# A RECURRENCE RESTRICTED BY A DIAGONAL CONDITION: GENERALIZED CATALAN ARRAYS


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## 1. Introduction

For a real sequence $\{a_k\}_{k \geq 0}$, $a_0 \neq 0$, and $\mu \in \mathbb{N}$, consider the real array $A(k, n, \mu)$, $(k, n) \in \mathbb{Z} \times \mathbb{Z}$, which satisfies the recurrence

$$A(k, n, \mu) = \sum_{j \geq 0} a_j A(k - j, n - 1, \mu) \tag{1.1}$$

subject to the *diagonal* condition

$$A(k, \mu k, \mu) = 0 \text{ for } k > 0, \tag{1.2}$$

and the conditions

$$A(0, 0, \mu) = 1 \quad \text{and} \quad A(k, n, \mu) = 0 \text{ for } k < 0. \tag{1.3}$$

We wish to use lattice path combinatorics to obtain known formulas for $A(k, n, \mu)$. Collectively these constitute a Lagrange inversion formula. Others have made similar studies; our explanations are influenced by those of Raney [18] and Gessel [9]. We examine specific examples of recurrences and their solutions, the generalized Catalan arrays. We illustrate our approach by enumerating certain plane trees.

For the given sequence $\{a_k\}_{k \geq 0}$, let $\alpha(x; \mu)$ denote

$$\sum_{k \geq 0} A(k, \mu k + 1, \mu) x^k,$$

which we view as a *diagonal series*. In particular, let

$$a(x) = \alpha(x; 0) = \sum_{k \geq 0} a_k x^k \text{ (the } \textit{initial series}\text{)},$$

and let

$$\alpha(x) = \alpha(x; 1) \text{ (the } \textit{principal} \text{ diagonal series)}.$$

For any power series, let $[x^k]\Sigma f_j x^j$ denote the coefficient $f_k$. Let

$$\alpha_k = [x^k]\alpha(x) = A(k, k + 1, 1) \text{ [the } \textit{principal diagonal} \text{ of } A(k, n, 1)].$$

It is immediate from (1.1) and elementary properties of formal power series (see [3], [12]) that

$$A(k, n, 0) = [x^k]a^n(x). \tag{1.4}$$

The following record solutions to (1.1, 1.2, and 1.3).

*Propositions:* For $m, n \in \mathbb{Z}$ and $k, \lambda \in \mathbb{N}$:

1. $A(k, n, \mu) = \dfrac{n - \mu k}{n} A(k, n, 0), \; n \neq 0.$ (1.5)

2. $A(k, n, \mu) = \sum_{j \geq 0} (1 - \mu j) a_j A(k - j, n - 1, 0); \; A(k, 1, \mu) = (1 - \mu k) a_k.$ (1.6)

3. $A(k,\ m + n + \mu k,\ \lambda) = \sum_{j \geq 0} A(j,\ m + \mu j,\ \lambda) A(k - j,\ n + \mu(k - j),\ \mu).$ (1.7)

4. $A(k,\ n + \mu k,\ \mu) = [x^k]\alpha^n(x;\ \mu).$ (1.8)

5. $\phi(x) = \alpha(x;\ \mu)$ is a unique series satisfying $\phi(x) = a(x\phi^{\mu}(x)).$ (1.9)

These are proven in Sections 3 and 4. In proving (1.5), we interpret the factor $(n - \mu k)/n$. For (1.6) we interpret $\{A(k,\ 1,\ \mu)\}_{k \geq 0}$. Proposition (1.7) is a Vandermonde-type convolution; (1.8) shows that $A(k,\ n,\ \mu)$ is a convolution array; (1.9) gives a functional relationship between $\alpha(x;\ \mu)$ and $a(x)$ which immediately yields $x\alpha(x)$ as the compositional inverse of $x/a(x)$. Correspondingly, (1.9) with (1.4) and (1.5) yields a Lagrange inversion formula; another is given in Section 5.

A lattice path is a directed path in the Cartesian plane with vertices the lattice points (integer pairs) and with steps (directed edges) of the form $((x,\ y),\ (x + u,\ y + v))$. There will be various restrictions on $(u,\ v)$; the set of permitted $(u,\ v)$'s is called the *step set*. A lattice path from $(0,\ 0)$ to $(k,\ n)$ which lies strictly above the line $y = \mu x$ for $0 < x \leq k$ is called a $(k,\ n,\ \mu)$-*path*. If we restrict to steps of the form $((x,\ y),\ (x + j,\ y + 1))$ with weight $a_j$, and if the weight of a path is the product of the weights of its steps, then we shall show that $A(k,\ n,\ \mu)$ is the sum of the weights of the $(k,\ n,\ \mu)$-paths for $n \geq \mu k$.

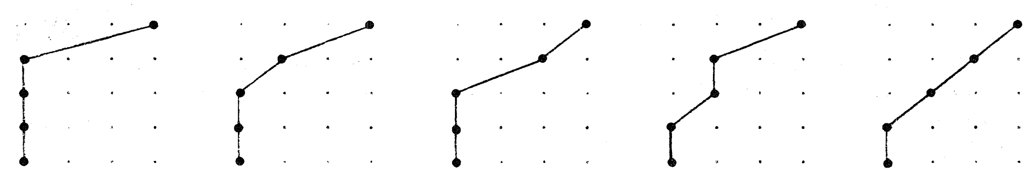

**FIGURE 1**

$A(3,\ 4,\ 1)$ counts the $(3,\ 4,\ 1)$-paths with step set $\{(j,\ 1) : j \in \mathbb{N}\}$ and $a_j = 1$ for $j \in \mathbb{N}$. $[A(3,\ 4,\ 1) = C(3,\ 4,\ 1) = \gamma_3$ of Example 2B).]

## 2. Examples of Recurrences and Their Solutions, the Catalan Arrays

The recurrences are defined by their initial series. $A(k,\ n,\ 0)$ and $\alpha = A(k,\ k + 1,\ 1)$ (often in [25]) are found from (1.4) and (1.5). For reference $b(x)$, $c(x)$, etc., $B(k,\ n,\ \mu)$, $C(k,\ n,\ \mu)$, etc., and $\beta$, $\gamma$, etc. denote the specific $a(x)$, $A(k,\ n,\ \mu)$, and $\alpha$. Here "PA," "CA," and "CN" abbreviate Pascal's array, Catalan's array (see [21], [24]) and the Catalan numbers [11]: 1, 1, 2, 5, 14, 42, ... . These examples are unnecessary for Sections 3 and 4.

