RETROGRADE RENEGADES AND THE PASCAL CONNECTION: REPEATING DECIMALS REPRESENTED BY FIBONACCI AND OTHER SEQUENCES APPEARING FROM RIGHT TO LEFT

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Repeating decimals show a surprisingly rich variety of number sequence patterns when their repetends are viewed in retrograde fashion, reading from the rightmost digit of the repeating cycle towards the left. They contain geometric sequences as well as Fibonacci numbers generated by an application of Pascal's triangle. Further, fractions whose repetends end with successive terms of F_{nm}, $m = 1, 2, \ldots$, occurring in repeating blocks of k digits, are completely characterized, as well as fractions ending with F_{nm+p} or L_{nm+p}, where F_n is the n^{th} Fibonacci number,

$$F_1 = 1,\ F_2 = 1,\ F_{n+1} = F_n + F_{n-1},$$

and L_n is the n^{th} Lucas number,

$$L_1 = 1,\ L_2 = 3,\ L_{n+1} = L_n + L_{n-1}.$$

1. The Pascal Connection

It is no surprise that 1/89 contains the sum of successive Fibonacci numbers in its decimal expansion [2], [3], [4], [5], as

```
1/89 = .012358
            13
             21
              34
               ...
```

However, 1/89 can also be expressed as the sum of successive powers of 11, as

```
1/89 = .01
       .0011
       .000121
       .00001331
       .0000014641
       ...
```

where

$$1/89 = 1/10^2 + 11/10^4 + 11^2/10^6 + \ldots,$$

which is easily shown by summing the geometric progression. If the array above had the leading zeros removed and was left-justified, we would have Pascal's triangle in a form where the Fibonacci numbers arise as the sum of the rising diagonals. Notice that 11^k generates rows of Pascal's triangle, and that the columns of the array expressing 1/89 are the diagonals of Pascal's triangle. We call this relationship "the Pascal connection."

Now, 1/89 leads to the Fibonacci numbers by summing diagonals of Pascal's triangle. Since $89 = 10^2 - 11$, consider $9899 = 10^4 - 101$. By summing the geometric series,

$$1/9899 = 1/10^4 + 101/10^8 + 101^2/10^{12} + \ldots .$$

However, 101^k generates rows of Pascal's triangle where the columns are interspersed with columns of zeros. By the Pascal connection, we obtain Fibonacci numbers in every second place, as

$$1/9899 = .0001010203050813\ldots .$$

The Pascal connection also gives us

$$1/998999 = .000001001002003005008013\ldots,$$

since $998999 = 10^6 - 1001$. In general,

$$1/(10^{2k} - 100\ldots01),$$

where $(k - 1)$ zeros appear between the two 1's, gives successive Fibonacci numbers at every k^{th} place by the Pascal connection.

Looking again at $89 = 10^2 - 11$, observe that $889 = 10^3 - 111$, and summing a geometric series gives

$$1/889 = 1/10^3 + 111/10^6 + 111^2/10^9 + \ldots,$$

where

```
1/889 = .001            = .001124859...
        .000111
        .000012321
        .000001367631
        ...
```

and we generate the Tribonacci numbers

$$0, 1, 1, 2, \ldots, T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

by the Pascal connection, since 111^k generates rows of the trinomial triangle, and the sums of the rising diagonals of the trinomial triangle yield the Tribonacci numbers [1].

The results of expressing 1/89, 1/9899, 1/998999 in terms of Fibonacci numbers have been developed by other methods by Long [2] and by Hudson & Winans [3]. Winans [4] also gives 1/109 and 1/10099 as a reverse diagonalization of sums of Fibonacci numbers, reading from the far right of the repeating cycle, where 1/109 ends in

```
     13853211
    21
   34
  55
 89
...
```

We next apply the Pascal connection to repeating decimals, looking out to the far end of the repetend and reading from right to left. 1/109 has a period length of 108, and 1/109 ends in powers of 11, as

$$1/10^{108} + 11/10^{107} + 11^2/10^{106} + \ldots,$$

or, a reverse diagonalization of powers of 11,

```
        1
       11
     121
   1331
 14641
...
```

