A NOTE CONCERNING THOSE n FOR WHICH $\phi(n) + 1$ DIVIDES n

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In [3, p. 52], Richard Guy gives the following problem of Schinzel: If p is an odd prime and $n = 2$ or p or $2p$, then $(\phi(n) + 1) | n$, where ϕ is Euler's totient function. Is this true for any other n?

We shall show that this question is closely related to a much older problem due to Lehmer [4]: whether or not there exist composite n such that $\phi(n) | (n - 1)$. It will turn out that if there are no such composite n, then Schinzel's are the only solutions of his problem; if there are other solutions of Schinzel's problem, then they have at least 15 distinct prime factors. Let $\omega(n)$ denote the number of distinct prime factors of n. More specifically, we shall prove the following.

Theorem: Let n be a natural number and suppose $(\phi(n) + 1) | n$. Then one of the following is true.

(i) $n = 2$ or p or $2p$, where p is an odd prime.
(ii) $n = mt$, where $m = 3$, 4, or 6, $\gcd(m, t) = 1$, and $t - 1 = 2\phi(t)$ [so that $\omega(t) \geq 14$].
(iii) $n = mt$, where $\gcd(m, t) = 1$, $\phi(m) = j \geq 4$, and $t - 1 = j\phi(t)$ [so that $\omega(t) \geq 140$].

Proof: Since $(\phi(n) + 1) | n$, we have

$$m(\phi(n) + 1) = n \tag{1}$$

for some natural number m. Let $t = \phi(n) + 1$ and $d = \gcd(m, t)$. Then, using (1) and an easy and well-known result (Apostol [1, p. 28]),

$$\phi(n) = \phi(mt) = \frac{\phi(m)\phi(t)d}{\phi(d)}. \tag{2}$$

Since $d|m$, we have $\phi(d) | \phi(m)$ so that $\phi(m)/\phi(d)$ is an integer. Then, from (2), $d | \phi(n)$; but, by definition, $d | (\phi(n) + 1)$. Hence $d = 1$. Thus, we have $n = mt$, where

$$t = \phi(n) + 1 = \phi(mt) + 1 = \phi(m)\phi(t) + 1.$$

We cannot have $t = 1$. Also, t is prime if and only if $\phi(m) = 1$. In this case, $m = 1$ or 2, and we have Schinzel's solutions, in (i).

Suppose now that t is composite. If $\phi(m) = 2$, then $m = 3$, 4, or 6 and $t - 1 = 2\phi(t)$. Cohen and Hagis [2] showed in this case that $\omega(t) \geq 14$. These are the solutions in (ii). It is impossible to have $\phi(m) = 3$, so the only remaining possibility is that $\phi(m) \geq 4$, so $t - 1 = j\phi(t)$, say, with $j \geq 4$. For this equation to hold, Lehmer [4] pointed out that t must be odd and squarefree, and Lieuwens [5] showed that $\omega(t) \geq 212$ if $3 | t$. (This latter remark applies also to the solution $n = 4t$ in (ii).] Suppose $3 \nmid t$, and write

$$t = \prod_{i=1}^{u} p_i, \quad 5 \le p_1 < p_2 < \cdots < p_u,$$

where $p_1, p_2, \ldots, p_u$ are primes. Then $p_2 \ge 7$, $p_3 \ge 11$, ... If $u \le 139$,

$$4 \le j = \frac{t - 1}{\phi(t)} < \frac{t}{\phi(t)} = \prod_{i=1}^{u} \frac{p_i}{p_i - 1} \le \frac{5}{4}\,\frac{7}{6}\,\frac{11}{10} \cdots \frac{811}{810} < 4.$$

(There are 139 primes from 5 to 811, inclusive.) This contradiction shows that $u = \omega(t) \ge 140$ in this case, giving (iii) and completing the proof.

Using the above and results of Pomerance [6, esp. the Remark] and [7], it is not difficult to show that the number of natural numbers n such that $n \le x$, $(\phi(n) + 1)|n$ and n is not a prime or twice a prime, is

$$O\left(x^{1/2} (\log x)^{3\ 4} (\log \log x)^{-5/6}\right).$$

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(Continued from page 282)

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