*Example 2A:* $b(x) = 1 + x$. $B(k,\ n,\ 0) = \binom{n}{k}$, PA, where, for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$\binom{n}{k} = (n)(n - 1)\cdots(n - k + 1)/k!$ if $k > 0$ and $\binom{n}{0} = 1$.

$B(k,\ n,\ 1) = \binom{n - 1}{k}$, another PA, and $\beta_k = 1$ for $k \geq 0$.

$B(k,\ n,\ 2) = \frac{n - 2}{n}\binom{n}{k}$, CA (see Table 1).

$[x^k]\beta(x;\ 2) = B(k,\ 2k + 1,\ 2) = \frac{1}{2k + 1}\binom{2k + 1}{k}$, CN (marked ← in Table 1).

$\beta(x;\ 2) = 1 + x\beta^2(x;\ 2)$ by (1.9). The step set $\{(0,\ 1),\ (1,\ 1)\}$ yields $\binom{n}{k}$ as the number of $(k, n, 0)$-paths.

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 5 | 1 | 3 | 2← | −2 | −3 |
| 4 | 1 | 2 | 0 | −2 | −1 |
| 3 | 1 | 1← | −1 | −1 | 0 |
| 2 | 1 | 0 | −1 | 0 | 0 |
| 1 | 1← | −1 | 0 | 0 | 0 |
| 0 | 1 | −2 | 2 | −2 | 2 |
| −1 | 1 | −3 | 5 | −7 | 9 |
| −2 | 1 | −4 | 9 | −16 | 25 |

**TABLE 1**
A section of $B(k, n, 2)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 5 | 1 | 4 | 9 | 14 | 14← |
| 4 | 1 | 3 | 5 | 5← | 0 |
| 3 | 1 | 2 | 2← | 0 | −5 |
| 2 | 1 | 1← | 0 | −2 | −5 |
| 1 | 1← | 0 | −1 | −2 | −3 |
| 0 | 1 | −1 | −1 | −1 | −1 |
| −1 | 1 | −2 | 0 | 0 | 0 |
| −2 | 1 | −3 | 2 | 0 | 0 |

**TABLE 2**
A section of $C(k, n, 1)$

*Example 2B:* $c(x) = \sum_{k \ge 0} x^k = (1 - x)^{-1}$.

$$C(k, n, 0) = \binom{n + k - 1}{k}, \text{ PA.}$$

$C(k, n, 1)$, CA, are the ballot numbers [3], [16], see Table 2.

$$\gamma_k = \frac{1}{k+1}\binom{2k}{k}, \text{CN (marked ← in Table 2).}$$

$x\gamma^2(x) - \gamma(x) + 1 = 0$ by (1.9). $C(k, n, 0)$ counts the $(k, n, 0)$-paths with step set $\{(0, 1), (1, 0)\}$ with $(0, 1)$ as the initial step. $C(k, n, 0)$ also counts the $(k, n, 0)$-paths with step set $\{(j, 1) : j \in \mathbb{N}\}$; see Figure 1.

*Example 2C:* $\hat{b}(x) = 1 + x^\nu = b(x^\nu)$, where $\nu \in \mathbb{N}$, $\nu > 0$.

$$\hat{B}(k, n, 0) = \binom{n}{K} \text{ if } k = \nu K \text{ and } = 0 \text{ otherwise, a variant PA.}$$

$$\hat{B}(k, n, 1) = \frac{n-k}{n}\binom{n}{K} = \frac{n - \nu K}{n}\binom{n}{K} = B(K, n, \nu) \text{ if } k = \nu K \text{ and } = 0 \text{ otherwise.}$$

*Example 2D:* $\check{c}(x) = 1 + x^2$.

$$\check{C}(k, n, 0) = \binom{n}{k/2} \text{ for } k \text{ even and } = 0 \text{ otherwise, a variant PA.}$$

$$\check{\gamma}_k = \frac{1}{k+1}\binom{k+1}{k/2} = 1, 0, 1, 0, 2, \ldots, \text{ zero-interspersed CN.}$$

*Example 2E:* $\tilde{c}(x) = 1 + 2x + x^2$.

$$\tilde{C}(k, n, 0) = \binom{2n}{k}, \text{ a PA with every other row missing.}$$

$$\tilde{C}(k, n, 1) = B(k, 2n, 2).$$

$$\tilde{\gamma}_k = B(k, 2k + 2, 2) = \frac{1}{k+1}\binom{2k+2}{k}, \text{ CN with first entry missing.}$$

Note that $[x^k]\tilde{\gamma}(x) = [x^k]\beta^2(x; 2) = [x^k]x^{-1}(\beta(x; 2) - 1)$.

*Example 2F:* $\tilde{m}(x) = 1 + x + x^2$.

$$\tilde{M}(k,\ n,\ 0) = [x^k](1 - x^3)(1 - x)^{-1} = \sum_{i,\,j:3j+i=k} (-1)^j \binom{n}{j}\binom{n+i-1}{i}.$$

Also

$$\tilde{M}(k,\ n,\ 0) = [x^k]((1 + x^2) + x)^n = \sum_{i,\,j:n-j+2i=k} \binom{n}{j}\binom{j}{i}.$$

$\tilde{\mu}_k = \tilde{M}(k,\ k + 1,\ 1) = 1,\ 1,\ 2,\ 4,\ 9,\ 21,\ \ldots$, named for Motzkin [17], who found them to count the ways of placing nonintersecting cords between $k$ points on a circle. Note

$$\tilde{\mu}_k = \frac{1}{k+1}[x^k]((1 + x^2) + x)^{k+1} = \frac{1}{k+1}\sum_{j\geq 0}\binom{k+1}{j}[x^{j-1}](1 + x^2)^j$$

$$= \sum_{j\geq 0}\binom{k}{j-1}\check{\gamma}_{j-1}$$

and

$$\tilde{\gamma}_k = \frac{1}{k+1}[x^k]((1 + x + x^2) + x)^{k+1} = \frac{1}{k+1}\sum_{j\geq 0}\binom{k+1}{j}[x^{j-1}](1 + x + x^2)^j$$

$$= \sum_{j\geq 0}\binom{k}{j-1}\tilde{\mu}_{j-1}.$$

See Example 7A and [4], [5], and [14].