Summing the geometric progression,

$$\sum_{i=1}^{108} \frac{11^{i-1}}{10^{109-i}} = \frac{11^{108} \cdot 10^{108} - 1}{10^{108} \cdot (110 - 1)} = \frac{11^{108} - 1}{109} + \frac{10^{108} - 1}{10^{108} \cdot 109}$$

Now, $109 \mid (11^{108} - 1)$ because 109 is prime, so that the left term is an integer. The rightmost term represents one cycle of the repetend of 1/109, since 109 has period length 108. Thus, 1/109 gives F_n, $n = 1, 2, \ldots$, reading from the right, by the Pascal connection.

Notice that 109 = 11(10) - 1, and 109 is prime with period 108. Now,

$$1109 = 111(10) - 1,$$

where 1109 is prime with period 1108. We can generate the last digits of the repeating cycle for 1/1109 in exactly the same way by writing

$$1/10^{1108} + 111/10^{1107} + 111^2/10^{1106} + \ldots .$$

By the Pascal connection, 1/1109 ends in the Tribonacci sequence, ...74211.

Generalizing 109 in another way, 10099 is a prime with period length 3366, where $10099 = 101(10^2) - 1$, so that 1/10099 can be expressed in terms of powers of 101 from the far right. As before, 101^k generates rows of Pascal's triangle where the columns are interspersed by zeros, so that the Pascal connection shows 1/10099 ending in ...0503020101. Similarly, $1000999 = 1001(10^3) - 1$, and 1000999 is prime with period length 500499 [6], so that, by the Pascal connection, 1/1000999 must end in F_n appearing as every third entry, as ..005003002001001.

We can immediately write fractions which generate the Lucas numbers L_n from the right. Since 1/109 ends in F_n, $n = 1, 2, \ldots$, reading from the right, and $L_n = 2F_{n-1} + F_n$, multiplying 1/109 by 21 in effect adds $2F_{n-1} + F_n$ in the expansion except for the rightmost digit. But because the digit on the right of F_1 is indeed 0, the last digit also fits the pattern, so that 21/109 ends in L_{n-1} from the right. Also, multiplying 1/109 by 101 in effect adds $F_{n-1} + F_{n+1}$ to make L_n except for the rightmost digit. Thus, 101/109 ends in L_n except for the rightmost digit. That is, 101/109 ends in ...74311, and L_n reads from the right to left beginning at the 107^{th} digit. Since 1/10099 gives F_n, $n = 1, 2, \ldots$, reading from the right with every second digit, 201/10099 ends in L_n from the right as ...181107040301. Similarly, 2001/1000999 ends in L_n as every third digit. Finally, 10001/10099 ends in ...18110704030101 while 1000001/1000999 ends in ...018011007004003001001.

We will eventually prove these notions, but to enjoy these relationships one needs an easy way to write the far right-hand digits in these long repeating cycles. If $(A, 10) = 1$, $A > 1$, then $A \cdot 1/A = 1 = .99999\ldots$. To generate 1/109 from the right, simply fill in the digits to make a product of ...9999999:

```
     109
...53211
     109
    109
   218
  327
 545
...999999
```

The last digit of the next partial product must be 2 to make the next digit in the product be 9. So the digit preceding 5 in the multiplier must be an 8. One proceeds thusly, filling in the digits of the multiplier one at a time. The multiplier gives successive digits of 1/109 as read from the right.

2. Retrograde Renegades: Repeating Decimals that Contain Geometric Series

Any repeating decimal can itself be considered as a geometric series, but here we want to study repeating decimals which contain geometric series within their repetends. First, we list some general known results in Lemma 1 [7], [8].

Lemma 1: Let n be an integer, $(n, 10) = 1$, $n > 1$. Then $L(n)$, the length of the period of n, is given by

(i) $10^{L(n)} \equiv 1 \pmod{n}$,

where $L(n)$ is the smallest exponent possible to solve the congruence; if $R(n)$ denotes the repetend of n, then $R(n)$ has $L(n)$ digits and

(ii) $R(n) = (10^{L(n)} - 1)/n$;

the remainder B after A divisions by n in finding $1/n$ is given by

(iii) $10^A \equiv B \pmod{n}$,

and

(iv) $m^{L(n)} \equiv 1 \pmod{n}$, $(m, n) = 1$.

