PASCAL'S TRIANGLE MODULO 4

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Introduction

Pascal's triangle has a seemingly endless list of fascinating properties. One such property which has been extensively studied is the fact that the number of odd entries in the n^{th} row is equal to 2^t where t is the number of ones in the base two representation of n (see [1], [2], and [3]).

Generalizations of this property seem surprisingly difficult. For a prime modulus, Hexel & Sachs [4] obtain a rather involved expression for the number of occurrences of each residue. Explicit formulas are obtained for $p = 3$ and 5. In particular, for a prime modulus p, the number of occurrences for a given residue in row n depends only on the number of times each digit appears in the base p representation of n. However, it is easily seen that composite moduli do not satisfy this property. In this article we consider Pascal's triangle modulo 4 and obtain explicit formulas for the number of occurrences of each residue modulo 4.

Notation and Conventions

The letters n, j, k, ℓ will denote nonnegative integers. The letter n will typically refer to an arbitrary row of Pascal's triangle. We will need detailed information on the base two representation of n. The following definitions will be useful.

Let

$$n = \sum_{i=0}^{k} a_i 2^i, \text{ where } a_i = 0 \text{ or } 1, \text{ and } B(n) = \sum_{i=0}^{k} a_i.$$

We also define

$$c_i = 1 \text{ if and only if } a_{i+1} = 1 \text{ and } a_i = 0, \text{ where } a_{k+1} = 0.$$

We then define

$$C(n) = \sum_{i=0}^{k} c_i.$$

Similarly, we define

$$d_i = (a_{i+1})(a_i) \text{ and } D(n) = \sum_{i=0}^{k} d_i.$$

Clearly, $B(n)$ is the number of "1"; $C(n)$ is the number of "10"; and $D(n)$ is the number of "11" blocks, not necessarily disjoint, in the base two representation of n.

For our purposes,

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

is defined for integer values of n and j; further,

$$\binom{n}{j} = 0 \text{ if } j < 0 \text{ or } j > n.$$

We define
$$\left\langle {n \atop j} \right\rangle = r \text{ if and only if } \binom{n}{j} \equiv r \pmod 4.$$

Let $N(n) = (a,\ b,\ c)$, where $N_1(n) = a$ is the number of ones, $N_2(n) = b$ is the number of twos, and $N_3(n) = c$ is the number of threes in the n^{th} row of Pascal's triangle.

We will make use of several well-known results found in Singmaster [5].

Lemma 1: $p^e \Big\| \binom{n}{j}$ if and only if the p-ary subtraction $n - j$ has e borrows.

Lemma 2: The number of odd binomial coefficients in the n^{th} level of Pascal's triangle is $2^{B(n)}$.

We begin our work with an easy result which we prove for completeness.

Lemma 3: $N(2^k) = (2,\ 1,\ 0)$ when $k \geq 1$.

Proof: Clearly
$$\left\langle {2^k \atop 0} \right\rangle = \left\langle {2^k \atop 2^k} \right\rangle = 1$$
so $N_1(2^k) \geq 2$. By Lemma 2,
$$N_1(2^k) + N_3(2^k) = 2.$$
So $N_1(2^k) = 2$ and $N_3(2^k) = 0$. Further, for $0 < j < 2^{k-1}$, $2^k - j$ will have at least two borrows when performed in base two. Thus,
$$4 \Big| \binom{2^k}{j}; \text{ hence, } \left\langle {2^k \atop j} \right\rangle = 0.$$
Similarly, for $2^{k-1} < j < 2^k$. Noticing
$$\left\langle {2^k \atop 2^{k-1}} \right\rangle = 2,$$
we conclude $N_2(2^k) = 1$. □

Lemma 4: Let $n = 2^k + \ell$, where $0 < \ell < 2^k$.

(i) If $\ell < j < 2^{k-1}$, then $\left\langle {n \atop j} \right\rangle = 0$.

(ii) If $\ell < j < 2^k$, then $\left\langle {n \atop j} \right\rangle = 0$ or 2.

Proof: In case (i), we must borrow at least twice in subtracting $n - j$, and in case (ii), at least one borrow must take place.