## 3. Lattice Path Analysis for Propositions 1 and 2

We use weighted paths with steps of the form $((x,\ y),\ (x + j,\ y + 1))$, denoted by $\langle j\rangle$ and assigned the weight $a_j$, $j \geq 0$. $\langle j_{1:n}\rangle$ denotes an arbitrary path $\langle j_1\rangle\langle j_2\rangle \cdots \langle j_n\rangle$ and $\prod_{i=1}^n a_{j_i}$ denotes its weight. $P(k,\ n,\ \mu)$ denotes the set of all $(k,\ n,\ \mu)$-paths and $|P(k,\ n,\ \mu)|$ denotes the sum of the weights of the paths in $P(k,\ n,\ \mu)$. When appropriate, $|A|$ denotes the cardinality of $A$.

Since all $(k,\ n,\ \mu)$-paths pass through $\{(k - j,\ n - 1) : 0 \leq j \leq k\}$ exactly once,

$$|P(k,\ n,\ \mu)| = \sum_{\langle j_{1:n}\rangle \in P(k,n,\mu)} \prod_{i=1}^n a_{j_i}$$

$$= \sum_{0\leq j\leq k}\ \sum_{\langle j_{1:n-1}\rangle \in P(k-j,n-1,\mu)} \prod_{i=1}^{n-1} a_{j_i} a_j$$

$$= \sum_{0\leq j\leq k} a_j |P(k - j,\ n - 1,\ \mu)|.$$

$|P(0,\ 0,\ \mu)| = 1$ and $|P(k,\ \mu k,\ \mu)| = 0$. Hence, $|P(k,\ n,\ \mu)|$ satisfies (1.1), (1.2), and (1.3) for $n \geq \mu k$. Thus,

*Proposition 6:* $|P(k,\ n,\ \mu)| = A(n,\ k)$ for $n \geq \mu k$. (3.1)

We next determine $|P(k,\ n,\ \mu)|$ by a "radiation" scheme, which extends the method used by Dvoretzky and Motzkin [7] on Barbier's ballot problem of counting the $(k,\ n,\ \mu)$-paths with a two element step set. Grossman [13], [16] reformulated their technique as "penetrating analysis." See also [18].

Each path $\langle j_{1:n}\rangle \in P(k,\ n,\ 0)$ determines a sequence of cyclic permutations, each being a path in $P(k,\ n,\ 0)$:

$$\langle j_{1:n}\rangle,\ \langle j_{2:1}\rangle = \langle j_2\rangle\langle j_3\rangle \cdots \langle j_n\rangle\langle j_1\rangle,\ \langle j_{3:2}\rangle = \langle j_3\rangle\langle j_4\rangle \cdots \langle j_1\rangle\langle j_2\rangle,$$
$$\ldots,\ \langle j_{n:n-1}\rangle = \langle j_n\rangle\langle j_1\rangle \cdots \langle j_{n-2}\rangle\langle j_{n-1}\rangle. \quad (3.2)$$

Let $p$ be the period of $\langle j_{1:n}\rangle$. $[\langle j_{1:n}\rangle]$ denotes the cyclic permutation class $\{\langle j_{1:n}\rangle, \langle j_{2:1}\rangle, \ldots, \langle j_{p:p-1}\rangle\}$, the set of distinct paths in (3.2).

Let $\langle j_{1:n}\rangle$ be a fixed [fixed until (3.5)] path in $P(k, n, 0)$ of period $p$.

$$\langle j_{1:n}\rangle = \langle j_1\rangle\langle j_2\rangle \cdots \langle j_p\rangle \in P(kp/n, p, 0),$$

is the initial subpath of $\langle j_{1:n}\rangle$. Each path in $[\langle j_{1:n}\rangle]$ is the concatenation of $n/p$ copies of a cyclic permutation of $\langle j_{1:p}\rangle$. Distinguish the steps in $\langle j_{1:p}\rangle$ by their index. Thus, each step in $\langle j_{1:p}\rangle$ initiates a unique path in $[\langle j_{1:n}\rangle]$. A step $\langle j\rangle$ is called a zero step if $j = 0$; otherwise it is called positive. Let $J_+ = \{i : j_i > 0 \text{ and } i \le p\}$, the index set of the positive distinguished steps of $\langle j_{1:p}\rangle$. See Figure 2.

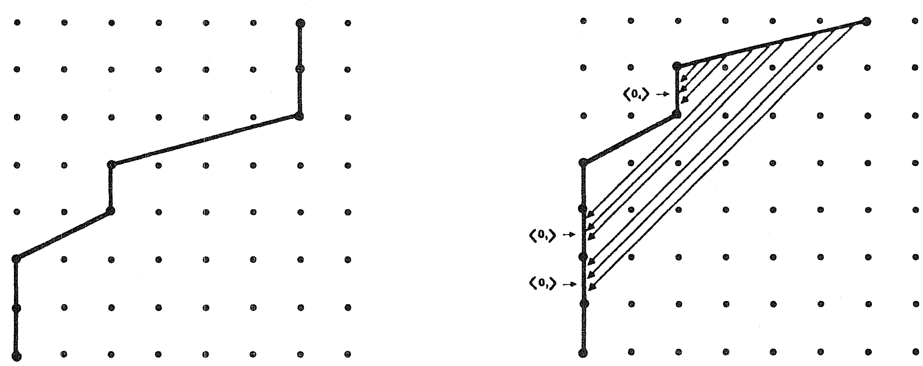


**FIGURE 2**

If $\mu = 1$ and $\langle j_{1:7}\rangle = \langle 0_1\rangle\langle 0_2\rangle\langle 2_3\rangle\langle 0_4\rangle\langle 4_5\rangle\langle 0_6\rangle\langle 0_7\rangle$ (the subscripts distinguish the steps), then $J_+ = \{3, 5\}$. $Z_3 = \{2\}$, $Z_5 = \{1, 4, 7\}$, $Z_c = \{6\}$. $\langle j_{1:7}\rangle$ and $\langle j_{6:5}\rangle$ are shown with 3 subswaths of rays indicated on $\langle j_{6:5}\rangle$.