While $L(n)$ can be calculated as in Lemma 1(i), Yates [6] has calculated period lengths for all primes through 1370471.

We first look at repetends which contain powers of numbers reading left to right, or right to left, such as $1/97 = .01030927\ldots$ and $1/29$, which ends in $\ldots 931$, both of which seem to involve powers of 3.

Lemma 2: The decimal expansion of $1/(100 - k)$, $(100, k) = 1$, contains powers of k from left to right, $k < 100$.

Proof: Summing the geometric series,

$$1/10^2 + k/10^4 + k^2/10^6 + \cdots = 1/(100 - k).$$

Lemma 3: The repetend of $1/(10k - 1)$ contains powers of k as seen from the right.

Proof: Let $n = 10k - 1$. Then the sum after $L(n)$ terms of

$$S = \frac{1}{10^{L(n)}} + \frac{k}{10^{L(n)-1}} + \frac{k^2}{10^{L(n)-2}} +$$

is given by summing the geometric progression for $L(n)$ terms as

$$S = \frac{1}{10^{L(n)}} \cdot \frac{(10^{L(n)}k^{L(n)} - 1)}{(10k - 1)}$$

$$= \frac{1}{10^{L(n)}} \cdot \frac{[10^{L(n)}k^{L(n)} - 10^{L(n)} + 10^{L(n)} - 1]}{(10k - 1)} = \frac{k^{L(n)} - 1}{n} + \frac{10^{L(n)} - 1}{10^{L(n)}n},$$

where the left-hand term is an integer and the right-hand term gives one cycle of $1/n$ following the decimal point, both by Lemma 1.

Notice that $1/89$ has powers of 11 or Fibonacci numbers as seen from the left and powers of 9 from the right, while $1/109$ has powers of (-0.09) from the left (where the initial term is 0.01), and powers of 11 or Fibonacci numbers as seen from the right. Also, $1/889$ has Tribonacci numbers as seen from the left, and powers of 89 on the right, since $889 = 10 \cdot 89 - 1$.

Next, consider pairs of fractions whose repeating decimal representations end in each other. For example, 31 appears as the rightmost two digits of 1/29 (period length 28), and 29 is the last pair of digits of 1/31 (period length 15). Now, $29 \cdot 31 = 9 \cdot 10^2 - 1$, and the digits in the two cycles, reading from the right, can be represented as

$$1/29: \quad 31/10^{28} + 9 \cdot 31/10^{26} + 9^2 \cdot 31/10^{24} + \ldots;$$

$$1/31: \quad 29/10^{15} + 9 \cdot 29/10^{13} + 9^2 \cdot 29/10^{11} + \ldots .$$

Further, 1/29 ends in ...137931, and 1/31 ends in ...29, 1/931 in ...029, 1/7931 in ...0029, 1/37931 in ...00029, and, finally, $1/B = 0.000\ldots29$ (26 zeros in the repetend), where B is the entire repetend of 1/29. Also, there are many representations of a fraction reading from the right, such as, for 1/59 with its 58-digit period length, ending in ...779661, we have

$$1/10^{58} + 6/10^{57} + 6^2/10^{56} + \ldots,$$

$$61/10^{58} + 36 \cdot 61/10^{56} + 36^2 \cdot 61/10^{54} + \ldots,$$

$$661/10^{58} + 39 \cdot 661/10^{55} + 39^2 \cdot 661/10^{52} + \ldots,$$

$$9661/10^{58} + 57 \cdot 9661/10^{54} + 57^2 \cdot 9661/10^{50} + \ldots,$$

where

$$10^{58} \equiv 1 \pmod{59}, \quad 10^{57} \equiv 6 \pmod{59}, \quad 10^{56} \equiv 36 \pmod{59},$$

$$10^{55} \equiv 39 \pmod{59}, \text{ and } 10^{54} \equiv 57 \pmod{59}.$$

Notice that the multipliers are the remainders in reverse order in the division to obtain 1/59.

