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UNITARY HARMONIC NUMBERS

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1. INTRODUCTION

Ore [2] investigated the harmonic mean $H(n)$ of the divisors of n, and showed that

$$H(n) = n\tau(n)/\sigma(n),$$

where, as usual, $\tau(n)$ and $\sigma(n)$ denote, respectively, the number and sum of the divisors of n. An integer n is said to be *harmonic* if $H(n)$ is an integer. For example, 6 and 140 are harmonic, since

$$H(6) = 2 \quad \text{and} \quad H(140) = 5.$$

Ore proved that any perfect number (even or odd) is harmonic, and that no prime power is harmonic. Pomerance [3] proved that any harmonic number of the form $p^a q^b$, with p and q prime, must be an even perfect number. Ore also conjectured that there is no odd $n > 1$ which is harmonic, and Garcia [1] verified Ore's conjecture for $n < 10^7$; however, since Ore's conjecture implies that there are no odd perfect numbers, any proof must be quite deep.

A divisor d of an integer n is a *unitary divisor* if g.c.d. $(d, n/d) = 1$, in which case we write $d \| n$. Let $\tau^*(n)$ and $\sigma^*(n)$ be, respectively, the number and sum of the unitary divisors of n. If n has $\omega(n)$ distinct prime factors, it is easy to show that

$$\tau^*(n) = 2^{\omega(n)} \quad \text{and} \quad \sigma^*(n) = \prod_{p^e \| n} (1 + p^e),$$

both functions being multiplicative.

Let $H^*(n)$ be the harmonic mean of the unitary divisors of n. It follows that

$$H^*(n) = n\tau^*(n)/\sigma^*(n) = \prod_{p^e \| n} \frac{2p^e}{1 + p^e}$$

We say that n is *unitary harmonic* if $H^*(n)$ is an integer.

In this paper we outline the proofs of two results:

Theorem 1

There are 23 unitary harmonic numbers n with $\omega(n) \leqslant 4$ (see Table 1).

Theorem 2

There are 43 unitary harmonic numbers $n \leqslant 10^6$. These numbers, which include all but one of those in Theorem 1, are given in Table 2.

TABLE 1

$\omega(n)$	$H^*(n)$	n	$\omega(n)$	$H^*(n)$	n
0	1	1	4	10	$9{,}100 = 2^2 5^2 7 \cdot 13$
2	2	$6 = 2 \cdot 3$	4	10	$31{,}500 = 2^2 3^2 5^3 7$
2	3	$45 = 3^2 5$	4	10	$330{,}750 = 2 \cdot 3^3 5^3 7$
3	4	$60 = 2^2 3 \cdot 5$	4	11	$16{,}632 = 2^3 3^3 7 \cdot 11$
3	4	$90 = 2 \cdot 3^2 5$	4	12	$51{,}408 = 2^4 3^3 7 \cdot 17$
3	7	$15{,}925 = 5^2 7^2 13$	4	12	$66{,}528 = 2^5 3^3 7 \cdot 11$
3	7	$55{,}125 = 3^2 5^3 7^2$	4	12	$185{,}976 = 2^3 3^4 7 \cdot 41$
4	7	$420 = 2^2 3 \cdot 5 \cdot 7$	4	12	$661{,}500 = 2^2 3^3 5^3 7^2$
4	7	$630 = 2 \cdot 3^2 5 \cdot 7$	4	13	$646{,}425 = 3^2 5^2 13^2 17$
4	9	$3{,}780 = 2^3 3^3 5 \cdot 7$	4	13	$716{,}625 = 3^2 5^3 7^2 13$
4	9	$46{,}494 = 2 \cdot 3^4 7 \cdot 41$	4	15	$20{,}341{,}125 = 3^4 5^3 7^2 41$
4	10	$7{,}560 = 2^3 3^3 5 \cdot 7$			

