ON A CONJECTURE BY HOGGATT WITH EXTENSIONS TO HOGGATT SUMS AND HOGGATT TRIANGLES

Daniel C. Fielder and Cecil O. Alford
Georgia Institute of Technology, Atlanta, Georgia 30332-0250
(Submitted April 1987)

1. Introduction

In letters [1] to one of us (Fielder) in mid-1977, the late Verner Hoggatt conjectured that the third diagonal of Pascal's triangle could be used in a simple algorithm to generate rows of integers whose row sums equaled correspondingly indexed Baxter permutation values (see [3], [4]). Later, in 1978, Chung, Graham, Hoggatt, and Kleiman produced a remarkable paper [2] in which they derived a general solution for Baxter permutation values.

In planning an extension of Hoggatt's work, we searched for, but never found, a proof of Hoggatt's conjecture or even a documented statement of the conjecture. Reference [2] did, however, state that Hoggatt had found a simple way of finding the first ten Baxter permutation values but, again, without giving the conjecture. In this note, we formalize Hoggatt's conjecture, derive formulas for the values predicted by the conjecture, and then prove the conjecture. As new material, we extend Hoggatt's conjecture to *all* Pascal diagonals. In so doing, we will introduce structures called *Hoggatt triangles* and integers called *Hoggatt sums*. These names were the explicit choice of one of us (Fielder) as a tribute to Verner Hoggatt for his work with Pascal triangles and, in some small way, to express gratitude for Vern's guidance, help, and friendship through the years. Finally, we report briefly on a computer-aided experiment to obtain recursion formulas for selected Hoggatt sums.

2. Hoggatt's Conjecture

Whereas Hoggatt chose a column representation to demonstrate his algorithm, we use a diagonal format. There is, of course, no conceptual or computational difference.

Hoggatt's conjecture may be phrased as follows: "Select the zeroth[1] and third right diagonal of Pascal's triangle and let them become, respectively, the zeroth and first right diagonal of a new triangle with as yet undetermined values for the entries of the other diagonals. For $m = 2, 3, 4, \ldots$, in succession, compute the m^{th} row sum and m^{th} row entries for the new triangle as

$$\text{Row}_m\,\text{sum} = 1 + \binom{m+2}{3}\left(\frac{(R_{m-1})_0}{D_0} + \frac{(R_{m-1})_1}{D_1} + \cdots + \frac{(R_{m-1})_{m-1}}{D_{m-1}}\right) \tag{1}$$

where the (R_{m-1})'s are the $(m-1)^{\text{th}}$ row integers starting with $q = 0$ at the left and the D's are the first diagonal integers starting with $q = 0$ at the top right. Then the m^{th} row sum as given by (1) is identically the m^{th} Baxter permutation value S_m."

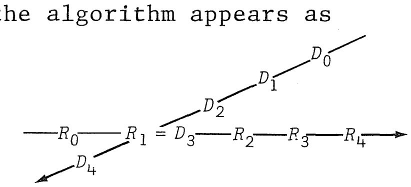
---

[1]Unless stated otherwise, the counting of indices, rows, columns, diagonals, etc. in this note starts with zero as the first encountered.

In order to visualize the algorithm of (1), assume that integers of the rows through the four have been found through successive application of the right side of (1). The diagrams below illustrate how the fifth row is constructed. (Note that the first two integers of any row are always known.)

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&4&&1&&& \\ &&1&&10&&10&&1&& \\ &1&&20&&50&&20&&1& \\ 1&&35&&X&&X&&X&&X \end{array} \tag{2}$$

By using R_s for the fourth row entries and D_s for the first diagonal entries, a graphic preparation for the algorithm appears as



(3)

When generated by (1), the fifth row becomes

$$R_0,\ D_4 \times \frac{R_0}{D_0},\ D_4 \times \frac{R_1}{D_1},\ D_4 \times \frac{R_2}{D_2},\ D_4 \times \frac{R_3}{D_3},\ D_4 \times \frac{R_4}{D_4}, \tag{4}$$

with calculated values, 1, 35, 175, 175, 35, 1. The row sum is 422, which equals Baxter permutation S_5. The rows completed prior to row five have sums equal to S_0, S_1, S_2, S_3, S_4, respectively. In anticipation of later work, the new triangle will be called a *Hoggatt triangle of order three*.