Both of these examples of retrograde renegades are explained by Theorem 1.

Let A and B be integers, $(A, 10) = 1$, $(B, 10) = 1$, $A > 1$. Let $L(A)$ be the number of digits in the period of A. If $1/A$ ends in B, then the end of $1/A$ can be expressed as

$$B/10^{L(A)} + KB/10^{L(A)-k} + K^2B/10^{L(A)-2k} + \ldots,$$

where

$$AB + 1 = K \cdot 10^k,$$

and the number of terms is $L(A)/k$ if k divides $L(A)$, or $[L(A)/k] + 1$ otherwise, where $[x]$ is the greatest integer in x.

Proof: $AB + 1 = K \cdot 10^k$ because $K \cdot 10^k$ is a partial dividend where A is the divisor, B is the quotient, and 1 is the remainder, in the long division process to find $1/A$. By Lemma 1,

$$10^{L(A)} \equiv 1 \pmod{A} \quad \text{and} \quad 10^{L(A)-k} \equiv K \pmod{A}.$$

Case 1. Let $k|L(A)$. Sum the geometric progression with $L(A)/k$ terms to obtain

$$S = \frac{B}{10^{L(A)}} \cdot \frac{(K^{L(A)/k} \cdot 10^{L(A)} - 1)}{(K \cdot 10^{k-1})}$$

$$= \frac{B}{10^{L(A)}} \cdot \frac{(K^{L(A)/k} \cdot 10^{L(A)} - 10^{L(A)} + 10^{L(A)} - 1)}{AB}$$

$$= \frac{K^{L(A)/k} - 1}{A} + \frac{10^{L(A)} - 1}{A \cdot 10^{L(A)}}$$

Now, the right-hand term represents one cycle of the repetend of $1/A$ following the decimal point, by Lemma 1. Next, if the left-hand term is an integer, we are done. By Lemma 1,

$$10^{L(A)-k} \equiv K \pmod{A},$$

so

$$K^{L(A)/k} \equiv (10^{L(A)-k})^{L(A)/k} \equiv (10^{L(A)})^{(L(A)-k)/k} \equiv 1 \pmod{A},$$

which means that the left-hand term is an integer.

Case 2. If k does not divide $L(A)$, then $L(A) = km + r$, $0 < r < m$, and there are $(m + 1)$ terms. Then, summing as before,

$$S = \frac{B}{10^{L(A)}} \cdot \frac{K^{m+1} \cdot 10^{k(m+1)} - 1}{K \cdot 10^{k} - 1}$$

$$= \frac{K^{m+1} \cdot 10^{k(m+1)} - 10^{L(A)}}{10^{L(A)} \cdot A} + \frac{10^{L(A)} - 1}{10^{L(A)} \cdot A}$$

Notice that the right-hand term is the same as in Case 1. If the left-hand term is an integer, then Case 2 is done. The left-hand term is equivalent to

$$(K^{m+1} \cdot 10^{k(m-1)-L(A)} - 1)/A$$

so we have an integer if

$$K^{m+1} \cdot 10^{k(m+1)-L(A)} \equiv 1 \pmod{A}.$$

But $K \equiv 10^{L(A)-k} \pmod{A}$, and substituting above,

$$(10^{L(A)-k})^{m+1} \cdot 10^{k(m+1)-L(A)} = 10^{L(A)(m+1)-k(m+1)+k(m+1)-L(A)}$$

$$= 10^{mL(A)} = (10^{L(A)})^{m} \equiv 1 \pmod{A}$$

and we are done.

Corollary (due to G. E. Bergum): Let A be a prime with k digits. If B is the integer formed by writing the last i digits of the repetend of $1/A$, $L(A) \geq 1 \geq k$, then $1/B$ ends in $\ldots 000\ldots A$, where A is preceded by $(i - k)$ zeros.

