THE ALPHA AND THE OMEGA OF THE WYTHOFF PAIRS

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1. Introduction

The Wythoff number pairs have been much discussed in the literature on Fibonacci integers (see [1] for references up to 1978). And in [2] and [3] M. Bicknell-Johnson treats generalizations of Wythoff numbers which provide number triples with many interesting properties. In this paper we present three different ways to generate the Wythoff pairs, and, with some trepidation in view of the extent of the literature on them, claim that these are "new." We emphasize the notion of "generation" (in contrast to "giving a formula"), and introduce Fibonacci word patterns [3] as a tool to define n-tuple generating processes.

A determinantal relation for the Wythoff pairs is described, which makes further use of the word-pattern tools.

In the final section we show how similar methods can be used to generate and study sequences of integer triples. Three examples are given, and each is an attempt to generalize aspects of the Wythoff pairs-sequence.

It is clear to us that these tools and methods hold much promise for developing a general theory of sequences of integer n-tuples which have structures related to Fibonacci word patterns.

2. Notation and Definitions

The main tool to be used below is the Fibonacci word pattern, which we developed in [3]. We shall also use an operation of merging two integer sequences, and its inverse; we shall use the terms *addmerge* and *submerge* for these two operations.

Definitions and examples

To keep the exposition brief and readable, we now give somewhat informal definitions of the operations and concepts we wish to use. The examples will make the intended operations perfectly clear.

(i) *Fibonacci word patterns* $(W_1W_2W_3 \ldots W_n \ldots)$
A *word pattern* is a concatenation of a sequence of words W_1, W_2, W_3, $\ldots$, W_n, $\ldots$. The words are formed using characters from a given *letter set* such as $\{0, 1\}$ or $\{a, b, c\}$. The basic word pattern is obtained by repeatedly using the concatenation recurrence

$$W_{n+2} = W_nW_{n+1}, \text{ with } W_1 = A,\ W_2 = B.$$

We shall denote the resulting pattern by $F(A, B)$. Then:

$$F(A, B) = A, B, AB, BAB, ABBAB, \ldots .$$

(N.B. The commas on the right should be removed; they are inserted to show the boundaries of successive words in the pattern.)

Examples to be used below:

$\alpha = (1, 0) = 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots$ (using $A = 1$, $B = 0$);

$\omega = (1, 01) = 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \ldots$ (using $A = 1$, $B = 01$).

These two binary word patterns are, respectively, the *alpha* and the *omega* referred to in the title of this article. The ω pattern is named after Wythoff, for reasons which will become abundantly clear as the paper develops.

We shall also use the tribonacci word pattern (with $W_1 = A$, $W_2 = B$, $W_3 = C$)

$F(A, B, C) = A, B, C, ABC, BCABC, \ldots$ $(W_{n+3} = W_n W_{n+1} W_{n+2})$.

(ii) *Set-sequences*

In [4] we introduced the following construction (though with a slightly different notation). Let $\{S_n\}$ be a sequence of sets, and let $\{a_n\}$ be a sequence of integers. The set-sequence is formed using the following recurrence

$$S_{n+2} = S_n \cup S_{n+1} + a_n,$$

with S_1, S_2 being any given sets. The + operation is to be carried out as indicated by

$$\{s_1, s_2, \ldots, s_i, \ldots\} + a = \{s_1 + a, s_2 + a, \ldots, s_i + a, \ldots\}.$$

(iii) *Addmerging and submerging*

A merging operation, and its inverse, should be clear from the following definition and example.

Let S and T be any monotone increasing sequences. Then the *addmerge* of S and T (written $S \frown T$) is obtained by taking the multi-union of the two sequences and sorting them into monotonic increasing order. By "multi-union" we mean that integer repetitions are to be allowed.

Example

Let $S = \{1, 3, 5, 7, 9, \ldots\}$ and $T = \{2, 5, 8, 11, \ldots\}$. Then

$$S \frown T = \{1, 2, 3, 5, 5, 7, 8, 9, 11, \ldots\}.$$

The inverse of addmerge is *submerge*, which we shall write $S \smile T$. This operation simply removes the sequence S from the sequence S (all elements of T, that is, which happen to be in S).

