# NIVEN REPUNITS AND $10^n \equiv 1 \pmod{n}$

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## 1. Introduction

A positive integer which is divisible by its digital sum is called a Niven number. This concept was introduced in [1], and was investigated in greater detail in [2], [3], and [4]. A variety of results and open questions were presented in these articles. One problem, however, that was not completely resolved was a characterization of Niven repunits. That is, Niven numbers whose decimal representation is all ones. For example, the repunits 1, 111, 111111111, and 111111111111111111111111111 (27 ones) are the first four Niven repunits. Here, we will give a complete characterization of such integers. In addition, it will be pointed out how all Niven repunits can be constructed from a certain list of primes.

To facilitate the following discussion, we use the notation $R(n)$ to represent the repunit made up of $n$ ones. Thus

$$R(n) = \frac{1}{9}(10^n - 1)$$

and so, we wish to determine under which conditions

$$R(n) \equiv 0 \pmod{n}. \tag{1.1}$$

## 2. A Useful Lemma

A particular instance of the following lemma will be useful in proving a characterization theorem for Niven repunits.

*Lemma 2.1:* Let $a$, $b$, $m$, and $n$ be positive integers. If $a \equiv b \pmod{m^n}$, then

$$a^{m^k} \equiv b^{m^k} \pmod{m^{k+n}}$$

for each nonnegative integer $k$.

*Proof:* By observing the factorization,

$$a^{m^{k+1}} - b^{m^{k+1}} = (a^{m^k} - b^{m^k})[(a^{m^k})^{m-1} + (a^{m^k})^{m-2}(b^{m^k}) + \cdots + (b^{m^k})^{m-1}],$$

the proof follows by induction on $k$.

For convenience, we state a special case of Lemma 2.1 as Lemma 2.2.

*Lemma 2.2:* Let $m$, $n$, and $t$ be positive integers. Then $10^t \equiv 1 \pmod{m^n}$ implies that

$$(10^t)^{m^k} \equiv 1 \pmod{m^{k+n}}$$

for each nonnegative integer $k$.

## 3. The Characterization Theorem

Using Lemma 2.2, we can now prove the following theorem, which gives necessary and sufficient conditions in order that (1.1) is true.

*Theorem:* Let $n$ and 10 be relatively prime. Denote the order of 10 (mod $n$) by $e_n(10)$. Then the following statements are equivalent.

(1) $R(n)$ is a Niven repunit.

(2) $10^n \equiv 1 \pmod{n}$.

(3) $n \equiv 0 \pmod{e_n(10)}$.

(4) $n \equiv 0 \pmod{e_p(10)}$ for each prime factor $p$ of $n$.

*Proof:* That $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ is a direct application of congruence arithmetic and Fermat's Theorem. Thus, we need only prove that $(4) \Rightarrow (1)$.

Suppose that $n \equiv 0 \pmod{e_p(10)}$ for each prime factor $p$ of $n$. Let $m$ be the least prime factor of $n$. Then, since $e_m(10) < m$ and, by the hypothesis $e_m(10)$ is also a factor of $n$, we have that $e_m(10)$ must be 1. This can only occur when $m = 3$.

So, we may write the prime factorization of $n$ in the form

$$3^k \prod_{i=1}^{t} p_i^{k_i}, \text{ where } 3 < p_1 < p_2 < p_3 < \cdots < p_t.$$

Thus, $n \equiv 0 \pmod{e_{p_i}(10)}$ for $i = 1, 2, 3, \ldots, t$ and since

$$10^{e_{p_i}(10)} \equiv 1 \pmod{p_i}$$

for each $i$, we have that $10^n \equiv 1 \pmod{p_i}$ for each $i$. But by Fermat's Theorem,

$$10^{p_i - 1} \equiv 1 \pmod{p_i}$$

and so, $e_{p_i}(10)$ divides $p_i - 1$ for each $i$. Thus,

$$10^{(n,\, p_i - 1)} \equiv 1 \pmod{p_i}$$

for each $i$ where, as usual, $(n, p_i - 1)$ denotes the greatest common factor of $n$ and $p_i - 1$. But, since $(n, p_i - 1)$ is a factor of $n/p_i^{k_i}$, we have

$$10^{n/p_i^{k_i}} \equiv 1 \pmod{p_i}$$

for each $i$. Noting that

$$10^{n/3^k} \equiv 1 \pmod{3^2},$$

we have, by Lemma 2.2, that

$$(10^{n/p_i^{k_i}})^{p_i^{k_i}} \equiv 1 \pmod{p_i^{k_i+1}}$$

for each $i$, and

$$(10^{n/3^k})^{3^k} \equiv 1 \pmod{3^{k+2}}.$$

Therefore,

$$10^n \equiv 1 \pmod{p_i^{k_i}}$$

for each $i$, and

$$10^n \equiv 1 \pmod{3^{k+2}}.$$

It follows that $10^n \equiv 1 \pmod{3^2 n}$ and so

$$\frac{1}{9}(10^n - 1) \equiv 0 \pmod{n}.$$

Therefore, $R(n)$ is a Niven repunit, and we have that $(4) \Rightarrow (1)$.