3. Formulas for Row Sums, Row Integers, and Proof of the Conjecture

The development of formulas for the row sums is presented by using the third right diagonal of Pascal's triangle. (If the entries are in the binomial coefficient form, the procedure is easy to follow.) This, in turn, is used as the first diagonal of a third-order Hoggatt triangle. Apply (1) as before, but retain the accumulated binomial coefficients in the row construction. The construction of rows one and two is shown.

$$S_1 = 1 + \binom{3}{3}\left[\frac{1}{\binom{3}{3}}\right] = 1 + \frac{\binom{3}{3}}{\binom{3}{3}}, \tag{5}$$

$$S_2 = 1 + \binom{4}{3}\left[\frac{1}{\binom{3}{3}} + \frac{\frac{\binom{3}{3}}{\binom{3}{3}}}{\binom{4}{3}}\right] = 1 + \frac{\binom{4}{3}}{\binom{3}{3}} + \frac{\binom{4}{3}\binom{3}{3}}{\binom{3}{3}\binom{4}{3}}, \tag{6}$$

The obvious pattern of the development can be generalized by summations in which the total is the general m^{th} row sum and the individual terms of a summation are the m^{th} row values of a third-order Hoggatt triangle.

$$S = 1 + \sum_{h=0}^{m-1} \prod_{k=0}^{h} \frac{\binom{m+2-k}{3}}{\binom{3+k}{3}} = 1 + \sum_{h=0}^{m-1} \prod_{k=0}^{h} \frac{(m+2-k)^{(3)}}{(3+k)^{(3)}}. \tag{7}$$

The general t^{th} term, $0 \le t \le m$ of our development for S_m in (7) can be shown as:[2]

$$\frac{(m+2)^{(3)}(m+1)^{(3)}(m)^{(3)}(m-1)^{(3)} \ldots (m-t+3)^{(3)}}{(3)^{(3)}(4)^{(3)}(5)^{(3)}(6)^{(3)} \ldots (t+2)^{(3)}}$$

$$= \frac{(m+2)^{(t)}(m+1)^{(t)}(m)^{(t)}}{(t+2)^{(t)}(t+1)^{(t)}(t)^{(t)}}. \tag{8}$$

In reference [2], the successful derivation of a compact expression for Baxter permutation values appears as $B(n)$ in equation (1) of [2] and also on page 392 of [2]. In [2], index n starts at one, while our index starts at zero (as does Hoggatt's original index). For compatibility with our index, $B(n)$ of [2] becomes

$$B(m+1) = \binom{m+2}{1}^{-1}\binom{m+2}{2}^{-1}\sum_{k=1}^{m+1}\binom{m+2}{k-1}\binom{m+2}{k}\binom{m+2}{k+1}. \tag{9}$$

The general t^{th} term, $0 \le t \le m$, from (9) is

$$\frac{2\binom{m+2}{t}\binom{m+2}{t+1}\binom{m+2}{t+2}}{(m+2)^2(m+1)} = \frac{2(m+2)^{(t)}(m+2)^{(t+1)}(m+2)^{(t+2)}}{(m+2)^2(m+1)(t+2)!(t+1)!(t)!}. \tag{10}$$

To prove Hoggatt's conjecture, all we need do is show that $B(m+1)$ in (9) and our S_m in (7) have identical t^{th} terms. By restructuring the right side of (10) and canceling like numerator-denominator terms as shown below

$$\frac{\cancel{2}(m+2)^{(t)}\cancel{(m+2)}(m+1)^{(t)}\cancel{(m+2)}\cancel{(m+1)}(m)^{(t)}}{\cancel{(m+2)^2}\cancel{(m+1)}(t+2)^{(t)}\cdot\cancel{2}\cdot 1(t+1)^{(t)}\cdot 1(t)^{(t)}}, \tag{11}$$

we have identically the right side of (8).