(iv) *Sequence notations*

We shall use either Greek letters or underlined, lowercase Roman letters to denote sequences; and will use or omit subscripts on individual sequence elements as is appropriate. The following examples illustrate our notation, and will be needed below.

$\underline{n} = \{1, 2, 3, \ldots\}$ the natural numbers

$\underline{n}^+ = \{0, 1, 2, \ldots\}$ the natural numbers with zero

$\underline{f} = \{1, 1, 2, 3, 5, \ldots\}$ the Fibonacci integers $\{F_n\}$

$\underline{f}' = \{1, 2, 3, 5, \ldots\} = \{F_{n+1}\}$, $n = 1, 2, \ldots$

$\underline{f}'' = \{2, 3, 5, 8, \ldots\} = \{F_{n+2}\}$, $n = 1, 2, \ldots$

$\underline{\omega_1} = \{1, 3, 4, 6, 8, \ldots\}$ first members of Wythoff pairs, equals $\{[n\alpha]\}$ where $\alpha = \frac{1}{2}(1 + \sqrt{5})$.

$\underline{\omega_2} = \{2, 5, 7, 10, 13, \ldots\}$ second members of Wythoff pairs, equals $\{[n\alpha^2]\}$.

$$\underline{\omega} = \begin{pmatrix}\underline{\omega_1}\\ \underline{\omega_2}\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix}\begin{pmatrix}4\\7\end{pmatrix}\begin{pmatrix}6\\10\end{pmatrix}\begin{pmatrix}8\\13\end{pmatrix} \ldots \text{ Wythoff pair-sequence.}$$

(v) *Binary word pattern representations*

Let B be a sequence of 0's and 1's, say,

$$B = b_1, b_2, b_3, \ldots, b_n, \ldots, \text{ all } b_i \in \{0, 1\}.$$

And let $\underline{s} = s_1, s_2, s_3, \ldots, s_n, \ldots$ be an integer sequence. Then,

$\underline{s} * B \equiv$ the subsequence from $\underline{s}$ whose elements are in the positions where the 1's occur in B.

For example, if $B = 0, 1, 0, 1, 0, 1, \ldots$, then,

$$\underline{n} * B = 2, 4, 6, 8, \ldots .$$

3. Three Ways of Generating the Wythoff Pairs

We now use our word patterns and sequence notations to give three different ways to generate the Wythoff pairs.

(i) *Use of the omega sequence*

Recall from Section 2 that the binary word pattern omega is

$$\omega = F(1, 01) = 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \ldots$$

It may be observed that the 1's occur in positions $\underline{\omega_1} = 1, 3, 4, 6, 8, \ldots$; and the 0's occur in positions $\underline{\omega_2} = 2, 5, 7, 10, 13, \ldots$. Thus, ω contains all the information needed for producing the Wythoff pairs. Using the notations of 2(iv) and 2(v), we can write:

$$\underline{\omega} = \begin{pmatrix}\underline{\omega_1}\\ \underline{\omega_2}\end{pmatrix} = \begin{pmatrix}\underline{n} * F(1, 01)\\ \underline{n} * F(0, 10)\end{pmatrix}$$

Note that we can do certain algebraic operations with sequences and our new notation.

Thus, for example:

$$\underline{n} = \underline{\omega_1} \frown\!\!\frown \underline{\omega_2} = \underline{n} * [F(1, 01) + F(0, 10)]$$

where + is mod 2 addition of elements.

In [6] two methods of proof are given to demonstrate that omega [i.e., $F(1, 01)$] does in fact generate the Wythoff sequences

$$\underline{\omega_1} = \{[n\alpha]\} \quad \text{and} \quad \underline{\omega_2} = \{[n\alpha^2]\}$$

as claimed.

