

Holter's	# Rows in Pascal's Triangle	Reiter's
1.5681	198	
	256	**1.577785**
1.5716	500	
	512	**1.580738**
1.5738	1000	
	1024	**1.582439**
	2048	**1.583437**

4. GENERALIZATION TO MULTINOMIAL ANALOG OF PASCAL'S TRIANGLE

Now we will generalize to multinomial coefficients and the fractals generated by them. Using a method similar to that in Theorem 2, the dimension of the fractals generated by the multinomial coefficients modulo p will be determined and proved.

Theorem 3—Multinomial Dimension Theorem: Consider a prime p and a k-dimensional pyramid consisting of multinomial coefficients. The fractal formed when the entries which are not divisible by a particular prime p are shaded has fractal dimension equal to $\ln\binom{p-1+k}{k}/\ln p$.

Proof: In entries $\binom{c}{a^1, a^2, \ldots, a^k}$, let c denote the sum of the $a^i, i = 1, 2, \ldots, k$. Let $c = c_0p^0 + c_1p^1 + c_2p^2 + c_3p^3 + \cdots + c_mp^m$, where $c_j \le p-1$. According to Theorem 1, $\binom{c}{a^1, a^2, \ldots, a^k}$ is divisible by p if and only if at least one carry occurs in the summing of the base p expressions of the a^i. In any set of a^i for which $\binom{c}{a^1, a^2, \ldots, a^k}$ is not divisible by p, there must not be a carry when adding the a^i in base p. If no carries occur, then $c_j = a_j^1 + a_j^2 + a_j^3 + \cdots + a_j^k$ for each j. Since $c_j \le p-1$, we can write $p-1 = a_j^1 + a_j^2 + a_j^3 + \cdots + a_j^k + z$, where $z = (p-1) - c_j$ is a non-negative integer. Thus, we are partitioning $p-1$ units into $k+1$ base p digits. These $k+1$ digits are the k possible a^i and the z which "takes up the slack" in each digit. If we consider values of c which are only one digit in base p, then each a^i is only one digit, so there are $\binom{p-1+k}{k}$ choices for the set of a^i. This follows from the observation that there are $\binom{p-1+k}{k}$ solutions among nonnegative integers to the equation $x_1 + x_2 + \cdots + x_k + z = p-1$. For each increase by one in the number of digits in the base p expression of c, the number of entries which are not divisible by p increases by a factor of $\binom{p-1+k}{k}$. The digits are not interdependent because we know there are no carries. Increasing the number of digits increases the number of rows and rescales the image by a factor of p. Thus, the dimension of the fractal corresponding to the pyramid of multinomial reduced modulo p is equal to

$$\lim_{m\to\infty} \frac{\ln\binom{p-1+k}{k}^m}{\ln p^m} = \frac{\ln\binom{p-1+k}{k}}{\ln p}.$$

Notice that when $k = 2$, this agrees with the result in Theorem 2.

5. DISCUSSION

As early as 1972, W. A. Broomhead [4] noted the self-similar nature of Pascal's triangle reduced modulo a prime. A great deal of study has been done on the specific case of mod 2, which generates Sierpinski's triangle. No work has been done on the dimension of the multinomial coefficients as defined here.

There are many extensions of this work which deserve further study. When the entries are reduced to their least residue mod n, where n is an integer other than a prime, the result is a pattern with fractional dimension, but which is not strictly self-similar. The determination of the dimension of such a fractal is a natural extension. Because these fractals are the union of two fractals with different dimensions, they are not strictly self-similar. I conjecture that the dimension of such a fractal is equal to the dimension of the fractal corresponding to the largest prime factor of n. Recent work [5] done on the divisibility of entries in Pascal's triangle by products of primes could be the basis for rigorous proof. Other cellular automata and the fractals which they generate are also likely candidates for this type of dimensional study.

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