In the following $n \ge \mu k$. Fix $i \in J_+$ and consider the geometrical configuration of $\langle j_{i+1:i}\rangle \in [\langle j_{1:p}\rangle]$ where each step is a line segment ($i + 1$ is replaced by 1 if $i = p$). From the points on the last step of $\langle j_{i+1:i}\rangle$, namely $\langle j_i\rangle$, draw rays in the direction of the ray from (0, 0) through $(-1, -\mu)$. See Figure 2. Since the terminal vertex of $\langle j_{i+1:i}\rangle$ is above or on the line $y = \mu x$, all rays must strike and be absorbed on the right side of $\langle j_{i+1:i}\rangle$ by the zero steps, positive steps being too inclined to be hit. By examining a triangle with vertices (0, 0), $(0, \mu j - 1)$, and $(j, \mu j)$, we see that the vertical width of the swath of rays from $\langle j_i\rangle$ is $\mu j_i - 1$. This swath can be partitioned into $\mu j_i - 1$ equal parallel subswaths. Since each subswath passes between vertically adjacent lattice points, each subswath must irradiate the entire interior of a zero step. If $Z_i$ denotes the set of indices with respect to $\langle j_{1:p}\rangle$ of the zero steps which are irradiated by the rays from $\langle j_i\rangle$, then

$$|Z_i| = \mu j_i - 1. \tag{3.3}$$

We claim that the $Z_i$, $i \in J_+$, are disjoint from one another. Suppose there is a zero step that is irradiated by both $\langle j_i\rangle$ and $\langle j_{i'}\rangle$, where the zero step appears earlier in, say, $\langle j_{i'+1:i'}\rangle$. But the configuration of $\langle j_{i'+1:i'}\rangle$ shows that the step $\langle j_i\rangle$ will shield this zero step from the irradiation of $\langle j_{i'}\rangle$.

Let $Z_c$ be the index set of *clean* (nonirradiated) zero steps in $\langle j_{1:p}\rangle$. By considering the vertical and the horizontal dimensions of $\langle j_{1:p}\rangle$,

$$p = \sum_{i \in J_+} |Z_i| + |J_+| + |Z_c|$$

and

$$\mu kp/n = \mu \sum_{i \in J_+} j_i = \sum_{i \in J_+} (\mu j_i - 1) + |J_+| = \sum_{i \in J_+} |Z_i| + |J_+|,$$

and thus,

$$|Z_c| = \frac{p(n - \mu k)}{n}. \tag{3.4}$$

As noted, each path in $[\langle j_{1:n}\rangle]$ is uniquely determined by its initial step which is a distinguished step of $\langle j_{1:p}\rangle$. A path beginning with a positive step touches or is below $y = \mu x$ by the first step. A path beginning with a zero step touches $y = \mu x$ for the first time on its radiating positive step. Thus, the paths beginning with a clean zero step are precisely those belonging to $P(k, n, \mu)$. By (3.4), we have

*Lemma:* The number of paths in $[\langle j_{1:n}\rangle] \cap P(k, n, \mu)$ is

$$|Z_c| = \frac{p(n - \mu k)}{n} = \frac{n - \mu k}{n}|[\langle j_{1:n}\rangle]|. \tag{3.5}$$

Since every path in a cyclic permutation class has the same weight and since the classes are disjoint with union $P(k, n, 0)$,

$$A(k, n, \mu) = |P(k, n, \mu)| = \sum \frac{n - \mu k}{n}|[\langle j_{1:n}\rangle]|\prod_{i=1}^{n} a_{j_i} \quad \text{(sum over all c.p. classes)}$$

$$= \frac{n - \mu k}{n}\sum |[\langle j_{1:n}\rangle]|\prod_{i=1}^{n} a_{j_i} = \frac{n - \mu k}{n}|P(k, n, 0)|$$

$$= \frac{n - \mu k}{n} A(k, n, 0).$$

Thus, a formula for $A(k, n, \mu)$ has been constructed for $n \geq \mu k$. Simple arithmetic shows that this formula satisfies (1.1), (1.2), and (1.3) for $n \neq 0$ and $k \geq 0$; hence (1.5) is valid.

A second realization of the contribution of each cyclic permutation class to $|P(k, n, \mu)|$ establishes (1.6). By (3.3) and (3.5), the weight contributed by $[\langle j_{1:n}\rangle]$ for $n \geq \mu k$ is

$$|Z_c|\prod_{i=1}^{n} a_{j_i} = \sum_{t \in Z_c} (1 - \mu j_t)a_{j_t}\prod_{i \neq t} a_{j_i} + \sum_{t \in J_+}\left\{\left[\sum_{s \in Z_t}(1 - \mu j_s)a_{j_s}\prod_{i \neq s} a_{j_i}\right]\right.$$

$$\left. + (1 - \mu j_t)a_{j_t}\prod_{i \neq t} a_{j_i}\right\} \quad \begin{matrix}\text{(since } j_t = j_s = 0 \text{ for zero steps and} \\ \text{since the term in } \{\ \} \text{ is } 0)\end{matrix}$$

$$= \sum_{t=1}^{p} (1 - \mu j_t)a_{j_t}\prod_{i \neq t} a_{j_i}.$$

Summing over all cyclic permutation classes yields

$$A(k, n, \mu) = |P(k, n, \mu)| = \sum_{\langle j_{1:n}\rangle \in P(k, n, 0)} (1 - \mu j_1)a_{j_1}\prod_{i=2}^{n} a_{j_i} \quad \text{(sum over all paths)}$$

$$= \sum_{j \geq 0} (1 - \mu j) a_j \sum_{\langle j_{2:n} \rangle} \prod_{i=2}^{n} a_{j_i}$$ [Since all paths pass through the line $y = 1$, the second sum is over all paths from $(j, 1)$ to $(k, n)$.]

$$= \sum_{j \geq 0} (1 - \mu j) a_j |P(k - j, n - 1, 0)| \quad \text{(weights are transition invariant)}$$

$$= \sum_{j \geq 0} (1 - \mu j) a_j A(k - j, n - 1, 0).$$

Thus we have constructed the formula of (1.6) for $n \geq \mu k$. If $\bar{A}(k, n)$ momentarily denotes this formula, then it is easily shown that $\bar{A}(k, n)$ satisfies (1.1) for $(k, n) \in \mathbb{N} \times \mathbb{Z}$. Since $\bar{A}(k, n)$ and $A(k, n, \mu)$ agree when $n \geq \mu k$, they agree for all $(k, n) \in \mathbb{N} \times \mathbb{Z}$, yielding (1.6).

