THE GENERAL SOLUTION TO THE DECIMAL FRACTION OF FIBONACCI SERIES

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1. INTRODUCTION

After Stancliff [1] noted that 1/89 can be represented as the sum of Fibonacci Series, Long [2] and Hudson & Winans [1] also perceived that there are some other numbers which can be represented as the sum of Fibonacci Series or Lucas Series. Hudson & Winans [1] gave the solution of the series

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}.$$

Long [2] gave some particular solutions for the series

$$\sum_{i=0}^{\infty} (\pm 10)^{-k(i+1)} F_i$$

and for

$$\sum_{i=0}^{\infty} (\pm 10)^{-k(i+1)} L_i.$$

In this paper, a method similar to that employed by Hudson & Winans is used to obtain the general solution for all such series.

2. THE SERIES $\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}$

According to Hoggatt [4], the n^{th} Fibonacci number and the n^{th} Lucas number can be represented, respectively, by

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{1}$$

and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2}$$

Note that we have

$$L_n + \sqrt{5}F_n = 2^{1-n}(1+\sqrt{5})^n, \tag{3}$$

and

$$L_n - \sqrt{5}F_n = 2^{1-n}(1-\sqrt{5})^n. \tag{4}$$

Using these, we obtain:

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i} = \sum_{i=1}^{\infty} 10^{-k(i+1)} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{\alpha i} - \left(\frac{1-\sqrt{5}}{2}\right)^{\alpha i}\right]$$

(continued)

$$= \sum_{i=1}^{\infty} 10^{-k(i+1)}\frac{1}{\sqrt{5}}\left[\left(\frac{L_\alpha + \sqrt{5}F_\alpha}{2}\right)^i - \left(\frac{L_\alpha - \sqrt{5}F_\alpha}{2}\right)^i\right]$$

$$= \sum_{i=1}^{\infty} \frac{1}{10^k\sqrt{5}}\left[\left(\frac{L_\alpha + \sqrt{5}F_\alpha}{2 \cdot 10^k}\right)^i - \left(\frac{L_\alpha - \sqrt{5}F_\alpha}{2 \cdot 10^k}\right)^i\right].$$

Since $r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^n$, $\sum_{n=1}^{\infty} r^n$ converges to $\frac{r}{1-r}$ iff $|r| < 1$.

Consequently, for values of α and k for which the series converges, we have

$$\sum_{i=1}^{\infty} 10^{-k(i+1)}F_{\alpha i} = \frac{4 \cdot 10^k \cdot F_\alpha\sqrt{5}}{10^k\sqrt{5}[4 \cdot 10^{2k} - 4 \cdot 10^k \cdot L_\alpha + L_\alpha^2 - 5F_\alpha^2]}$$

$$= \frac{4F_\alpha}{4[10^{2k} - 10^k \cdot L_\alpha + (-1)^\alpha]}$$

$$= \frac{F_\alpha}{10^{2k} - 10^k \cdot L_\alpha + (-1)^\alpha} \tag{5}$$

Equation (5) agrees with (1.1) and (1.2) of [1] obtained by Hudson, noting that $(\alpha + 1)/2$ in (1.2) of [1] is a misprint and should read $(\alpha - 2)/2$.

Using the same method, we have

$$\sum_{i=0}^{\infty} 10^{-k(i+1)}L_{\alpha i} = \frac{2 \cdot 10^k - L_\alpha}{10^{2k} - 10^k \cdot L_\alpha + (-1)^\alpha} \tag{6}$$

$$\sum_{i=1}^{\infty} (-10^{-k})^{i+1}F_{\alpha i} = \frac{F_\alpha}{10^{2k} + 10^k \cdot L_\alpha + (-1)^\alpha} \tag{7}$$

$$\sum_{i=0}^{\infty} (-10^{-k})^{i+1}L_{\alpha i} = -\frac{2 \cdot 10^k + L_\alpha}{10^{2k} + 10^k \cdot L_\alpha + (-1)^\alpha} \tag{8}$$

(k, α, i, n are integers).