(ii) *Use of Fibonacci set-sequences*

In 2(ii) above, we explained the recurrence for generating Fibonacci set-sequences, viz.,

$$S_{n+2} = S_n \cup S_{n+1} + a_n.$$

Let $S_{-1} = \{0\}$ and $S_0 = \{0\}$, and $\{a_n\} \equiv \underline{f}' = 1, 2, 3, 5, \ldots$. Then,

$$S_1 = \{1\},\ S_2 = \{3\},\ S_3 = \{4, 6\}, \text{ etc.};$$

and it soon becomes clear that

$$\bigcup_{n=1}^{\infty} S_n = \underline{\omega}_1.$$

Similarly, if $\{a_n\} = \underline{f}'' = 2, 3, 5, 8, \ldots$ and the same S_{-1}, S_0 are used, the infinite union

$$\bigcup_{n=1}^{\infty} S_n = \underline{\omega}_2.$$

Proofs of these assertions are given in [6].

(iii) *Use of Fibonacci magic matrices*

In [5] we decided that square matrices all of whose elements were Fibonacci integers, whose diagonal, row, and column sums were Fibonacci integers and, moreover, whose powers also possessed these properties, deserved to be called *magic*. The spectral radius of these matrices is α, the golden mean. The simplest such matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \quad \text{Note that } A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is $\alpha^2 - \alpha - 1$; and of A^2 is $\lambda^2 - 3\lambda + 1$, which has maximum root $\lambda = \alpha^2$ with $\alpha = \frac{1}{2}(1 + \sqrt{5})$.

Many properties of A have been noted in the literature, but the following relationships with the Wythoff pairs may possibly be new. We give them without full proof.

Proposition (generation of $\underline{\omega}$, the Wythoff pairs sequence):

(A) $\underline{\omega} = \left\{[nA + m_n I]\binom{1}{1}\right\}$,

where I is the 2×2 identity matrix, $\underline{n}$ is the natural number sequence, and

$$\underline{m} = \{m_n\} = 0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, \ldots .$$

(N.B. The generation of $\underline{m}$ by a Fibonacci word pattern is given below.)

(B) $\underline{\omega} = \left\{[r_n A + m_n A^2]\binom{1}{1}\right\}$, where

$$\underline{r} = \{r_n\} = 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, \ldots .$$

Note that (B) follows from (A) since $\underline{m} + \underline{r} = \underline{n}$ and $A^2 = A + I$.

Generation of $\underline{m}$ *and* $\underline{r}$

The sequence $\underline{m}$ is generated as follows. Take the Fibonacci word pattern

$$F(1, 2) = 1, 2, 12, 212, 12212, \ldots .$$

We can use the elements of this sequence as *frequencies*, drawing elements from the sequence $\underline{n} = 0, 1, 2, 3, 4, \ldots$ with these frequencies. This gives a natural extension to the star operation which we defined in 2(v), in connection with binary words. Thus, we get

$$\underline{m} = \underline{n}^+ * F(1, 2) = 0, 11, 2, 33, 44, 5, 66, \text{etc.}$$

as required. With this very useful extended operation (which includes the earlier one, if 0 frequency is interpreted as "leave out"), we see that the sequence $\underline{r}$ in proposition (B) for $\underline{\omega}$ is:

$$\underline{r} = 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, \ldots = \underline{n} * F(2, 3).$$

Corollary: $\underline{n}^+ * F(1, 2) + \underline{n} * F(2, 3) = \underline{n}$, since $\underline{m} + \underline{r} = \underline{n}$.

The attractiveness of the method of generation of the Wythoff pairs just given lies in comparisons that can be made with the classical generation of the individual sequences. To spell these out, we note that $\underline{\omega}_1 = 1, 3, 4, 6, 8, \ldots$ is generated by $[n\alpha]$, and $\underline{\omega}_2 = 2, 5, 7, 10, 13, \ldots$ is generated by $[n\alpha^2]$, where α is the golden mean. By comparison, the generation formula given in (A) above for

$$\underline{\omega} = \binom{1}{2}\binom{3}{5}\binom{4}{7}\binom{6}{10}\binom{8}{13} \ldots$$

uses only nA, where A is a matrix having α as spectral radius, and a sequence $\underline{n}^+ * F(1, 2)$: and the sequence $F(1, 2)$ has the same pattern as that other "α," the basic Fibonacci word pattern referred to in the title of our paper.