Equation (1.6) yields a nice interpretation for $A(k, n, \mu)$ on both sides of $y = \mu x$ and $n \geq 1$. Retaining the definitions of this section, reassign the weight of $(1 - \mu j)a_j$ to the initial steps $((0, 0), (j, 1))$, $j \geq 0$. Then $A(k, n, \mu)$ is the sum of the modified weights of *all unrestricted* paths from $(0, 0)$ to $(n, k)$.

With $a'(x)$ denoting the usual formal derivative, immediately (1.6) is equivalent to (similar to a result in [1])

$$A(k, n, \mu) = [x^k](a(x) - \mu x a'(x)) a^{n-1}(x) \text{ for } n \in \mathbb{Z}. \tag{3.6}$$

## 4. The Proofs of Propositions 3, 4, and 5

We establish (1.7), a useful generalized Vandermonde-type convolution [10], [16]. Then, using the tractable notation of series, we reformulate both the convolution and (1.1) in terms of diagonal series.

First we give a lattice path proof of (1.7) for $m, n \geq 0$ and $m + \mu k \geq \lambda k$. Since $((x, y), (x+j, y+1))$, $j \geq 0$, is the form of the lattice steps, any path in $P(k, m + n + \mu k, \lambda)$ must intersect the line $M = \{(j, m + \mu j) : 0 \leq j \leq k\}$. Since the weight of a path is invariant under translation, the sum of the weights of the paths from $(j, m + \mu j)$ to $(k, m + n + \mu k)$ which remain above $M$ is $|P(k - j, n + \mu(k - j), \mu)|$. Hence the sum of the weights of the paths in $P(k, m + n + \mu k, \lambda)$ that pass through $M$ *for a last time* at $(j, m + \mu j)$ is the product

$$|P(j, m + \mu j, \lambda)| |P(k - j, n + \mu(k - j), \mu)|.$$

Summing over $M$ and putting $A(x, y, \mu) = |P(x, y, \mu)|$ yields (1.7) in this case.

Now for $m \in \mathbb{Z}$ and $n > 0$, (1.7) can be proved by induction on the value of $n + \mu k$ by observing that, for $n + \mu i - 1 < n + \mu k$,

$$\sum_{i \geq 0} a_i A(k - i, m + n + \mu i - 1 + \mu(k - i), \lambda)$$

$$= \sum_{i \geq 0} \sum_{j \geq 0} a_i A(j, m + \mu j, \lambda) A(k - i - j, n + \mu i - 1 + \mu(k - i - j), \mu)$$

$$= \sum_{j \geq 0} A(j, m + \mu j, \lambda) \sum_{i \geq 0} a_i A(k - j - i, n - 1 + \mu(k - j), \mu).$$

The case for $m \in \mathbb{Z}$ and $n \leq 0$ can be proved by induction with respect to $-n$ upon noting that (1.1) yields

$$A(k, m + (n - 1) + \mu k, \mu)$$

$$= A(k, m + n + \mu k, \mu) - \sum_{j \geq 0} a_j A(k - j, m + (n - 1 + \mu j) + \mu(k - j), \mu)$$

and that $-n + 1 - \mu j \leq -n$.

Equivalent to (1.8) is

$$\alpha^n(x;\ \mu) = \sum_{k \ge 0} A(k,\ \mu k + n,\ \mu)x^k \text{ for } n \in \mathbb{Z}.$$

For $n \ge 0$, this can be proved inductively since, by (1.7),

$$\begin{aligned}\alpha^{n+1}(x;\ \mu) = \alpha(x;\ \mu)\alpha^n(x;\ \mu) &= \sum_{k \ge 0} A(k,\ \mu k + 1,\ \mu)x^k \sum_{k \ge 0} A(k,\ \mu k + n,\ \mu)x^k \\ &= \sum_{k \ge 0} A(k,\ \mu k + n + 1,\ \mu)x^k.\end{aligned}$$

The case for $n \ge 0$ and (1.7) yields

$$\begin{aligned}\alpha^{-n}(x) &= \alpha^{-n}(x) \sum_{k \ge 0} A(k,\ \mu k + n,\ \mu)x^k \sum_{k \ge 0} A(k,\ \mu k - n,\ \mu)x^k \\ &= \alpha^{-n}(x)\alpha^n(x) \sum_{k \ge 0} A(k,\ \mu k - n,\ \mu)x^k = \sum_{k \ge 0} A(k,\ \mu k - n,\ \mu)x^k.\end{aligned}$$

As in [9], equation (1.8) has the following meaning for $n \ge \mu k$: Since each $(k,\ \mu k + n,\ \mu)$-path must sequentially intersect *for a last time* each of the lines $y = \mu x + i$ for $1 \le i \le n$, each $(k,\ \mu k + n,\ \mu)$-path is an $n$-fold concatenation of $(k,\ \mu j + 1,\ \mu)$-paths for various $j$. Correspondingly, the total weight of the $(k,\ \mu k + n,\ \mu)$-paths is a coefficient of an $n$-fold convolution of $\alpha(x;\ \mu)$.