We note that:

$$F_\alpha = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^\alpha - \left(\frac{1-\sqrt{5}}{2}\right)^\alpha\right],$$

$$F_\alpha^2 = \frac{1}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2\alpha} + \left(\frac{1-\sqrt{5}}{2}\right)^{2\alpha} - 2\left(\frac{1+\sqrt{5}}{2}\right)^\alpha\left(\frac{1-\sqrt{5}}{2}\right)^\alpha\right].$$

$$\therefore\ 5F_\alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^{2\alpha} + \left(\frac{1-\sqrt{5}}{2}\right)^{2\alpha} + 2\left(\frac{1+\sqrt{5}}{2}\right)^\alpha\left(\frac{1-\sqrt{5}}{2}\right)^\alpha - 4\left(\frac{1+\sqrt{5}}{2}\right)^\alpha\left(\frac{1-\sqrt{5}}{2}\right)^\alpha$$

$$= L_\alpha^2 - 4(-1)^\alpha.$$

$$\therefore\ L_\alpha^2 - 5F_\alpha^2 = 4(-1)^\alpha.$$

3. SOME PARTICULAR VALUES FOR THE ABOVE SERIES

TABLE 1. Some Values of $\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{89}$	$\frac{1}{71}$	$\frac{2}{59}$	$\frac{3}{31}$						
2	$\frac{1}{9899}$	$\frac{1}{9701}$	$\frac{2}{9599}$	$\frac{3}{9301}$	$\frac{5}{8899}$	$\frac{8}{8201}$	$\frac{13}{7099}$	$\frac{21}{5301}$	$\frac{34}{2399}$	
3	$\frac{1}{998999}$	$\frac{1}{997001}$	$\frac{2}{995999}$	$\frac{3}{993001}$	$\frac{5}{988999}$	$\frac{8}{982001}$	$\frac{13}{970999}$	$\frac{21}{953001}$	$\frac{34}{923999}$	$\frac{55}{877001}$
	$\frac{89}{800999}$	$\frac{144}{678001}$	$\frac{233}{478999}$	$\frac{377}{157001}$						

TABLE 2. Some Values of $\sum_{i=1}^{\infty} 10^{-k(i+1)} L_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$\frac{19}{89}$	$\frac{17}{71}$	$\frac{16}{59}$	$\frac{13}{31}$						
2	$\frac{199}{9899}$	$\frac{197}{9701}$	$\frac{196}{9599}$	$\frac{193}{9301}$	$\frac{189}{8899}$	$\frac{182}{8201}$	$\frac{171}{7099}$	$\frac{153}{5301}$	$\frac{124}{2399}$	
3	$\frac{1999}{998999}$	$\frac{1997}{997001}$	$\frac{1996}{995999}$	$\frac{1993}{993001}$	$\frac{1989}{988999}$	$\frac{1982}{982001}$	$\frac{1971}{970999}$	$\frac{1953}{953001}$	$\frac{1924}{923999}$	$\frac{1877}{877001}$
	$\frac{1801}{800999}$	$\frac{1678}{678001}$	$\frac{1479}{478999}$	$\frac{1157}{157001}$						

TABLE 3. Some Values of $\sum_{i=1}^{\infty} (-10)^{-k(i+1)} F_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{109}$	$\frac{1}{131}$	$\frac{2}{139}$	$\frac{3}{171}$						
2	$\frac{1}{10099}$	$\frac{1}{10301}$	$\frac{2}{10399}$	$\frac{3}{10701}$	$\frac{5}{11099}$	$\frac{8}{11801}$	$\frac{13}{12899}$	$\frac{21}{14701}$	$\frac{34}{17599}$	
3	$\frac{1}{1000999}$	$\frac{1}{1003001}$	$\frac{2}{1003999}$	$\frac{3}{1007001}$	$\frac{5}{1010999}$	$\frac{8}{1018001}$	$\frac{13}{1028999}$	$\frac{21}{1047001}$	$\frac{34}{1075999}$	$\frac{55}{1123001}$
	$\frac{89}{1198999}$	$\frac{144}{1322001}$	$\frac{233}{1520999}$	$\frac{377}{1843001}$						

TABLE 4. Some Values of $\sum_{i=0}^{\infty} (-10)^{-k(i+1)} L_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8	9	10
1	$-\frac{21}{109}$	$-\frac{23}{131}$	$-\frac{24}{139}$	$-\frac{27}{171}$						
2	$-\frac{201}{10099}$	$-\frac{203}{10301}$	$-\frac{204}{10309}$	$-\frac{207}{10701}$	$-\frac{211}{11099}$	$-\frac{218}{11801}$	$-\frac{229}{12899}$	$-\frac{247}{14701}$	$-\frac{276}{17599}$	
3	$-\frac{2001}{1000999}$	$-\frac{2003}{1003001}$	$-\frac{2004}{1003999}$	$-\frac{2007}{1007001}$	$-\frac{2011}{1010999}$	$-\frac{2018}{1018001}$	$-\frac{2029}{1028999}$	$-\frac{2047}{1047001}$	$-\frac{2076}{1075999}$	$-\frac{2123}{1123001}$
	$-\frac{2199}{1198999}$	$-\frac{2322}{1322001}$	$-\frac{2521}{1520999}$	$-\frac{2843}{1843001}$						