4. A Determinantal Relation for the Wythoff Pairs

The following interesting relationship is reminiscent of the well-known Fibonacci relation

$$\begin{vmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{vmatrix} = (-1)^{n+1}.$$

Consider the Wythoff pair-sequence

$$\underline{\omega} = \binom{\underline{\omega}_1}{\underline{\omega}_2}.$$

To simplify the notation, we write $\underline{\omega}_1 = \underline{a} = \{a_i\}$ and $\underline{\omega}_2 = \underline{b} = \{b_i\}$. Then,

$$\underline{\omega}_1 = \binom{a_1}{b_1}\binom{a_2}{b_2}\binom{a_3}{b_3} \cdots = \binom{1}{2}\binom{3}{5}\binom{4}{7} \cdots$$

Let us pass a 2×2 window along this sequence and compute determinants as we go. Thus,

$$\left\{\begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix}\right\} = \begin{array}{l} -1, 1, -2, -2, 3, -3, 4, -4, -4, \\ 6, -5, -5, 8, -6, 9, -7, -7, 11, -8, \ldots . \end{array}$$

There is clearly an interesting pattern to the sequence, but how can we capture it in a formula? It is here that our word pattern notation becomes really useful. Let us *submerge* the negative and the positive elements, to find:

$$-1, -2, -2, -3, -4, -4, -5, -5, -6, -7, -7, -8, \ldots$$

and

$$1, 3, 4, 6, 8, 9, 11, \ldots .$$

Now we see that the negative sequence can be written as $(-\underline{n}) * F(1, 2)$. And the positive sequence is just $\underline{\omega}_1$.

Thus, we state finally

Proposition:

(i) The determinants of successive Wythoff pairs are given by

$$\Delta\underline{\omega} \equiv \left\{ \begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix} \right\} = \underline{\omega}_1 \curlywedge [(-\underline{n}) * F(1, 2)],$$

(with the addmerge ignoring minus signs). □

It might be said that to complete the above proposition we must give a precise formula for the n^{th} term of $\Delta\underline{\omega}$, whereas we have given only a sequence generator. We shall do this later. As we said in the Introduction, we wish first to emphasize ways in which our notation can describe the generation of interesting sequences. Picking out particular values of a sequence is always harder to do. In [3] and [6] are given many formulas for making that task easier, being results concerning counts of runs and runlengths of particular letters or integers in given Fibonacci word patterns.

(ii) Formulas for the n^{th} term in the sequence of determinants are:

$$(\Delta\underline{\omega})_n = \begin{cases} \underline{\omega}_{1i}, & \text{when } n \in \underline{\omega}_2 \text{ with } n = [i\alpha^2], \\ & \text{where } \alpha \text{ is the golden ratio;} \\ -i, & \text{when } [(i-1)\alpha^2] < n < [i\alpha^2]. \end{cases}$$

Proof: It may be seen that the positive terms in the sequence of determinants, namely,

$$\underline{\omega}_1 = 1, 3, 4, 6, 8, 9, 11, \ldots,$$

occur at positions

$$\underline{\omega}_2 = 2, 5, 7, 10, 13, 15, 18, \ldots .$$

This is fascinating in itself, and immediately explains the given formulas, because the positive terms occur when $n = [i\alpha^2]$. □

[N.B. Because of the fact just noted in the proof, we could give the determinant sequence as

$$\Delta\underline{\omega} = \underline{n} * F(0, 10) \curlywedge (-\underline{n}) * F(1, 2).$$

An immediate corollary of the fact that $\Delta\underline{\omega}$ includes the sequence

$$(-\underline{n}) * F(1, 2)$$

is the following proposition on the representation of the natural numbers in terms of the Wythoff numbers.

(iii) In terms of the Wythoff numbers, every integer N can be represented as follows:

either uniquely as $N = a_{i+1}b_i - b_{i+1}a_i$ using Wythoff pairs;

or in two ways using a run of three consecutive Wythoff pairs thus: $N = a_{i+1}b_i - b_{i+1}a_i = a_{i+2}b_{i+1} - b_{i+2}a_{i+1}$.