By Lemmas 3 and 4, it is clear that Pascal's triangle modulo 4 has the following form:

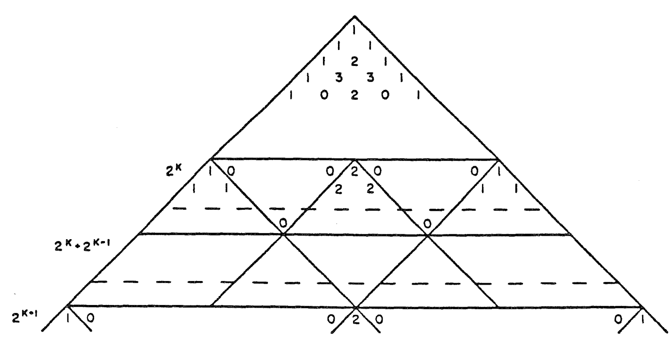

Figure 1

The standard identity

$$\left\langle {n \atop j-1} \right\rangle + \left\langle {n \atop j} \right\rangle = \left\langle {n+1 \atop j} \right\rangle$$

shows that any row in Figure 1 completely determines all subsequent rows. This identity and Lemma 3 yield the following recursive relations.

Part 1: If $n = 2^k + \ell$, where $0 \le \ell < 2^{k-1}$ (see upper dashed line in Fig. 1):

(i) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle$ for $0 \le j \le \ell$;

(ii) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle = 0$ for $\ell + 1 \le j < 2^{k-1}$;

(iii) $\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle$ for $2^{k-1} \le j \le 2^{k-1} + \ell$;

(iv) $\left\langle {n \atop j} \right\rangle = 0$ for $2^{k-1} + \ell < j < 2^k$;

(v) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j - 2^k} \right\rangle$ for $2^k \le j \le n$.

Part 2: If $n = 2^k + \ell$, where $2^{k-1} \le \ell < 2^k$ (see lower dashed line in Fig. 1):

(vi) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle$ for $0 \le j < 2^{k-1}$;

(vii) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle + 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle$ for $2^{k-1} \le j \le \ell$;

(viii) $\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle$ for $\ell < j < 2^k$;

(ix) $\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle + \left\langle {\ell \atop j - 2^k} \right\rangle$ for $2^k \le j \le \ell + 2^k$

(x) $\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j - 2^k} \right\rangle$ for $2^{k-1} + \ell < j \le n$.

All of the expressions above are considered modulo 4.

We are now in a position to count the number of ones and threes modulo 4. Recall that $D(n) > 0$ if and only if the base two representation of n has a "11" block.

Theorem 5: If $D(n) = 0$, then $N_1(n) = 2^{B(n)}$ and $N_3(n) = 0$.

Proof: We use induction on n. The theorem is true for $n \le 3$. Since $D(n) = 0$, we know $n = 2^k + \ell$, where $\ell < 2^{k-1}$ and $D(\ell) = 0$. Using (iii) of the recursion, we have

$$\left\langle {n \atop j} \right\rangle \equiv 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle \pmod 4$$

for $2^{k-1} \le j < 2^k$. Thus, there are no threes in this section of the n^{th} row of Pascal's triangle. By (i) and (v), we see

$$\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle \text{ for } j < 2^{k-1} \quad \text{and} \quad \left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j - 2^{k-1}} \right\rangle \text{ for } j > 2^k.$$

Thus, $N_3(n) = 2N_3(\ell)$. But by induction, $N_3(\ell) = 0$. The theorem now follows from Lemma 2. □

Theorem 6: If $D(n) > 0$, then $N_1(n) = N_3(n) = 2^{B(n)-1}$.

Proof: The result is clear for $n \le 4$.

Case 1: $n = 2^k + \ell$, where $\ell < 2^{k-1}$. Clearly, $D(\ell) > 0$. When considering $\langle {n \atop j} \rangle$, by the recursion, we need only consider $j \le \ell$ or $2^k \le j$. For $0 \le j \le \ell$, there are as many ones and threes as in row ℓ. By symmetry, there are as many for $2^k \le j$. Thus, $N_1(n) = 2N_1(\ell)$ and $N_3(n) = 2N_3(\ell)$, so the result holds by induction.