3. Fractions that Contain F_{nm} in Their Decimal Representations

Hudson & Winans [3] completely characterized decimal fractions which can be represented in terms of F_{nm}, reading from the left. In particular, they give

$$1/71 = \sum_{i=1}^{\infty} F_{2i}/10^{i+1}.$$

Winans [4] gives 9/71 as ending in Fibonacci numbers with odd subscripts. Since 9/71 also begins with F_{2m-1} reading from the left and

$$L_{2m} = F_{2m-1} + F_{2m+1},$$

we write $11 \cdot 9 = 99 \equiv 28 \pmod{71}$ and 28/71 begins with L_{2m}, $m = 1, 2, \ldots$. Since we find that 19/71 ends in F_{2m-3}, and

$$L_{2m-2} = F_{2m-1} + F_{2m-3},$$

$19/71 + 9/71 = 28/71$ ends in L_{2m-2}, $m = 1, 2, \ldots$, reading from the right. Further, Hudson & Winans [3] give

$$1/9701 = .000103082156\ldots,$$

where F_{2m} appears in groups of two digits. We note that $9701 = 89 \cdot 109$, with 1188 digits in its repeating cycle. It turns out that

$$99/9701 = .0102051334\ldots$$

and that 99/9701 ends in ...893413050201, where F_{2m-1} appears in groups of two digits, reading either from the left or from the right. Since

$$L_{2m} = F_{2m-1} + F_{2m+1},$$

and

$$101 \cdot 99 = 9999 \equiv 298 \pmod{9701},$$

we should have 298/9701 both beginning and ending in Lucas numbers with even subscripts. In fact,

$$298/9701 = .03071848\ldots$$

and ends with ...4718070302, or begins with L_{2m} and ends with L_{2m-2}, $m = 1, 2, \ldots$, moving in blocks of two.

Next, we give a description of fractions with a decimal representation using F_{nm}, reading from right to left.

Theorem 2: The decimal representation of

$$\frac{F_n}{10^{2k} + L_n \cdot 10^k - 1}, \quad n \text{ odd},$$

ends in successive terms of F_{nm}, $m = 1, 2, \ldots$, reading from the right end of the repeating cycle, and appearing in groups of k digits.

Proof: Change the sum written in (i) to geometric progressions by using the Binet form for F_n,

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \text{ where } \alpha = (1 + \sqrt{5})/2 \text{ and } \beta = (1 - \sqrt{5})/2.$$

Then sum the geometric progressions, making use of $\alpha\beta = -1$ and $L_n = \alpha^n + \beta^n$. After sufficient algebraic patience, one can write, for $k > 0$,

$$\text{(i)} \qquad \sum_{i=1}^{L} 10^{k(i-1)} F_{ni} = \frac{(-1)^{n+1} 10^{k(L+1)} F_{nL} + 10^{kL} F_{n(L+1)} - F_n}{(-1)^{n+1} 10^{2k} + L_n \cdot 10^k - 1}.$$

Notice that the sum is a positive integer at this point, and dividing by 10^y, $y > 0$, will move the decimal point y places to the left. Let

$$M = (-1)^{n+1} 10^{2k} + L_n \cdot 10^k - 1,$$

where $M > 0$ when n is odd, and let $L(M)$ be the length of the period of M. The number of terms L in the sum must be chosen so that $L \geq L(M)/k$. We divide both sides of (i) by $10^{L(M)}$, and add $(F_n - F_n)$ to the numerator on the right-hand side, making

$$\text{(ii)} \qquad \sum_{i=1}^{L} 10^{k(i-1)-L(M)} F_{ni} = \frac{10^{kL-L(M)}((-1)^{n+1} 10^k F_{nL} + F_{n(L+1)}) - F_n}{M} + \frac{F\,(10^{L(M)} - 1)}{10^{L(M)} M}.$$

Since $kL \geq L(M)$, $10^{kL-L(M)} \geq 1$, and the decimal point has been shifted $L(M)$ places left. Now, the rightmost term is F_n times one cycle of the repetend of $1/M$. Thus, when n is odd,

$$M = 10^{2k} + L_n \cdot 10^k - 1, \text{ and } F_n/M \text{ has the needed form.}$$

Now, if n is even,

$$M = (-1)^{n+1}10^{2k} + L_n \cdot 10^k - 1$$

is negative, and we have to modify Theorem 2.