There any many other interesting things that could be said about the sequence $\Delta\underline{\omega}$. One more will have to suffice. Suppose we mark the sequence into words, each of which ends at a positive integer thus:

$$(-1, 1)\ (-2, -2, 3)\ (-3, 4)\ (-4, -4, 6)\ (-5, -5, 8)\ (-6, 9)$$
$$(-7, -7, 11)\ (-8, 12) \text{ etc.}$$

The *lengths* of these words have the pattern $F(2, 3)$. And their *totals* follow the pattern

$$0, -1, 1, 2, -2, -2, 3, -3, 4, -4, -4, 6, -5, -5, 8, \ldots .$$

The first 0 indicates that the sum of the first two determinants is:

$$(\Delta\underline{\omega})_1 + (\Delta\underline{\omega})_2 = a_1b_2 + a_2(b_3 - b_1) - a_3b_2 = 0.$$

If we consider the sequence of word totals, it appears that it will oscillate with increasing amplitude; and that the sum

$$\sum_{i=1}^{n} (\Delta\underline{\omega})_i$$

will equal zero infinitely often; but we have not established proofs of these observations.

5. Generation of Other Pair-Sequences

Any Fibonacci word-pattern which uses a binary letter-set can be used to generate a pair-sequence. For example, the *alpha* sequence

$$\alpha = F(1, 0) = 1, 0, 10, 010, 10010, 01010010, \ldots$$

generates the following:

$$\underline{\alpha} = \begin{pmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}\begin{pmatrix} 8 \\ 7 \end{pmatrix} \begin{matrix} \ldots \text{ the 1 positions } \ldots \\ \ldots \text{ the 0 positions } \ldots . \end{matrix}$$

A question of interest now is whether $\underline{\alpha}$ can be expressed in terms of the Wythoff pairs, and vice-versa. Using our word-pattern tools, we find as follows:

$$\begin{aligned} \underline{\alpha}_1 &= 1\ 3\ 6\ 8\ 11\ 14\ 16\ 19\ \ldots \\ &= (1\ 3\ 4\ 6\ 8\ 9\ 11\ 12\ 14\ 16\ 17\ 19) \smile\smile (4\ 9\ 12\ 17\ \ldots) \\ &= \underline{\omega}_1 \smile\smile [\underline{\omega}_1 * F(0, 01)] = \underline{\omega}_1 * F(1, 10); \end{aligned}$$

and

$$\begin{aligned} \underline{\alpha}_2 &= 2\ 4\ 5\ 7\ 9\ 10\ 12\ 13\ 15\ 17\ 18\ \ldots \\ &= (2\ 5\ 7\ 10\ 13\ 15\ \ldots) \frown\frown (4\ 9\ 12\ 17\ \ldots) \\ &= \underline{\omega}_2 \frown\frown [\underline{\omega}_1 * F(0, 01)]. \end{aligned}$$

Thus, we have:

$$\underline{\alpha} = \begin{pmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \underline{\omega}_1 \smile\smile [\underline{\omega}_1 * F(0, 01)] \\ \underline{\omega}_2 \frown\frown [\underline{\omega}_1 * F(0, 01)] \end{pmatrix}.$$

By similar methods we can invert these equations thus:

$$\left.\begin{aligned} \underline{\omega}_1 &= \underline{\alpha}_1 \frown\frown [\underline{\alpha}_2 * F(0, 10)] \\ \underline{\omega}_2 &= \underline{\alpha}_2 \smile\smile [\underline{\alpha}_2 * F(0, 10)] \end{aligned}\right\}.$$

And so the *alpha* pair-sequence can be expressed in terms of the *omega* (Wythoff) pair-sequence; and vice-versa.

It is evident that by such means an infinite number of pair-sequences can be generated, and their properties studied by establishing relationships between them and the fundamental Wythoff pairs. A new kind of number theory could be developed, based upon the sequences $\underline{\omega}_1$ and $\underline{\omega}_2$, and related to the "ordinary" number theory based on $\underline{n}$ through the functions $[n\alpha]$ and $[n\alpha^2]$.