An immediate corollary to the theorem is that $R(3^t)$ is a Niven repunit for every nonnegative integer $t$. This follows from the fact that $e_3(10) = 1$ and statement (4) of the theorem. In fact, statement (4) gives the most useful characterization of Niven repunits.

## 4. Generation of Niven Repunits

Using statement (4) of the theorem, we can construct all $n$ such that $R(n)$ is Niven by determining which primes, $p$, are such that every prime factor of $e_p(10)$ also satisfies the condition of statement (4). For example, since $e_3(10) = 1$, as has already been pointed out every power of 3 is a Niven repunit. But since $e_7(10) = 6$ has a factor of 2, it follows that no multiple of 7 can satisfy statement (4). That is, $R(7m)$ can never be a Niven repunit. In fact, the first prime larger than 3 that can be a factor of an $n$ that satisfies statement (4) is 37. This follows because $e_{37}(10) = 3$ and, as stated above, 3 is a prime that must be a factor of every $n$ that satisfies statement (4) of the theorem.

Similarly, it is found that the next two primes, after 37, which could possibly be factors of an $n$ such that $R(n)$ is Niven are 163 and 757 since

$$e_{163}(10) = 3^4 \quad \text{and} \quad e_{757}(1) = 3^3.$$

The first column in the following table gives all primes, less than 50,000, which could possibly be factors of an $n$ that satisfies statement (4). The second column gives the corresponding $e_p(10)$.

**TABLE 4.1**

| prime $p$ | $e_p(10)$ |
|---|---|
| 3 | 1 |
| 37 | 3 |
| 163 | $81 = 3^4$ |
| 757 | $27 = 3^3$ |
| 1999 | $999 = (3^3)(37)$ |
| 5477 | $1369 = 37^2$ |
| 8803 | $1467 = (3^2)(163)$ |
| 9397 | $81 = 3^4$ |
| 13627 | $6813 = (3^2)(757)$ |
| 15649 | $489 = (3)(163)$ |
| 36187 | $18093 = (3)(37)(163)$ |
| 40879 | 757 |

It should be noted that an infinitude of such primes exist, since $e_p(10)$ is a power of 3 infinitely often. As an example, suppose that 757 is the largest prime factor of $n$. Then in order that $R(n)$ be a Niven repunit, $n$ would have to be of the form

$$3^{n_1}\,37^{n_2}\,163^{n_3}\,757^{n_4}$$

where the exponents are necessarily interdependent. That is, if $n_3 \neq 0$, then, by inspection of the right column of Table 4.1, $n_1 \geq 4$. So, a list of generators of Niven repunits can be continuously constructed by consideration of Table 4.1. The first few of such a list is given in Table 4.2.

TABLE 4.2

| |
|---|
| 3 |
| (3)(37) |
| $(3^4)(163)$ |
| $(3^3)(757)$ |
| $(3^3)(37)(1999)$ |
| $(3)(37^2)(5477)$ |
| $(3^4)(163)(8803)$ |
| $(3^4)(9397)$ |
| $(3^3)(757)(13627)$ |
| (3)(163)(15649) |

For example, the product $(3^4)(163)(8803)$ is in the list given by Table 4.2 because

$$e_{8803}(10) = 1467 = (3^2)(163)$$

and each of its prime factors is in the list given by Table 4.1. So, if 8803 is the largest admissible prime factor of $n$, 163 would also have to be a factor which, in turn, forces $3^4$ to be a factor of $n$. The phrase, ". . . generators of Niven repunits . . ." is used because *increasing the exponents of any of the prime factors of the least common multiple of any collection chosen from the list given in Table 4.2 will be an* n *such that* R(n) *is a Niven repunit.* For example

$$\text{lcm}((3^4)(163), (3^3)(757), (3^3)(757)(13627)) = (3^4)(163)(757)(13627)$$

and so

$$R(3^{n_1}163^{n_2}757^{n_3}13627^{n_4})$$

will be a Niven repunit for any $n_1 \geq 4$, $n_2 \geq 1$, $n_3 \geq 1$, and $n_4 \geq 1$.

## 5. Concluding Remarks

As pointed out, the list of primes given by Table 4.1 can be extended by inspecting $e_p(10)$ for primes $p$. A useful reference for finding such primes has been published by Yates [5]. For example, he has calculated that, for the prime 333667,

$$e_{333667}(10) = 9 = 3^2.$$

Hence, 333667 may be added to the list given by Table 4.1 since 3 is already listed in Table 4.1.

Finally, it should be mentioned that since, for any decimal digit $d \neq 0$,

$$\frac{d}{9}(10^n - 1) \equiv 0 \pmod{dn}$$

if and only if $R(n) \equiv 0 \pmod{n}$, the characterization theorem for repunits also gives a complete characterization for what could be called "Niven repdigits."

## References

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5. S. Yates. *Prime Period Lengths*. 1975.

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