Theorem 3: The decimal representation of

$$\frac{M - F_n}{M}, \; M = 10^{2k} - L_n \cdot 10^k + 1, \; n \text{ is even,}$$

ends in successive terms of F_{nm}, $m = 1, 2, \ldots$, reading from the right end of the repeating cycle and appearing in groups of k digits, if 1 is added to the rightmost digit.

Proof: Return to (ii) in the proof of Theorem 2. When n is even, both numerator and denominator of the left-hand term are negative, so we still have a positive term there. Since M is negative when n is even, rewrite the right-hand term as

$$-F_n(10^{L(M)} - 1)/10^{L(M)} M$$

for adjusted M,

$$M = 10^{2k} - L_n \cdot 10^k + 1.$$

Then write

$$\frac{-F_n(10^{L(M)} - 1)}{10^{L(M)} M} = \frac{-F_n(10^{L(M)} - 1) + (M(10^{L(M)} - 1)) - (M(10^{L(M)} - 1))}{10^{L(M)} M}$$

$$= \frac{(M - F_n)(10^{L(M)} - 1)}{10^{L(M)} M} + \frac{1}{10^{L(M)}} - 1$$

The fractional part represents $(M - F_n)$ times one cycle of the repetend of $1/M$, with 1 added to the rightmost digit, which finishes Theorem 3.

Further, notice that if F_n/M is represented in terms of F_{nm}, then other fractions with the same denominator will have representations in terms of F_{nm+r} and L_{nm+r}, $r = 0, 1, \ldots, n - 1$. For example, for $n = 2$, $k = 1$ and $m = 1, 2, \ldots$.

$2/139$ ends in F_{3m}, $20/139$ in F_{3m-3}, $11/139$ in F_{3m-1}, $13/139$ in F_{3m+1};

$24/139$ ends in L_{3m}, $31/139$ in L_{3m-2}, $41/139$ in L_{3m+2}.

In general, for $n = 3$, $m = 1, 2, \ldots$, and $M = 10^{2k} + 4 \cdot 10^k - 1$, we have

$2/M$ ends in F_{3m}; $\quad 2 \cdot 10^k/M$ ends in F_{3m-3}.

Since $F_{3m} + F_{3m-3} = 2F_{3m-1}$, and $F_{3m+1} = F_{3m} + F_{3m-1}$, we find that

$(10^k + 1)/M$ ends in F_{3m-1}; $\quad (10^k + 3)/M$ ends in F_{3m+1}.

Then $L_{3m} = F_{3m+1} + F_{3m-1}$ and $L_{3m-2} = F_{3m-3} + F_{3m-1}$, give us that

$(2 \cdot 10^k + 4)/M$ ends in L_{3m}; $\quad (3 \cdot 10^k + 1)/M$ ends in L_{3m-2}.

Lastly, $L_{3m+2} = F_{3m} + 3F_{3m+1}$ means that

$(3 \cdot 10^k + 11)/M$ ends in L_{3m+2},

where all of the above occur in groups of k digits.

The even examples are both more difficult and more entertaining. For $n = 2$, $m = 1, 2, \ldots$, $M = 10^{2k} - 3 \cdot 10^k + 1$, the following occur in blocks of k digits from the right:

$(10^k - 3)/M$ ends in F_{2m+2}, $(10^k - 2)/M$ ends in F_{2m+1};

$(2 \cdot 10^k - 3)/M$ ends in L_{2m}, $(10^k - 4)/M$ ends in L_{2m+1}.

For $n = 4$, $m = 1, 2, \ldots$, $M = 10^{2k} - 7 \cdot 10^k + 1$, the following occur in blocks of k digits from the right:

$(10^k - 5)/M$ ends in F_{4m+1}, $(10^k - 8)/M$ ends in F_{4m+2};

$(3 \cdot 10^k - 18)/M$ ends in L_{4m+2}, $(4 \cdot 10^k - 29)/M$ ends in L_{4m+3}.

These are by no means exhaustive. Fibonacci and Lucas numbers abound but encountering negative numerators causes addition of multiples of M to write a fraction with a positive numerator and the same repetend, and there will be adjustments to the last digit in the representation.