Finally, we give an indication of how these methods can be extended to study sequences of triples.

6. Generation of Triple-Sequences

We shall show, proceeding by examples and comments upon them, how to generate *triple-sequences* in three different ways. The first uses Fibonacci word-pattern with letter-set $\{a, b, c\}$; the second uses a tribonacci word pattern with letter-set $\{a, b, c\}$; and the third uses a "magic" tribonacci matrix.

(i) *Use of a Fibonacci word pattern*

Consider the following word pattern:

$$F(a, bc) = a, bc, abc, bcabc, abcbcabc, \ldots .$$

Listing the *positions* of a, b, and c, respectively, produces the following triple-sequence:

$$\begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 8 \end{bmatrix}\begin{bmatrix} 12 \\ 10 \\ 11 \end{bmatrix}\begin{bmatrix} 17 \\ 13 \\ 14 \end{bmatrix}\begin{bmatrix} 22 \\ 15 \\ 16 \end{bmatrix}\begin{bmatrix} 25 \\ 18 \\ 19 \end{bmatrix}\begin{bmatrix} 30 \\ 20 \\ 21 \end{bmatrix} \ldots$$

It will be noted that, as might be expected since the word pattern is Fibonacci, the three component sequences can each be expressed in terms of the Wythoff numbers. Thus:

$$\begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} \underline{\omega}_1 * \alpha \\ \underline{\omega}_2 \\ \underline{\omega}_2 + 1 \end{bmatrix}, \text{ where } \alpha = F(1, 0).$$

Then $\underline{a}_n = (\underline{\omega}_1)_i$, $\underline{b}_n = (\underline{\omega}_2)_n$, and $\underline{c}_n = (\underline{\omega}_2)_n + 1$, where

$$i = \underline{c}_{n-1} = (\underline{\omega}_2)_{n-1} + 1.$$

Note also that $\underline{a}$, $\underline{b}$, $\underline{c}$ are each strictly increasing sequences, they are non-intersecting, and their union equals $\mathbb{Z}^+$: all properties of the Wythoff pairs-sequence. Their proof is immediate from the way in which $F(a, bc)$ is expanded.

Another interesting point is that the parity of the terms in $\underline{a}$ is alternately odd and even. And then, since the sum $(b_n + c_n)$ is always odd, we have the sum $(a_n + b_n + c_n)$ also of alternating parity.

The parity patterns, and more generally mod 3, mod 4, etc., patterns of elements of multi-sequences generated from word patterns would seem to be worthy of study.

(ii) *Use of a tribonacci pattern*

Consider next the tribonacci expansion of $F(a, b, c)$ and the triple-sequence it generates through the positions of a, b, c in the resulting pattern.

$$F(a, b, c) = a, b, c, abc, bcabc, cabcbcabc, \ldots$$

$$\begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 8 \end{bmatrix}\begin{bmatrix} 13 \\ 10 \\ 11 \end{bmatrix}\begin{bmatrix} 18 \\ 14 \\ 12 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \\ 15 \end{bmatrix}\begin{bmatrix} 26 \\ 19 \\ 17 \end{bmatrix}\ldots$$

This triple-sequence again clearly has the property that each element sequence is monotone increasing, and the three sequences partition $\mathbb{Z}^+$.

When we first studied this sequence, we hoped that we would find simple relationships between $\underline{a}$, $\underline{b}$, and $\underline{c}$, respectively,

$$\{[n\tau]\}, \{[n\tau^2]\}, \text{ and } \{[n\tau^3]\},$$

where τ is the positive root of the tribonacci equation

$$x^3 - x^2 - x - 1 = 0.$$

This would have been an excellent generalization of the Wythoff pairs property whereby $\underline{a} = \{[n\alpha]\}$ and $\underline{b} = \{[n\alpha^2]\}$. Unfortunately, we have not been able to find such relationships, although there seems to be hope for relating Fibonacci word patterns to the sequences of first differences $\{\Delta[n\tau]\}$, etc. To encourage the reader to search for these, we show the first few tribonacci triples ($\tau \doteq 1.839$):

$$\begin{bmatrix} [n\tau] \\ [n\tau^2] \\ [n\tau^3] \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 18 \end{bmatrix}\begin{bmatrix} 7 \\ 13 \\ 24 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \\ 31 \end{bmatrix}\begin{bmatrix} 11 \\ 20 \\ 37 \end{bmatrix}\begin{bmatrix} 12 \\ 23 \\ 43 \end{bmatrix}\ldots$$