Case 2: $n = 2^k + \ell$, where $2^{k-1} \le \ell < 2^k$. Let $\ell = 2^{k-1} + r$. Consider the five sections of row n:

A. $0 \le j < 2^{k-1}$;

B. $2^{k-1} \le j \le \ell$;

C. $\ell < j < 2^k$;

D. $2^k \le j \le \ell + 2^{k-1}$;

E. $\ell + 2^{k-1} < j \le \ell + 2^k = n$.

By symmetry, $A = E$ and $B = D$. In section C, by (viii),

$$\left\langle {n \atop j} \right\rangle = 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle,$$

and there are no ones or threes in C.

In section A,

$$\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle \quad \text{for } 0 \le j < 2^{k-1}.$$

Since we are trying to count the number of times $\langle {\ell \atop j} \rangle = 1$ or 3, by Lemma 4, we need only consider $j \le r$.

In section B,

$$\left\langle {n \atop j} \right\rangle = \left\langle {\ell \atop j} \right\rangle + 2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle.$$

Now, by Lemma 1, $\langle {\ell \atop j} \rangle$ and $\langle {\ell \atop j - 2^{k-1}} \rangle$ are both odd or both even. We need only consider the case when they are both odd. Thus,

$$2\left\langle {\ell \atop j - 2^{k-1}} \right\rangle \equiv 2 \pmod 4.$$

Observing $x + 2 \equiv 3x$ if $x \equiv 1$ or 3 (modulo 4), we have

$$\left\langle {n \atop j} \right\rangle \equiv 3\left\langle {\ell \atop j} \right\rangle \equiv 3\left\langle {\ell \atop \ell - j} \right\rangle \pmod 4.$$

Since we are in section B, $2^{k-1} \le j \le \ell$, and recalling that $\ell = 2^{k-1} + r$, we see that $0 \le \ell - j \le r$, that is, $\langle {\ell \atop \ell - j} \rangle$ is in section A.

This implies the number of ones in section A equals the number of threes in section B and the number of threes in section A equals the number of ones in section B. Hence, there are an equal number of ones and threes in the combined sections of A and B; thus, $N_1(n) = N_3(n)$. The theorem now follows from Lemma 2. □

Theorem 7: $N_2(n) = C(n)2^{B(n)-1}$.

Proof: Recall that

$$\left\langle {n \atop j} \right\rangle = 2 \text{ if and only if } 2 \,\Big\|\, \binom{n}{j},$$

which occurs if and only if $n - j$ has exactly one borrow in base two. Thus, we wish to count the number of j's such that $n - j$ has exactly one borrow. Suppose the borrow occurs from position $i + 1$ to position i. If

$$n = \sum_{i=0}^{k} a_i 2^i \quad \text{and} \quad j = \sum_{i=0}^{k} b_i 2^i,$$

then $a_{i+1} = 1$ and $a_i = 0$, $b_{i+1} = 0$ and $b_i = 1$. Thus, if $C(n) = 0$, it follows that $N_2(n) = 0$.

So we assume $C(n) \geq 1$. To ensure no other borrow occurs, it must be the case that $b_\ell = 0$ when $a_\ell = 0$ for $\ell \neq i$. When $a_\ell = 1$, $\ell \neq i + 1$, b_ℓ may equal 0 or 1. So for each "10" in n's representation, there are $2^{B(n)-1}$ j's for which $\langle {n \atop j} \rangle = 2$. Thus, $N_2(n) = C(n)2^{B(n)-1}$. □

To summarize, we have

$$N(n) = \begin{cases} (2^{B(n)},\ C(n)2^{B(n)-1},\ 0) & \text{if } D(n) = 0, \\ (2^{B(n)-1},\ C(n)2^{B(n)-1},\ 2^{B(n)-1}) & \text{if } D(n) > 0. \end{cases}$$

Recurrences of the type used here are possible for other composite moduli, but they become increasingly complex. A complete characterization of the residues modulo 6 would be interesting, since 6 is not a prime power. Also, the question of general results for arbitrary composite moduli remains open.

References

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