If $(m-t)$ is substituted for t in the left side of (10), the same binomial coefficient product is obtained except for reverse order. This indicates equality between the $(m-t)^{\text{th}}$ and t^{th} terms of the sum and establishes symmetry of third-order Hoggatt triangles about a central vertical axis.

Thus, thanks in large measure to work [2] in which Hoggatt participated, a solid conjecture proof exists. We would like to think that Vern would be pleased to know that there are no longer any loose ends.

4. Hoggatt Sums and Hoggatt Triangles

A natural extension of Hoggatt's conjecture is to apply it to all right diagonals of Pascal's triangle. In this paper, the resultant row sums are called Hoggatt sums and the triangles formed by the successive row elements are called Hoggatt triangles. A particular row sum is identified by its index (0, 1, 2, ...) and its order. Order is equal to the index of the particular Pascal diagonal. Order of a Hoggatt triangle is similarly specified. The physical

[2]The terminology $(s)^{(p)} = p!\binom{s}{p}$ is a "partial" factorial, where

$$(s)^{(p)} = (s)(s-1) \ldots (s-p+1).$$

Because $0! = 1$, $(s)^{(0)} = (0)^{(0)} = 1$.

layout of a Hoggatt triangle is similar to that of Pascal's triangle in that each has the same number of row members. The k^{th} row of a Pascal triangle can be computed from the $(k-1)^{\text{st}}$ row. Hoggatt triangles share this attribute but additionally require data from the first diagonal to complete a new row.

The general Hoggatt development, including the proof of symmetry, is similar to that used earlier for the special case of $d = 3$. The row sum, $(S_d)_m$, becomes

$$(S_d)_m = (R_m)_0 + \sum_{i=0}^{m} (R_m)_{i+1}, \tag{12}$$

where

$$(R_m)_0 = 1, \quad (R_m)_{i+1} = \frac{\binom{m+d-1}{d}(R_{m-1})_i}{\binom{d+i}{d}} \tag{13}$$

and $\binom{d+i}{d}$ is the i^{th} element of the d^{th} Pascal diagonal.

In our terminology, d is the order and m is the index of the row sum. With the nucleus diagonals in place, operations similar to (5) and (6) lead to the summation forms

$$(S_d)_m = 1 + \sum_{h=0}^{m-1} \prod_{k=0}^{h} \frac{\binom{m+d-1-k}{d}}{\binom{d+k}{d}} = \sum_{k=1}^{m+1} \prod_{h=1}^{d} \frac{\binom{m+d-1}{k-2+h}}{\binom{m+d-1}{h-1}}. \tag{14}$$

The right expression in (14) is the reference [2] "analog" of the left expression in that, for $d = 3$, it reduces to (9).

Examples of Hoggatt triangles appear in Appendix A; Hoggatt sums in Appendix B. Although the extension of Hoggatt's conjecture is new, it is interesting to note that several of the resulting triangles or sums of orders zero through three are already well known. This actually enhances Hoggatt's work, since his conjecture and extensions introduce new ways of calculating the triangles and/or sums. For example, Hoggatt and Bicknell [5] point out that the array we designate as the Hoggatt triangle of order zero provides triangular numbers in base nine. Development of the Hoggatt triangle of order one introduces a new way of generating the time-honored Pascal triangle. Reference [5] anticipates the Hoggatt triangle of order two as an array of generalized binomial coefficients for the triangular numbers. Further, [5] demonstrates that Hoggatt sums of order two are identically the Catalan numbers, C_{n+1}. The equivalence of Hoggatt sums of order three and Baxter permutation values needs no further discussion.

5. A Computational Experiment

If a sequence of integers follows a linear index-invariant recursion, it is very easy to find the recursion formula. However, when the recursion is index-variant, the analytic difficulty increases dramatically. Reference [2] credits Paul S. Bruckman for equation (21) of [2], the linear, third-order, index-variant recursion formula for Baxter permutation values (Hoggatt sums of order three). When recast in our index m, Bruckman's formula is identically that which Hoggatt stated in [1]. Unfortunately, we have no way of knowing how Vern obtained this formula.