(iii) *Use of a tribonacci magic matrix*

Our third attempt to generalize the Wythoff pairs is to take a 3 × 3 matrix which generalizes the "magic" properties of the 2 × 2 matrix used in Section 3(iii), and attempt to generate with it a unique sequence of triples whose members partition $\mathbb{Z}^+$. Once again we must confess that we have not found a fully satisfactory way of defining such a unique sequence; but in the spirit of the aims of this paper we believe it is worth presenting our attempt.

The tribonacci magic matrix we shall use is:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Note that the characteristic polynomial of T is $-(\lambda^3 - \lambda^2 - \lambda - 1)$ so its spectral radius is $\tau \doteq 1.839$.

Note further that the row sums of powers of T are

$$T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, T^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ldots, \text{ which give } \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \\ 17 \end{bmatrix}\begin{bmatrix} 9 \\ 17 \\ 31 \end{bmatrix}, \ldots,$$

with each element of the triples being in tribonacci sequence. These are the magic properties of T.

The question we ask now is the following. In 3(iii)(B) we generated the Wythoff pairs using only the 2×2 matrices A and A^2 and coefficients from the sequences $\underline{n} * F(2, 3)$ and $\underline{n}^+ * F(1, 2)$. Can we generate a unique sequence of *triples*, a *T-sequence*, using only the 3×3 matrices T, T^2, and T^3, together with coefficients from sequences which can be defined in terms of Fibonacci word patterns? Furthermore, can we require the three member sequences of the triple sequence to be strictly increasing, and to partition $\mathbb{Z}^+$? If we can find such a T-sequence uniquely, it will constitute an excellent generalization of the Wythoff pairs sequence.

Out attempt, down through the first twenty triples, is tabulated below, showing the triples horizontally for convenience.

Triples (a, b, c) Elements			Row sums of $(pT + qT^2 + rT^3)$ Coefficients		
a	b	c	p	q	r
1	3	5	0	1	0
2	4	8	1	1	0
6	10	20	2	1	1
7	11	23	3	1	1
9	15	31	4	2	1
12	22	31	5	4	1
13	25	49	5	5	1
14	26	52	6	5	1
16	30	60	7	6	1
17	33	65	7	7	1
18	34	68	8	7	1
19	35	71	9	7	1
21	39	79	10	8	1
23	43	87	11	9	1
24	46	92	11	10	1
27	51	101	11	10	2
28	54	106	11	11	2
29	55	109	12	11	2
32	62	122	13	13	2
36	70	136	13	14	3
37	73	141	13	15	3

It will be noted that we have succeeded in advancing $(\underline{a}, \underline{b}, \underline{c})$ thus far without increasing p, q, and r by more than 1 at each step. But, as we confessed above, we have not determined a formula for advancing the triple sequence indefinitely while satisfying all the requirements for generalizing the Wythoff pairs to triples.

7. Summary

In this paper we have defined word patterns, and various tools derived from them, in order to generate and study increasing sequences of integer pairs and integer triples.

A particular outcome of our study of pair-sequences as derived from Fibonacci binary word patterns was to show how all such sequences (and there is an infinite class of them) might be related to the Wythoff pairs.

It is hoped that we have convinced the reader that there is much scope for developing a number theory of integer pairs defined by binary sequence representations and using tools such as we have described. The title of our paper, namely, "The Alpha and the Omega of Wythoff Pairs," might suggest that all has now been said upon the pairs. In fact we claim the opposite—that this paper can mark a beginning of a broad development in their study and application.

The path to a general study of triple sequences would seem to be a much harder (but nevertheless a most interesting) one to seek.

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