Moreover, since each $(k,\ \mu k + 1,\ \mu)$-path intersects the line $y = \mu k$ only preceding its last step, the set of $(k,\ \mu k + 1,\ \mu)$-paths is the disjoint union

$$\begin{aligned}P(k,\ \mu k + 1,\ \mu) &= \bigcup_{j=0}^{k} \{\langle j_{1:\mu k}\rangle\langle j\rangle : \langle j_{1:\mu k}\rangle \text{ is a } (k - j,\ \mu(k - j) + \mu j,\ \mu)\text{-path}\} \\ &= \bigcup_{j=0}^{k} \{\langle j_{1:\mu k}\rangle\langle j\rangle : \langle j_{1:\mu k}\rangle \text{ is a } \mu j\text{-fold concatenation of various } (j,\ \mu j + 1,\ \mu)\text{-paths}\}.\end{aligned}$$

More precisely, we have that

$$\begin{aligned}\alpha(x;\ \mu) &= \sum_{k \ge 0} A(k,\ \mu k + 1,\ \mu)x^k \\ &= \sum_{k \ge 0}\sum_{j \ge 0} a_j x^j A(k - j,\ \mu(k - j) + \mu j,\ \mu)x^{k-j} \\ &= \sum_{j \ge 0} a_j x^j \sum_{k \ge 0} A(k - j,\ \mu(k - j) + \mu j,\ \mu)x^{k-j} \\ &= \sum_{j \ge 0} a_j x^j \alpha^{\mu j}(x;\ \mu) = \sum_{j \ge 0} a_j (x\alpha^{\mu}(x;\ \mu))^j.\end{aligned}$$

This establishes (1.9) since comparing coefficients shows the uniqueness.

As a consequence of (1.9), we have

*Proposition 7:* For each $\mu \in \mathbb{N}$, if $\alpha(x;\ \mu)$ is taken as the initial series, then $\alpha(x;\ \mu + 1)$ is the corresponding principal diagonal series.

*Proof:* If $\overline{\alpha}(x)$ denotes the principal diagonal for $\alpha(x;\ \mu)$, then by (1.9),

$$\begin{aligned}\overline{\alpha}(x) &= \alpha(x\overline{\alpha}(x);\ \mu) = a(x\overline{\alpha}(x)[\alpha(x\overline{\alpha}(x);\ \mu)]^{\mu}) \\ &= a(x\overline{\alpha}(x)[\overline{\alpha}(x)]^{\mu}) = a(x[\overline{\alpha}(x)]^{\mu+1}).\end{aligned}$$

But $\overline{\alpha}(x)$ must be $\alpha(x;\ \mu + 1)$ by the uniqueness in (1.9).

## 5. A Lagrange Inversion Formula

A common Lagrange inversion formula [3], [12], is included as it easily follows (1.4), (1.5), (1.6), and (1.9). See [3], [8], [9], [12], and [18] for more general formulas.

*Proposition 8:* For any initial series $a(x)$, there exists a unique series

$$\omega(x) = \sum_{k \geq 1} \omega_k x^k$$

such that $\omega(x) = xa(\omega(x))$. Moreover, if $f(x)$ is a formal Laurent series so that

$$f(x) = \sum_{k \geq t} f_k x^k \text{ for some } t \in \mathbb{Z},$$

$$[x^n]f(\omega(x)) = \begin{cases} \frac{1}{n}[x^{n-1}]f'(x)a^n(x) & \text{for } n \neq 0, \\ [x^0]f(x) + [x^{-1}]f'(x)\log(a(x)a^{-1}(0)) & \text{for } n = 0. \end{cases}$$

*Proof:* By (1.9), $\omega(x) = x\alpha(x)$ is the unique solution. It suffices to show the second part for $f(x) = x^k$, $k \in \mathbb{Z}$. For $n \neq 0$,

$$[x^n](x\alpha(x))^k = [x^{n-k}]\alpha^k(x) = A(n-k, n, 1) = [x^{n-k}]\frac{k}{n} a^n(x)$$
$$= \frac{1}{n}[x^{n-1}]kx^{k-1}a^n(x).$$

As noted in [12],

$$0 = [x^{-1}]\frac{d}{dx}(x^k \log(a(x)a^{-1}(0)))$$
$$= [x^{-1}]kx^{k-1}\log(a(x)a^{-1}(0)) + [x^{-1}]x^k a'(x)a^{-1}(x).$$

For $n = 0$, it follows from (3.6) that

$$[x^0]\omega^k(x) = [x^0]x^k\alpha^k(x) = [x^{-1}]\alpha^k(x) = A(-k, 0, 1)$$
$$= [x^{-k}](1 - xa'(x)a^{-1}(x)) = [x^{-1}]kx^{k-1}\log(a(x)a^{-1}(0)).$$

## 6. More Examples of Recurrences

*Example 6A:* $r(x) = 1 + (w+1)\sum_{k \geq 1} x^k = (1 + wx)(1 - x)^{-1}$.

$$R(k, n, 0) = [x^k] \sum_{j \geq 0} \binom{n}{j} w^j x^j \sum_{i \geq 0} \binom{n+i-1}{i} x^i$$
$$= \sum_{i \geq 0} w^{k-i} \binom{n}{k-i} \binom{n+i-1}{i}.$$

Note how $r_k$ relates to $a_k$ of Section 3. Also,

$$R(k, n, \mu) = R(k, n-1, \mu) + wR(k-1, n-1, \mu) + R(k-1, n, \mu),$$

and $R(k, n, \mu)$ is the sum of the weights of the $(k, n, \mu)$-paths with the step set $\{(0, 1), (1, 1), (1, 0)\}$ where $(1,1)$ has weight $w$. As shown in [21], [22], or from $R(k, n, 0)$, we have

$$R(k, n, 1) = \sum_{j \geq 0} w^j \binom{n+k-j-1}{j} C(k-j, n-j, 1).$$

For $w = 1$, see Table 3, where

$$\rho_k = \sum_{j \geq 0} \binom{2k - j}{j} \gamma_{k-j},$$

the $r$-Schröder numbers, are marked. See [19], [21], and [22]. Note that

$$\rho_k = 2\sigma_k \quad (k > 0);$$

see Example 6E. $x\rho^2(x) + (wx - 1)\rho(x) + 1 = 0$ by (1.9).