After a brief struggle with z-transform methods (see Jury's comments in [6], p. 59), we decided to attempt a nonanalytical determination of recursion formulas for second- and third-order Hoggatt sums as an experiment in digital computation. Because of the large, exact integers involved and the need for mixed symbolic and numeric operations, we chose to compute, in muMath, one of the currently available computer algebra systems (see [7], [8]). The experiment consisted essentially of a brute-force calculation of the coefficients of a recursion formula using simultaneous linear equations. After each run through the experiment, any false, inconsistent, or arbitrary values were either deleted or reassigned and the run repeated with fewer equations.

Surprisingly, we could never duplicate the coefficients of Bruckman's formula. A significant result, however, was that we could obtain an infinite number of sets of coefficients for formulas which were correct for all m values except one. For Hoggatt sums of order three (or Baxter permutation values), S_7 was always indeterminate. While the presence of arbitrary coefficients was responsible for the infinite number of sets of valid coefficients, the indeterminancy of S_7 was independent of the arbitrary coefficients. The results for the second-order Hoggatt sums were similar except that the sole indeterminant value occurred for $m = 2$, i.e., S_2 was indeterminant.

From the experiment we can ask, "Is Bruckman's analytical solution the only solution with no indeterminant S_m's? Also, does the above behavior hold for $d = 4, 5, 6, \ldots$?"

For a more detailed account of the experiment as well as more complete derivations from within the main body of the paper, the reader is encouraged to contact the authors.

6. Summary

We have proved Hoggatt's conjecture and have extended it to all Pascal diagonals. Formulas for obtaining Hoggatt triangles and sums have been developed. We have shown that lower-order triangles and sums provide new ways to view previously known structures. A computational experiment produced an infinite number of restricted recursion formulas for several lower-order Hoggatt sums.

7. Acknowledgments

We are indebted to Marjorie Bicknell-Johnson and Paul S. Bruckman for sharing their recollections, calculations, and correspondence relative to the time when Vern Hoggatt conceived his conjecture. Specifically, Bicknell-Johnson alerted us to the Catalan connection, while Bruckman provided an outline of his analytic derivation of the recursion formula for Baxter permutation values.

We also wish to thank the referees for excellent suggestions which improved the content and readability of the paper. The first referee provided the right expression of (8), which simplified our calculations greatly. The second referee contributed a neat, condensed version of (12) and (13) and also found the errors from two embarrassing typos by the senior author.

APPENDICES

Appendix A: Hoggatt Triangles

1
1 1
1 3 1
1 6 6 1
1 10 20 10 1
1 15 50 50 15 1
1 21 105 175 105 21 1

ORDER TWO

1
1 1
1 4 1
1 10 10 1
1 20 50 20 1
1 35 175 175 35 1
1 56 490 980 490 56 1

ORDER THREE

1
1 1
1 5 1
1 15 15 1
1 35 105 35 1
1 70 490 490 70 1
1 126 1764 4116 1764 126 1

ORDER FOUR

1
1 1
1 6 1
1 21 21 1
1 56 196 56 1
1 126 1176 1176 126 1
1 252 5292 14112 5292 252 1

ORDER FIVE

Appendix B: Hoggatt Sums

SUMS	VALUE
S0	1
S1	2
S2	5
S3	14
S4	42
S5	132
S6	429
S7	1430
S8	4862
S9	16796
S10	58786
S11	208012
S12	742900
S13	2674440
S14	9694845
S15	35357670
S16	129644790
S17	477638700
S18	1767263190
S19	6564120420
S20	24466267020
S21	91482563640
S22	343059613650
S23	1289904147324
S24	4861946401452
S25	18367353072152
S26	69533550916004
S27	263747951750360
S28	1002242216651368
S29	3814986502092304
S30	14544636039226909
S31	55534064877048198
S32	212336130412243110
S33	812944042149730764
S34	3116285494907301262
S35	11959798385860453492
S36	45950804324621742364
S37	176733862787006701400
S38	680425371729975800390
S39	2622127042276492108820
S40	10113918591637898134020
S41	39044429911904443959240
S42	150853479205085351660700
S43	583300119592996693088040
S44	2257117854077248073253720
S45	8740328711533173390046320
S46	33868773757191046886429490
S47	131327898242169365477991900
S48	509552245179617138054608572
S49	1978261657756160653623774456
S50	7684785670514316385230816156
S51	29869166945772625950142417512
S52	116157871455782434250553845880
S53	451959718027953471447609509424
S54	1759414616608818870992479875972
S55	6852456927844873497549658464312
S56	26700952856774851904245220912664
S57	104088460289122304033498318812080
S58	405944995127576985730643443367112
S59	1583850964596120042686772779038896

