ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-682 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Let $T(n)$ be the triangular number $n(n + 1)/2$. Show that

$$T(L_{2n}) - 1 = \frac{1}{2}(L_{4n} + L_{2n}).$$

B-683 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Let $L(n) = L_n$ and $T_n = n(n + 1)/2$. Show that

$$L(T_{2n}) = L(2n^2)L(n) + (-1)^{n+1}L(2n^2 - n).$$

B-684 *Proposed by L. Kuipers, Sierre, Switzerland*

(a) Find a straight line in the Cartesian plane such that (F_n, F_{n+1}) and (F_{n+1}, F_{n+2}) are on opposite sides of the line for all positive integers n.

(b) Is the line unique?

B-685 *Proposed by Stanley Rabinowitz, Westford, Massachusetts, and Gareth Griffith, U. of Saskatchewan, Saskatoon, Saskatchewan, Canada*

For integers $n \geq 2$, find k as a function of n such that

$$F_{k-1} \leq n < F_k.$$

B-686 *Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada*

Let a and b be integers with $0 < a \le b$. Set $c_0 = a$, $c_1 = b$, and for $n \ge 2$ define c_n to be the least integer with $c_n/c_{n-1} > c_{n-1}/c_{n-2}$. Find a closed form for c_n in the cases:

(a) $a = 1$, $b = 2$; (b) $a = 2$, $b = 3$.

B-687 *Proposed by Jeffrey Shallit, U. of Waterloo, Ontario, Canada*

Let c_n be as in Problem B-686. Find a closed form for c_n in the case with $a = 1$ and b an integer greater than 1.

SOLUTIONS

Pell Parity Problem

B-658 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Prove that $Q_1^2 + Q_2^2 + \cdots + Q_n^2 \equiv P_n^2 \pmod 2$, where the P_n and Q_n are the Pell numbers defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;$$

$$Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1,\ Q_1 = 1.$$

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

More generally, it can be proved that

$$S = \sum_{i=1}^{n} Q_i^{k_i} \equiv P^h \pmod 2,$$

where $k_1, k_2, \ldots, k_n$ and h are arbitrary positive integers. Using the recurrence relation, it is readily seen that Q_i is odd for all i, so that $Q_i^{k_i}$ is. Therefore, S is odd (even) if n is odd (even). On the other hand, it is known that the Pell numbers P_n (and any power of them) are odd (even) if n is odd (even).

Also solved by Richard André-Jeannin, Charles Ashbacher, Wray Brady, Paul S. Bruckman, Russell Euler, Herta T. Freitag, C. Georghiou, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Nearest Integer

B-659 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

For $n \ge 3$, what is the nearest integer to $F_n\sqrt{5}$?

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, NY

For $n \ge 3$, L_n is the nearest integer to $F_n\sqrt{5}$, since

$$|F_n\sqrt{5} - L_n| = 2|\beta|^n \le 2|\beta|^3 < 1/2.$$

Also solved by Charles Ashbacher, Wray Brady, Paul S. Bruckman, Russell Euler, Piero Filipponi, Herta T. Freitag, C. Georghiou, Russell Jay Hendel & Sandra A. Monteferrante, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Binomial Expansions

B-660 *Proposed by Herta T. Freitag, Roanoke, VA*

Find closed forms for:

$$\text{(i)}\quad 2^{1-n}\sum_{i=0}^{[n/2]}\binom{n}{2i}5^i,\quad \text{(ii)}\quad 2^{1-n}\sum_{i=1}^{[(n+1)/2]}\binom{n}{2i-1}5^{i-1},$$

where $[t]$ is the greatest integer in t.

Solution by Lawrence Somer, Washington, D.C.

The answer to (i) is L_n; the answer to (ii) is F_n. These representations are obtained from the binomial expansions for

$$L_n = ((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^n$$

and

$$F_n = (1/\sqrt{5})[((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n],$$

respectively. The representation for F_n in (ii) was given by E. Catalan in 1857 in *Manuel des Candidats a l'Ecole Polytechnique*. A proof for the representation of L in (i) can be found in [2, p. 69]. Proofs for the representation of F in (ii) can be found in [1, p. 150] and [2, p. 68].

References

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*, 4th ed. London: Oxford University Press, 1960.
2. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. New York: Halsted Press, 1989.