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 4 | 1 | 6 | 16 | 22← | 0 |
| 3 | 1 | 4 | 6← | 0 | −22 |
| 2 | 1 | 2← | 0 | −6 | −16 |
| 1 | 1← | 0 | −2 | −4 | −6 |
| 0 | 1 | −2 | 0 | −2 | 0 |
| −1 | 1 | −4 | 6 | −8 | 10 |

**TABLE 3**
$R(k, n, 1)$ for $w = 1$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 4 | 1 | 3 | 7 | 11← | 0 |
| 3 | 1 | 2 | 3← | 0 | −22 |
| 2 | 1 | 1← | 0 | −6 | −28 |
| 1 | 1← | 0 | −2 | −8 | −24 |
| 0 | 1 | −1 | −3 | −7 | −15 |
| −1 | 1 | −2 | −3 | −4 | −5 |

**TABLE 4**
$S(k, n, 1)$ for $w = 1$

*General Example 6B:* Given any sequence $w_1, w_2, w_3, \ldots$, consider the step set $\{(0, 1)\} \cup \{(j, 0) : j > 0\}$ where $(0, 1)$ has weight 1 and $(j, 0)$ has weight $w_j$. If $A(k, n, \mu)$ is the sum of the weights of the $(k, n, \mu)$-paths (the initial step must be vertical), we have

$$A(k, n, \mu) = A(k, n - 1, \mu) + \sum_{j > 0} w_j A(k - j, n, \mu).$$

It follows inductively that this $A(k, n, \mu)$ satisfies (1.1) for $\{a_k\}_{k \geq 0}$ defined by $a_0 = 1$, $a_1 = w_1$, $a_2 = w_2 + w_1 w_1$, $a_3 = w_3 + w_2 w_1 + w_1 w_2 + w_1 w_1 w_1$, and in general

$$a_k = \sum_{i_1 + i_2 + \cdots + i_m = k} \prod_{1 \leq t \leq m} w_{i_t}.$$

Hence,

$$a(x) = \sum_{k \geq 0} a_k x^k = 1 + \sum_{k \geq 1} \Bigl( \sum_{i \geq 1} w_i x^i \Bigr)^k = \Bigl(1 - \sum_{i \geq 1} w_i x^i \Bigr)^{-1}. \quad \text{See [26].}$$

*Example 6C:* In Example 6B put $w_1 = w_2 = 1$ and $w_i = 0$ for $i > 2$.

$$a(x) = (1 - x - x^2)^{-1}.$$

Thus, $a_k = 1, 1, 2, 3, 5, \ldots$, the Fibonacci numbers. See 7B.

*Example 6D:* In Example 6B put $w_i = 1$ for $i = \nu$ and $= 0$ otherwise.

$$\hat{c}(x) = (1 + x^\nu)^{-1} = c(x^\nu) \text{ of } (2.2).$$

$$\hat{\gamma}_k = \hat{C}(k, k + 1, 1) = C(K, k + 1, \nu) = \frac{1}{\nu K + 1} \binom{(\nu + 1)K}{K}$$

for $k = \nu K$ and $= 0$ otherwise.

*Example 6E:* In Example 6B put $w_i = w$ for $i > 0$:

$$s(x) = 1 + \sum_{k > 0} (w + 1)^{k-1} x^k = (1 - wx)(1 - (w + 1)x)^{-1}.$$

$$S(k,\ n,\ 0) = \sum_{j \ge 0} (-1)^j w^j (w + 1)^{k-j} \binom{n}{j} \binom{n + k - j - 1}{k - j}.$$

When $w = 1$, see Table 4 for $S(k,\ n,\ 1)$ where $\sigma_k = S(k,\ k + 1,\ 1)$, the $s$-Schröder numbers [23], are marked; see Example 7B.

From $(w + 1)x\sigma^2(x) - (1 + wx)\sigma(x) + 1 = 0$ [by (1.9)] and the last identity of Example 6A, one can show:

(i) $(w + 1)(\sigma(x) - 1) = \rho(x) - 1$,

(ii) $(1 + wx\rho(x)) = (1 - wx\sigma(x))^{-1}$, and

(iii) $\sigma(x) = (1 - x\rho(x))^{-1}$.

These are illustrated in Tables 3 and 4: (i) relates the principal diagonals. (ii) and (iii) relate the partial row sums in the triangle above the zeros in one array to the principal diagonal in the other, as generalized in the following:

*Proposition 9:* If $t_n = \sum_{k=0}^{n-1} w^{n-k} A(k,\ n,\ 1) = \sum_{j \ge 0} w^j A(n - j,\ n,\ 1)$, a weighted partial row sum, then $t(x) = \sum_{n \ge 0} t_n x^n = (1 - wx\alpha(x))^{-1}$.

*Proof:* $$t(x) = \sum_{n \ge 0} \sum_{j \ge 0} w^j A(n - j,\ n,\ 1)x^n = \sum_{j \ge 0} w^j x^j \sum_{n \ge 0} A(n - j,\ n - j + j,\ 1)^{n-j}$$
$$= \sum_{j \ge 0} w^j x^j \alpha^j (x).$$

This extends a result in [20].

## 7. Enumerating Plane Trees

Informally, a rooted plane tree is an unlabeled tree which is oriented in the plane so that it branches upward from a root (a distinguished vertex which need not be univalent) to the leaves. Two plane trees are equal if one can be continuously transformed into the other in the plane so that the nonroot vertices remain above the level of the root. A more formal definition is given by Klarner [14] and [16]. A planted plane tree is a plane tree with univalent root. See Figure 3.

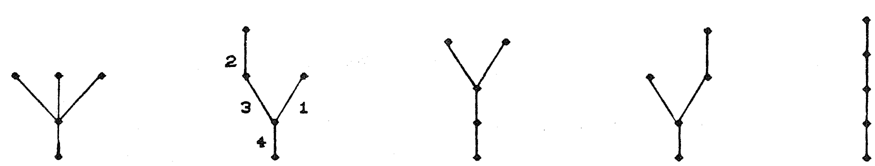


**FIGURE 3**

This illustrates the planted plane trees with 4 edges and no degree restriction. These trees are listed as they correspond to the paths of Figure 1 under Bijection A. The numbers indicate the order of growth.

Here we enumerate rooted plane trees with vertex degree restrictions by establishing bijections between the trees and previously counted lattice paths. Equivalently, one can establish directly a recurrence for the tree counts in the form of (1.1). The following examples are enumerated by other methods in [4], [5], [7], [12], [14], [15], [16], and [22]. One common method for planted plane trees is to establish a functional equation for a generating function and then solve the equation perhaps, and not surprisingly, by the Lagrange inversion formula as in [12].