ORDER TWO

SUMS	VALUE
S0	1
S1	2
S2	6
S3	22
S4	92
S5	422
S6	2074
S7	10754
S8	58202
S9	326240
S10	1882960
S11	11140560
S12	67329992
S13	414499438
S14	2593341586
S15	16458756586
S16	105791986682
S17	687782586844
S18	4517543071924
S19	29949238543316
S20	200234184620736
S21	1349097425104912
S22	9154276618636016
S23	62522506583844272
S24	429600060173571952
S25	2968354097506204352
S26	20616682170931488704
S27	143886306136373723072
S28	1008739441056488779984
S29	7101857696077190042814
S30	50197792010624790718274
S31	356134037157421426324858
S32	2535503283457453475113498
S33	18111330098002679241995204
S34	129775523667497672794119820
S35	932649996060323085135343660
S36	6721418743462792115061865000
S37	48568825344643221105258466964
S38	351844920522232388929981300716
S39	2554987813422078288794169298972
S40	18596055885560437500207978342572
S41	135644235608879594521014316895264
S42	991488035658098636545959755543168
S43	7261715593999548236305978326928768
S44	53286745759568455589698874494878272
S45	391734954014771562094562102701976912
S46	2884866707621100648995326107469142704
S47	21280832747254136400685727258623694064
S48	157235970697232109921578618634420133232
S49	1163558691573487855005674103586862832160
S50	8623270949913637637693313639417883473760
S51	63999829606711522650915748086714806055520
S52	475648020504874336968975846703558704767360
S53	3539736620746899551478214384426524560969920
S54	26376309482014901194800065543131184691392320
S55	196786571758072254774209654628466146096941120
S56	1469930377434643825117255656238830229231391040
S57	10992599534625333878995280114433052775213597440
S58	82298082996123210666432106893608345734255512320
S59	616806373541881093477734895753501754683667475200

ORDER THREE

Appendix B (continued)

SUMS	VALUE
S0	1
S1	2
S2	7
S3	32
S4	177
S5	1122
S6	7898
S7	60398
S8	494078
S9	4274228
S10	38763298
S11	366039104
S12	3579512809
S13	36091415154
S14	373853631974
S15	3966563630394
S16	42997859838010
S17	475191259977060
S18	5344193918791710
S19	61066078557804360
S20	707984385321707910
S21	8318207051955884772
S22	98936727936728464152
S23	1190144254132426538652
S24	14467503754920598547852
S25	177588968969030657062952
S26	2199766555762125604480244
S27	27479841323744789830066304
S28	346013356369921918769855929
S29	4389333539509515126591248594
S30	56070810203828991362664847534
S31	720991537747532706012643525026
S32	9328596513998279672146714203426
S33	121407761182708178024779745555236
S34	1588853327416452312225693971901886
S35	20902698473348916294574193083438576
S36	276366709279158375016777229713551178
S37	3671353895684626011348096048652533188
S38	48991879229954382412465500058360070428
S39	656578339509065473624710057081932405468
S40	8835422665626508141712557966494394806108
S41	119361980337149820156413158335452884741480
S42	1618555251833277417723413502651871963117380
S43	22026306046942304682421202107440636378252080
S44	300775665856985037635815504148162320960569030
S45	4120680721821174437200697187060554338727113380
S46	56632089950769630959003010091719578219572701768
S47	780672963674065363024657714942613611640651191668
S48	10792880714535509030956272898321515183823343600148
S49	149630114772321753565389670918869975981300480583368
S50	2080024562297436725383387627342232184290452724623868
S51	28989631221925585334377822573493132380111499239694256
S52	405042859452333599815966969539580644980304039216295996
S53	5672895639230230744501228216933481231786496342059764296
S54	79637499355923524957310381320358435277891452641832058656
S55	1120481796741420900139026353148731893246859107399301109816
S56	15799141364786589904575760510447056727208857922105968342104
S57	223240203381865382931261283307541517610831772674383845140304
S58	3160762512031293096204497160156094620737550686304124391199144
S59	44839790319506826307665601833880717528407912782175379485606144