When n is even, Theorem 3 gives the same denominators as found by Hudson & Winans [3] for the even case, in representations using F_{nm} from left to right. We find examples such as 9/71 and 99/9701, which both begin and end in F_{2m-1}, and 98/9301, which has F_{4m-3} from the left and F_{4m-1} from the right. We can write a corollary to Theorem 3.

Corollary: (i) $\dfrac{10^k - 1}{10^{2k} - 3 \cdot 10^k + 1}$ begins and ends with F_{2m-1},

(ii) $\dfrac{10^k - 2}{10^{2k} - 7 \cdot 10^k + 1}$ begins with F_{4m-3} and ends with F_{4m-1},

both appearing in blocks of k digits.

Proof: Case (i), where $n = 2$. From left to right, $1/M$ begins with F_{2m-2} and $10^k/M$ begins with F_{2m}, so subtracting gives $(10^k - 1)/M$ for F_{2m-1}. From right to left,

$$(M - 1)/M = (10^{2k} - 3 \cdot 10^k)/M \text{ ends in } F_{2m}$$

except for the last digit, so moving one block left,

$$(10^k - 3)/M \text{ ends in } F_{2m+2}.$$

Using $F_{2m-1} = F_{2m+2} - 2F_{2m}$, compute

$$(10^k - 3 - 2(-1))/M = (10^k - 1)/M,$$

where the numerator is positive, ending in F_{2m-1}.

Case (ii), where $n = 4$. From left to right, $3/M$ begins with F_{4m-4}, so $3 \cdot 10^k/M$ begins with F_{4m}. Since $3F_{4m-3} = F_{4m} - 2F_{4m-4}$, we find that

$$(10^k - 2)/M \text{ begins with } F_{4m-3}.$$

From right to left, except for the last digit,

$$(M - 3)/M \text{ ends in } F_{4m},$$

so that F_{4m+4} ends in

$$(M - 3)/10^kM \equiv (3M - 3)/10^kM = (3 \cdot 10^k - 21)/M.$$

Now, $3F_{4m-1} = F_{4m+4} - 5F_{4m}$ allows us to compute

$$(3 \cdot 10^k - 21 - 5(-3))/3M = (10^k - 2)/M,$$

where the numerator is positive, ending in F_{4m-1}.

Examining the proof of the corollary, we have seen several examples for $n = 2$ and $n = 4$ where

$$\frac{F_p \cdot 10^k - F_{p+n}}{10^{2k} - L_n \cdot 10^k + 1} \text{ ends in } F_{nm+p}$$

and some earlier examples for $n = 3$ and $n = 1$, where

$$\frac{F_p \cdot 10^k + F_{p+n}}{10^{2k} + L_n \cdot 10^k - 1} \text{ ends in } F_{nm+p}.$$

We write our final generalization as Theorem 4.

Theorem 4: The repeating cycle of

$$\frac{F_p \cdot 10^k + (-1)^{n+1}F_{p+n}}{10^{2k} + (-1)^{n+1}(L_n \cdot 10^k - 1)} \text{ ends in } F_{nm+p}$$

and the repeating cycle of

$$\frac{L_p \cdot 10^k + (-1)^{n+1}L_{p+n}}{10^{2k} + (-1)^{n+1}(L_n \cdot 10^k - 1)} \text{ ends in } L_{nm+p},$$

for $m = 1, 2, \ldots$, occurring in blocks of k digits, for positive integers k and n such that

$$10^{2k} + (-1)^{n+1}(L_n \cdot 10^k - 1) > F_p \cdot 10^k + (-1)^{n+1}F_{p+n} > 0.$$

The proof of the Fibonacci case follows from summing

$$\sum_{i=1}^{L} 10^{k(i-1)-L(M)} F_{ni+p},$$

using the techniques of Theorems 2 and 3. Since we force cases where the numerator and denominator are both positive, we can do the proof as one case, and the proof is fairly straightforward but very long and tedious. The Lucas case follows by adding the fractions which represent $F_{nm+(p-1)}$ and $F_{nm+(p+1)}$.

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