TABLE 2

$H^*(n)$	n	$H^*(n)$	n
1	1	9	$3{,}780 = 2^2 3^3 5 \cdot 7$
2	$6 = 2 \cdot 3$	13	$5{,}460 = 2^2 3 \cdot 5 \cdot 7 \cdot 13$
2	$45 = 3^2 5$	10	$7{,}560 = 2^3 3^3 5 \cdot 7$
3	$60 = 2^2 3 \cdot 5$	13	$8{,}190 = 2 \cdot 3^2 5 \cdot 7 \cdot 13$
3	$90 = 2 \cdot 3^2 5$	10	$9{,}100 = 2^2 5^2 7 \cdot 13$
7	$420 = 2^2 3 \cdot 5 \cdot 7$	7	$15{,}925 = 5^2 7^2 13$
7	$630 = 2 \cdot 3^2 5 \cdot 7$		

(continued)

TABLE 2 (continued)

$H^*(n)$	n	$H^*(n)$	n
11	$16{,}632 = 2^3 3^3 7 \cdot 11$	12	$185{,}976 = 2^3 3^4 7 \cdot 41$
15	$27{,}300 = 2^2 3 \cdot 5^2 7 \cdot 13$	15	$232{,}470 = 2 \cdot 3^4 5 \cdot 7 \cdot 41$
10	$31{,}500 = 2^2 3^2 5^3 7$	20	$257{,}040 = 2^4 3^3 5 \cdot 7 \cdot 17$
15	$40{,}950 = 2 \cdot 3^2 5^2 7 \cdot 13$	10	$330{,}750 = 2 \cdot 3^3 5^3 7^2$
9	$46{,}494 = 2 \cdot 3^4 7 \cdot 41$	20	$332{,}640 = 2^5 3^3 5 \cdot 7 \cdot 11$
12	$51{,}408 = 2^4 3^3 7 \cdot 17$	18	$464{,}940 = 2^2 3^4 4 \cdot 7 \cdot 41$
7	$55{,}125 = 3^2 5^3 7^2$	22	$565{,}448 = 2^4 3^3 7 \cdot 11 \cdot 17$
17	$64{,}260 = 2^2 3^3 5 \cdot 7 \cdot 17$	19	$598{,}500 = 2^2 3^2 5^3 7 \cdot 19$
12	$66{,}528 = 2^2 3^3 7 \cdot 11$	13	$646{,}425 = 3^2 5^2 13^2 17$
18	$81{,}900 = 2^2 3^2 5^2 7 \cdot 13$	12	$661{,}500 = 2^2 3^3 5^3 7^2$
16	$87{,}360 = 2^6 3 \cdot 5 \cdot 7 \cdot 13$	13	$716{,}625 = 3^2 5^3 7^2 13$
14	$95{,}550 = 2 \cdot 3 \cdot 5^2 7^2 13$	17	$790{,}398 = 2 \cdot 3^4 7 \cdot 17 \cdot 41$
19	$143{,}640 = 2^3 3^3 5 \cdot 7 \cdot 19$	18	$859{,}950 = 2 \cdot 3^3 5^2 7^2 13$
20	$163{,}800 = 2^3 3^2 5^2 7 \cdot 13$	33	$900{,}900 = 2^2 3^2 5^2 7 \cdot 11 \cdot 13$
19	$172{,}900 = 2^2 5^2 7 \cdot 13 \cdot 19$	20	$929{,}880 = 2^2 3^4 5 \cdot 7 \cdot 41$

The complete proofs of Theorems 1 and 2 are quite tedious, requiring many cases and subcases. However, the techniques are quite simple, and are adequately illustrated by the cases discussed here.

2. TECHNIQUES FOR THEOREM 1

If p and q are (not necessarily distinct) primes and $p^a < q^b$, then it is easy to show that $H^*(p^a) > H^*(q^b)$. This fact can be used, once $\omega(n)$ and $H^*(n)$ are specified, to find an upper bound for the smallest prime power unitary divisor of n; for each choice, the process is repeated to find choices for the next smallest prime power unitary divisor, and the process continues until all but one of the prime power unitary divisors is found; the largest prime power can then be solved for directly, without a search. Of course, this procedure is interrupted any time it becomes obvious that the as yet unknown portion of n must have more prime divisors than allowed by the prespecified size of $\omega(n)$.