ORDER FOUR

SUMS	VALUE
S0	1
S1	2
S2	8
S3	44
S4	310
S5	2606
S6	25202
S7	272582
S8	3233738
S9	41454272
S10	567709144
S11	8230728508
S12	125413517530
S13	1996446632130
S14	33039704641922
S15	566087847780250
S16	10006446665899330
S17	181938461947322284
S18	3393890553702212368
S19	64807885247524512668
S20	1264344439859632559216
S21	25157307567003414461132
S22	509758613701956725065312
S23	10504632497377410035604512
S24	219882344614457972071894112
S25	4670084185135656513206765312
S26	100545485811075166151802795824
S27	2192418844178243335833955155336
S28	48379667285208331243156909951858
S29	1079611110993258648130498445227930
S30	24347329288405445022766602579123442
S31	554566629846326336323633836780509714
S32	12750735363523736895224533482780247714
S33	295784841468452675270005420750848137236
S34	6919476264486250695584491663120163937904
S35	163169952940281696912145006005492340179568
S36	3877071820176178830433674797637159283033876
S37	92790578667967629170910388674462669088090860
S38	2236101047387592560288927021551097525450121020
S39	54241035539604690484028904444539438691470414804
S40	1323989240924397287678504113074504691152647841900
S41	32511753934173216440442934840169645923808825880160
S42	802940141706099352768612717751656935735641885154128
S43	19939180912572384650238245526841891336089453041203076
S44	497753077340750439345361117379228609393516565922287616
S45	12488499358131177277361272166359232020651763382248580116
S46	314835804201793426134718207824493669241936791219996967272
S47	7974948166936771934625468937473862232444337311576280767968
S48	202903531153561613979904796282755350111618310117591503533536
S49	5184669810730751738195690096846261341819375230915901136000064
S50	133031155163064817158937067974769173342349063013900835309293824
S51	3427055486216516092233309733621545807996225269108441463298272448
S52	88626309532650460632389871849045042148680549782040988260727829632
S53	2300485147484122546569992300460137733495781737369600781780516134976
S54	59928849983050369859897945129004440086248293188817578582162240727680
S55	1566605017513482961732955658845929295951027100154584705176100343447424
S56	41090373302605147887752883307921674253835971440718737659073287348513408
S57	1081259820998848048353209424742475697589922619283381601497939222715737088
S58	28541983181144917576594561989169677540337165840094612722197240073620315232
S59	755716976463771668194168330657640641261070871073397885785459539567999933788

ORDER FIVE

References

1. Personal letters from Verner E. Hoggatt, Jr., to Daniel C. Fielder, dated June 15, 1977, August 19, 1977, and October 28, 1977.
2. F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., & M. Kleiman. "The Number of Baxter Permutations." *Journal of Combinatorial Theory 24.3* (1978):382-394.
3. G. Baxter & J. T. Joichi. "On Permutations Induced by Commuting Functions, and an Imbedding Question." *Math. Scand. 13* (1963):140-150.
4. W. M. Boyce. "Generation of a Class of Permutations Associated with Commuting Functions." *Math. Algorithms 2* (1967):14-26.
5. V. E. Hoggatt, Jr., & M. Bicknell. "Triangular Numbers." *Fibonacci Quarterly 12.3* (1974):221-230.
6. E. I. Jury. *Theory and Application of the z-Transform Method.* New York: Wiley & Sons, 1964.
7. D. Small, J. Hosack, & K. Lane. "Computer Algebra Systems in Undergraduate Instruction." *The College Mathematics Journal* 17.5 (1987):423-433.
8. D. D. Shochat. "A Symbolic Mathematics System." *Creative Computing 8.10* (1982):26-33.

*****