4. EXTENSION TO GENERALIZED FIBONACCI NUMBERS

A general Fibonacci number can be represented as

$$T_n = aT_{n-1} + bT_{n-2} \text{ with } T_0 = c,\ T_1 = d. \tag{9}$$

Long [2] has given the form of the general Fibonacci number as

$$T_n = \left(\frac{c}{2} + \frac{2d - ca}{2\sqrt{a^2 + 4b}}\right)\left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n + \left(\frac{c}{2} - \frac{2d - ca}{2\sqrt{a^2 + 4b}}\right)\left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n \tag{10}$$

Here, if $c = 0$, $a = b = d = 1$, then T_n can be reduced to F_n, and if $c = 2$, $a = b = d = 1$, then T_n can be reduced to F_n.

Using the above method, we obtain

$$S_n \pm \sqrt{a^2 + 4b}R_n = 2^{1-n}(a \pm \sqrt{a^2 + 4b})^n \tag{11}$$

where

$$S_n = \left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n + \left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n \tag{12}$$

$$R_n = \frac{1}{\sqrt{a^2 + 4b}}\left[\left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n - \left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n\right] \tag{13}$$

Then we can get

$$\sum_{i=0}^{\infty} 10^{-k(i+1)} T_{\alpha i} = \frac{\frac{c}{2}(2 \cdot 10^k - S_\alpha) + \frac{2d - ca}{2}R_\alpha}{10^{2k} - 10^k \cdot S_\alpha + (-b)^\alpha} \tag{14}$$

$$\sum_{i=0}^{\infty} (-10)^{-k(i+1)} T_{\alpha i} = \frac{-\frac{c}{2}(2 \cdot 10^k + S_\alpha) + \frac{2d - ca}{2}R_\alpha}{10^{2k} + 10^k \cdot S_\alpha + (-b)^\alpha} \tag{15}$$

for values of α and k with

$$\frac{(a + \sqrt{a^2 + 4b})^\alpha}{2 \cdot 10^k} < 1,$$

or, equivalently, with

$$\frac{S_\alpha + \sqrt{a^2 + 4b}R_\alpha}{2 \cdot 10^k} < 1 \tag{16}$$

As an example, if $a = 1$, $b = 3$, $c = 2$, $d = 5$, some values of S_α, R_α, T_α, $\sum_{i=0}^{\infty} (10)^{-k(i+1)} T_{\alpha i}$, and $\frac{S_\alpha + R_\alpha\sqrt{13}}{2 \cdot 10^k}$ are shown in Tables 5, 6, and 7 for different α and k.

TABLE 5. Some Values of S_α, R_α, T_α

Series \ α	1	2	3	4	5	6	7	8	9	10
S_α	1	7	10	31	61	154	337	799	1810	4207
R_α	1	1	4	7	19	40	97	217	508	1159
T_α	5	11	26	59	137	314	725	1667	3842	8843

TABLE 6. Some Values of $\sum_{i=0}^{\infty} 10^{-k(i+1)} T_{\alpha i}$

k \ α	1	2	3	4	5	6	7	8
1	$\frac{23}{87}$	$\frac{17}{39}$						
2	$\frac{203}{9897}$	$\frac{197}{9309}$	$\frac{206}{8973}$	$\frac{197}{6981}$	$\frac{215}{3657}$			
3	$\frac{2003}{998997}$	$\frac{1997}{993009}$	$\frac{1916}{989973}$	$\frac{1997}{969081}$	$\frac{2015}{938757}$	$\frac{2006}{846729}$	$\frac{2051}{660813}$	$\frac{2069}{207561}$

TABLE 7. Some Values of $\frac{S_\alpha + R_\alpha\sqrt{13}}{2 \cdot 10^k}$

k \ α	1	2	3	4	5	6	7	8	9
1	0.2303	0.5303	1.2211						
2	0.0230	0.0530	0.1221	0.2812	0.6475	1.4911			
3	0.0023	0.0053	0.0122	0.0281	0.0648	0.1491	0.3434	0.7907	1.8208

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