With $\omega(n)$ and $H^*(n)$ given, the problem is to find n with

$$n/\sigma^*(n) = H^*(n)/\tau^*(n)$$

being a prespecified fraction, which in turn requires that any odd prime that divides $\sigma^*(n)$ must also divide n. Also, since $\tau^*(n)$ is a power of 2, any odd prime that divides $H^*(n)$ must also divide n. Several of the cases are shortened by using results of Subbarao and Warren [4] for the special case $\sigma^*(n) = 2n$ (i.e., for n being unitary perfect).

We present here the proof for the case $\omega(n) = 4$, $H^*(n) = 15$, one of the longer and subtler cases of Theorem 1. Throughout, let $n = pqrs$ with $p < q < r < s$ and p, q, r, and s powers of distinct primes (though not necessarily prime). Note that because $n/\sigma^*(n) = 15/16$, $3 \cdot 5 | pqrs$. Also, if n has a prime power unitary divisor which is congruent to 3 (mod 4), then n must be even.

If $p \geqslant 59$, then $n/\sigma^*(n) > 15/16$, so $p \leqslant 53$.

$p = 53$: $q < 61$, so $q = 59$, which requires that $2 \cdot 3 \cdot 5 | rs$, a contradiction.

$p = 49$: $q < 64$. But $q = 61$ implies $3 \cdot 5 \cdot 31 | rs$, and $q = 59$ requires $2 \cdot 3 \cdot 5 | rs$; both of these are impossible. If $q = 53$, then $r < 79$, but there are no powers of 3 or 5 between 53 and 79.

$p = 47$: $q < 67$ and $2 \cdot 3 \cdot 5 | qrs$, so $q = 64$, from which follows the impossibility $3 \cdot 5 \cdot 13 | rs$.

$p = 43$: $2 \cdot 3 \cdot 5 \cdot 11 | qrs$, a contradiction.

$p = 41$: $q < 71$ and $3 \cdot 5 \cdot 7 | qrs$. The only possibility is $q = 49$, which requires $r < 103$ and $3 \cdot 5 | rs$. This in turn forces $r = 81$, which implies $s = 125$. Thus we have a unitary harmonic number, since

$$H^*(3^4 5^3 7^2 41) = 15.$$

$p = 37$: $q < 79$ and $3 \cdot 5 \cdot 19 | qrs$, a contradiction.

$p = 32$: $q < 83$ and $3 \cdot 5 \cdot 11 | qrs$, so $q = 81$. But then $5 \cdot 11 \cdot 41 | rs$, a contradiction.

$p = 31$: $q < 89$ and $2 \cdot 3 \cdot 5 | qrs$. There are three unpalatable choices: $q = 81$ requires that $2 \cdot 5 \cdot 41 | rs$, and $q = 64$ implies $3 \cdot 5 \cdot 13 | rs$, while $q = 32$ forces $3 \cdot 5 \cdot 11 | rs$.

$p = 29$: $31 < q < 97$, and rs is divisible by at least three primes unless q is 89, 81, 59, or 49. If $q = 89$, then $r < 103$ and there are no powers of 3 or 5 between 89 and 103. If $q = 81$, then $r < 109$ and $5 \cdot 41 | rs$, a contradiction. If $q = 59$, then $r < 167$ and the only possible cases are $r = 125$, which implies $3 \cdot 7 | s$, and $r = 81$, which forces $5 \cdot 41 | s$. If $q = 49$, then $r < 193$, so either $r = 125$, which does not leave the required 5 in the numerator of $n/\sigma^*(n)$, or $r = 81$, which forces $5 \cdot 41 | s$. Thus $p = 29$ is impossible.

$p = 27$: $35 < q < 107$ and $2 \cdot 5 \cdot 7 | qrs$, so the only possible values for q are 64 and 49. If $q = 64$, then $5 \cdot 7 \cdot 13 | rs$. If $q = 49$, then $125 < r < 251$ and $2 \cdot 5 | rs$, so $r = 138$, whence $5 \cdot 43 | s$, a contradiction.