Also solved by Richard André-Jeannin, Wray Brady, Paul S. Bruckman, Piero Filipponi, C. Georghiou, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, Dan Redmond, H.-J. Seiffert, Sahib Singh, and the proposer.

Integral Divisor

B-661 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = n(n+1)/2$. In Problem B-646, it was seen that $T(n)$ is an integral divisor of $T(2T(n))$ for all n in $Z^+ = \{1, 2, \ldots\}$. Find the n in Z^+ such that $T(n)$ is an integral divisor of

$$\sum_{i=1}^{n} T(2T(i)).$$

Solution by C. Georghiou, University of Patras, Greece

We have $T(2T(i)) = (i + 2i^2 + 2i^3 + i^4)/2$ and, therefore,

$$\sum_{i=1}^{n} T(2T(i)) = T(n)\frac{(n^3 + 4n^2 + 6n + 4)}{5}.$$

But $n^3 + 4n^2 + 6n + 4 \equiv (n - 1)(n^2 + 1) \pmod{5}$, from which it follows that $T(n)$ is a divisor of the given sum iff $n \equiv 1, 2,$ or $3 \pmod{5}$.

Also solved by Richard André-Jeannin, Paul S. Bruckman, David M. Burton, Russell Euler, Piero Filipponi, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Smith, Gregory Wulczyn, and the proposer.

Congruences Modulo 9

B-662 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $H_n = L_nP_n$, where the L_n and P_n are the Lucas and Pell numbers, respectively. Prove the following congruences modulo 9:

(1) $H_{4n} \equiv 3n$; (2) $H_{4n+1} \equiv 3n + 1$;

(3) $H_{4n+2} \equiv 3n + 6$; (4) $H_{4n+3} \equiv 3n + 2$.

Solution by C. Georghiou, University of Patras, Greece

More generally, we show that for any integer m we have

$$H_{4n+m} \equiv L_mP_m - 3nL_{m+2}P_m - 6nL_mP_{m+2} \pmod{9}.$$

Indeed, we have

$$\begin{aligned} L_{4n+m} &= \alpha^{4n+m} + \beta^{4n+m} = \alpha^m(3\alpha^2 - 1)^n + \beta^m(3\beta^2 - 1)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}3^i(-1)^{n-i}[\alpha^{2i+m} + \beta^{2i+m}] \\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}3^iL_{2i+m} \\ &\equiv (-1)^n[L_m - 3nL_{m+2}] \pmod{9}. \end{aligned}$$

Similarly, if $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$, we have

$$\begin{aligned} P_{4n+m} &= (\gamma^{4n+m} - \delta^{4n+m})/2\sqrt{2} = [\gamma^m(6\gamma^2 - 1)^n - \delta^m(6\delta^2 - 1)^n]/2\sqrt{2} \\ &= 2^{-3/2}\sum_{i=0}^{n}\binom{n}{i}6^i(-1)^{n-i}[\gamma^{2i+m} - \delta^{2i+m}] \\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}6^iP_{2i+m} \\ &\equiv (-1)^n[P_m - 6nP_{m+2}] \pmod{9}, \end{aligned}$$

from which the assertion follows immediately.

Now, by setting $m = 0, 1, 2,$ and 3, we find congruences (1)-(4), respectively.

Also solved by Paul S. Bruckman, Piero Filipponi, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Lawrence Somer, Gregory Wulczyn, and the proposer.

Dense in an Interval

B-663 *Proposed by Clark Kimberling, U. of Evansville, Indiana*

Let $t_1 = 1$, $t_2 = 2$, and $t_n = (3/2)t_{n-1} - t_{n-2}$ for $n = 3, 4, \ldots$. Determine lim sup t_n.

Solution by Hans Kappus, Rodersdorf, Switzerland

Solving the given difference equation by standard techniques, one easily obtains

$$t_n = (32/7)^{1/2}\sin(na - b),$$

where

$$a = \arctan(\sqrt{7}/3),\ b = \arctan(\sqrt{7}/11).$$

Now, since $\cos a = 3/4$, we conclude that a is not a rational multiple of π, and hence (t_n) is not periodic. Therefore, by a well-known theorem, the numbers t_n are everywhere dense in the interval $|t| \le (32/7)^{1/2}$. It follows that

$$\limsup t_n = (32/7)^{1/2}.$$

Also solved by Richard André-Jeannin, Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, and the proposer.

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