*Bijection A: Counting rooted plane trees with respect to the number of edges*

Let $T(n, \rho)$ be the set of such trees with $\underline{n\ edges}$ and root degree $\rho$. $\underline{\textit{Let } D}$ be a specified set of permitted degrees for the nonroot vertices. Let $P(n - \rho, n; D)$ be the set of $(n - \rho, n, 1)$-paths with step set

$$\underline{\{(j,\ 1) : j + 1 \in D\}}.$$

A bijection from $P(n - \rho, n; D)$ to $T(n, \rho)$ is defined inductively. The trivial zero-length path in $P(0, 0; D)$ corresponds to the tree consisting of just a root. A path in $P(n - \rho, n; D)$ and the corresponding tree in $T(n, \rho)$ can be extended in two ways. (i) The path can be extended to a path in $P(n - \rho, n + 1; D)$ by attaching a new step $(0, 1)$, while the corresponding tree is extended to a tree in $T(n + 1, \rho + 1)$ by grafting a new left-most edge to the root. Such a new step corresponds to a new leaf. (ii) For $\underline{j + 1 \in D \text{ and } j \le \rho}$, the path can be extended to a path in $\underline{P(n - \rho + j, n + 1; D)}$ by attaching the $\underline{\textit{step}}$ $\underline{(j,\ 1)}$ while the corresponding tree is extended to a tree in $\underline{T(n + 1, \rho - j + 1)}$ by cutting at the root *the* $j$ left-most incident edges and then grafting the lower vertices of these edges to the upper vertex of a new left-most edge incident to the root. Thus, a $\underline{(j,\ 1)\ \textit{step}}$ corresponds to a new vertex of $\underline{\textit{degree } j + 1}$. Hence,

$$|T(n, \rho)| = A(n - \rho, n, 1) = \frac{\rho}{n}[x^{n-\rho}]\Big(\sum_{j+1\in D} x^j\Big)^n.$$

*Bijection B: Counting rooted plane trees with respect to the number of leaves*

Modify the scheme of Bijection A by replacing the underlined phrases sequentially by: $n$ leaves; let $D$ $(2 \notin D)$; $\{(0, 1)\} \cup \{(j, 0) : j + 2 \in D\}$; $j + 2 \in D$ and $j \le \rho - 1$; $P(n - \rho + j, n; D)$; step $(j, 0)$; $T(n, \rho - j)$; the $j + 1$; $(j, 0)$ step; degree $j + 2$. Thus, by Example 6B with $w_j = 1$ if $j + 2 \in D$ and $= 0$ otherwise,

$$|T(n, \rho)| = A(n - \rho, n, 1) = \frac{\rho}{n}[x^{n-\rho}]\Big(1 - \sum_{j+2\in D} x^j\Big)^{-n}.$$

*Example 7A: Applications of Bijection A*

If $D = \mathbb{N} - \{0\}$, no degree restriction, $|T(n, 1)| = C(n - 1, n, 1) = \gamma_{n-1}$; see Example 2B and Figure 3. For $D = \{1, 3\}$, trivalent planted trees, $|T(n, 1)| = \check{C}(n - 1, n, 1) = \check{\gamma}_{n-1}$ of 2D. For $D = \{1, \nu + 1\}$, use 2C. For $D = \{1, 2, 3\}$, no vertex has degree greater than 3, $|T(n, 1)| = \tilde{M}_1(n - 1, n, 1) = \tilde{\mu}_{n-1}$; see 2F.

If $D = \mathbb{N} - \{0, 2\}$, no bivalent nonroot vertices, let $d(x) = (1 - x)^{-1} - x$.

$$|T(n, \rho)| = \frac{\rho}{n}[x^{n-\rho}]d^n(x) = \frac{\rho}{n}\sum_{i\ge 0}(-1)^{n-i}\binom{n}{i}\binom{2i - \rho - 1}{i - \rho}.$$

$$|T(n, 1)| = \sum_{i\ge 0}(-1)^{n-i}\binom{n - 1}{i - 1}\delta_{i-1}.$$

$(x + 1)\delta(x) = (1 - x\delta(x))^{-1}$ by (1.9). One can show

$$(x + 1)(\delta(x) - 1) = x(\tilde{\mu}(x) - 1) \quad (\text{thus, } \delta_{n-1} + \delta_n = \tilde{\mu}_{n-1},\ n > 0).$$

Therefore, by Proposition 9,

$$\sum_{k=0}^{n-1} D(k, n, 1) = \tilde{\mu}_{n-1}.$$

Hence

$$\sum_{\rho \geq 1} |T(n, \rho)| = \tilde{\mu}_{n-1},$$

as in [4].

*Example 7B: Applications of Bijection B*

An immediate source for such trees is the problem [3] of counting the ways to bracket $n$ nonassociative, noncommutative factors so that the number of factors associated by a pair of brackets is restricted to some set $B$. If $B = \{2\}$, we have the problem of Catalan [2], 1838. There is a simple bijection between the usual pairwise bracketings on $n$ factors and the planted plane trees with $D = \{1, 3\}$ and $n$ leaves. For $D = \{1, 3\}$,

$$|T(n, \rho)| = \frac{\rho}{n}[x^{n-\rho}](1 - x)^{-n}.$$

$|T(n, 1)| = \gamma_{n-1}$, the appropriately named sequence of Example 2B.

If $B = \mathbb{N} - \{0, 1\}$, we have the problem of Schröder [23], 1870. If $n = 4$, the bracketings are

$$(a(b(cd))),\ (a((bc)d)),\ (a(bcd)),\ (((ab)c)d),\ ((a(bc))d),$$

$$((abc)d),\ ((ab)(cd)),\ (a(bc)d),\ (ab(cd)),\ ((ab)cd),\ (abcd).$$

There is a simple bijection between the unrestricted bracketings on $n$ factors and the planted trees with $n$ leaves and no bivalent vertices. For $D = \mathbb{N} - \{0, 2\}$, refer to Example 6E with $w = 1$:

$$|T(n, \rho)| = S(n - \rho, n, 1) \quad \text{and} \quad |T(n, 1)| = \sigma_{n-1}.$$

If $D = \{1, \nu + 2\}$,

$$|T(n, 1)| = \hat{C}(n - 1, n, 1) = \hat{\gamma}_{n-1};$$

refer to Example 6D. If $D = \{1, 3, 4\}$, refer to Example 6C.

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