$p = 25$: $39 < q < 121$ and rs is divisible by three or more primes except when q is 107, 103, 89, 81, 64, or 53. If $q = 107$, then $r < 125$, while $q = 103$ implies $r < 128$, and $q = 89$ forces $r < 149$; in each case, $3 \cdot 13 | rs$, a contradiction. If $q = 81$, then $r < 157$ and $13 \cdot 41 | rs$, which is impossible. If $q = 64$, then $r < 211$ and $3 \cdot 13 | rs$; thus $r = 169$, which forces $3 \cdot 17 | s$, or $r = 81$, in which case $13 \cdot 41 | s$. If $q = 53$, then $r < 307$ and $3 \cdot 13 | rs$, so r is 243, 169, or 81; each of these possibilities forces s to be divisible by two distinct primes.

$p = 23$: $45 < q < 137$ and $2 \cdot 3 \cdot 5 | qrs$. The possible values for q are 128, 125, 81, and 64, but each of these forces rs to be divisible by three or more primes, a contradiction in any event.

$p = 19$: $75 < q < 227$ and $2 \cdot 3 \cdot 5 | qrs$. Thus, q is 128, 125, or 81. Each of these possibilities is ruled out since rs cannot be divisible by three primes.

$p = 17$: $135 < q < 407$ and $3 \cdot 5 | qrs$. To be within the interval, q cannot be a power of 5, and $q = 243$ forces $r < 611$ and $5 \cdot 61 | rs$, a contradiction. Therefore, q is a prime power between 135 and 407, congruent to 1 (mod 4), and such that $q + 1$ has no odd prime factor other than 3's, 5's, and at most one 17. There are but two possibilities: $q = 269$ and $q = 149$. If $q = 269$, then $r < 544$ and $3 \cdot 5 | rs$, a contradiction. If $q = 149$, then $1446 < r < 283$ and $3 \cdot 5 | rs$, so $r = 2187$, whence $5 \cdot 547 | s$, a contradiction.

$p = 16$: $255 < q < 765$ and $3 \cdot 5 \cdot 17 | qrs$, so q is 729, 625, or 289, each of which would require that rs be divisible by three distinct primes.

Finally, if $q < 16$, then $n/\sigma^*(n) < 15/16$.

3. TECHNIQUES FOR THEOREM 2

Suppose that n is unitary harmonic, i.e., that

$$H^*(n) = n\tau^*(n)/\sigma^*(n)$$

is an integer. Suppose also that $n \leqslant 10^6$ and that $2^a \| n$. Since $\tau^*(n)$ is a power of 2, any odd prime that divides $\sigma^*(n)$ must also divide n. For $a > 0$, $\sigma^*(2^a) = 1 + 2^a$, so $2^a \| n$ implies $2^a(1 + 2^a) | n$, and hence $a < 10$.

Except for $a = 0$, the supposition that $2^a \| n$ requires that n be divisible by the largest prime dividing $1 + 2^a$, and the restriction that $n \leqslant 10^6$ can be used to determine how many times this prime divides n. This gives rise to newly known unitary divisors of n, and therefore (usually) newly known odd primes dividing $\sigma^*(n)$ and hence n. The procedure is repeated until all the possibilities are exhausted.

No particular difficulty arises with this procedure, except when one runs out of primes with which to work, and then the procedure breaks down completely. In such a case we write $n = Nk$, where $N \| n$ and k is unknown. In light of Theorem 1, we may require $\omega(n) > 4$, which imposes a lower bound on $\omega(k)$; and $n \leqslant 10^6$ imposes an upper bound on k and hence on $\omega(k)$. There are also divisibility restrictions on k and $\sigma^*(k)$ from N and $\sigma^*(N)$. See the $2 \cdot 3^2 5 \| n$ and $2 \cdot 3 \cdot 7 \| n$ cases in the discussion below.

The n odd ($a = 0$) case of Theorem 2 is somewhat easier to handle than the others since $p^b \| n$ implies $p^b \equiv 1 \pmod 4$ in order to avoid having too many 2's in the denominator of $H^*(n)$.

We present here the $a = 1$ (i.e., $2n$) case of Theorem 2:

Immediate size contradictions result if $3^{12} | n$ or if $3^b \| n$ for $6 \leqslant b \leqslant 11$. If $3^3 \| n$, then $61 | n$, so either $61^2 | n$ or $61 \| n$, in which case $31 | n$; both possibilities make $n > 10^6$.

If $3^4 \| n$, then $41 | n$. If $41^3 | n$ or $41^2 \| n$, then $n > 10^6$, so $41 \| n$. Then $7 | n$, and $n > 10^6$ if $7^3 | n$ or $7^2 \| n$. If $n = 2 \cdot 3 \cdot 7 \cdot 41k$, then $1 < k \leqslant 21$, $(2 \cdot 3 \cdot 7,\ k) = 1$ and $\sigma^*(k)$ 18, so k is 5 or 17. Thus we have located two unitary harmonic numbers:

$$H^*(2 \cdot 3^4 5 \cdot 7 \cdot 41) = 15,$$

$$H^*(2 \cdot 3^4 7 \cdot 17 \cdot 41) = 17.$$

If $3^3 \| n$, then $7 | n$. Size contradictions easily result if $7^6 | n$ or $7^5 \| n$ or $7^4 \| n$ or $7^3 \| n$. If $7^2 \| n$, then $5^2 | n$, and $n > 10^6$ if $5^4 | n$. If $5^3 \| n$, then $n = 2 \cdot 3^3 5^3 7^2$ since $n < 10^6$, but $\omega(n) = 4$. Therefore, $5^2 \| n$, so $13 | n$ and hence $13 \| n$, and another unitary harmonic number is found:

$$H^*(2 \cdot 3^3 5^2 7^2 13) = 18.$$

If $3^3 7 \| n$, then $n = 2 \cdot 3^3 7k$. It follows that $H^*(n) = 9H^*(k)/2$. But $H^*(k)$ does not have an even numerator after reduction, so $H^*(n)$ is not an integer.

If $3^2 \| n$, then $5 | n$. Size contradictions occur if $5^7 | n$ or $5^6 \| n$ or $5^4 \| n$, while there are too many 3's in the denominator of $H^*(n)$ if $5^5 \| n$ or $5^3 \| n$. Therefore, $5^2 \| n$ or $5 \| n$.

If $3^2 5^2 \| n$, then $13 | n$, and $n > 10^6$ if $13^4 | n$ or $13^3 \| n$ or $13^2 \| n$. Thus, $13 \| n$, so $7 | n$, but $n > 10^6$ if $7^3 | n$, and if $7^2 \| n$ there are too many 5's in the denominator of $H^*(n)$, so $7 \| n$. Therefore,

$$n = 2 \cdot 3^2 5^2 7 \cdot 13 \cdot k,$$

where $k \leqslant 24$, $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 13,\ k) = 1$ and $\sigma^*(k) | 30$. This locates another unitary harmonic number:

$$H^*(2 \cdot 3^2 5^2 7 \cdot 13) = 15.$$

If $2 \cdot 3^2 5 \| n$, then $n = 2 \cdot 3^2 5 \cdot k$ with $(2 \cdot 3 \cdot 5, k) = 1$, $k \leqslant 11{,}111$ and $(\sigma^*(k), 3 \cdot 5) = 1$, so k is composed of prime powers from the set

$$\{7, 13, 31, 37, 43, 61, 67, 73, 97, 103, 121, \ldots\}.$$

Since $\omega(n) \geqslant 5$, $\omega(k) \geqslant 2$. However, $\omega(k) \leqslant 3$ since

$$7 \cdot 13 \cdot 31 \cdot 37 > 11{,}111.$$

If $\omega(k) = 3$, then the smallest prime dividing k is 7, since

$$13 \cdot 31 \cdot 37 > 11{,}111.$$

Also, $37 \nmid k$ or else $19 | k$, which is impossible if $k \leqslant 11{,}111$. Thus, the only possibility with $\omega(k) = 3$ is $k = 7 \cdot 13 \cdot 31$, which forces $H^*(n)$ to be nonintegral. If $\omega(k) = 2$, then write $n = 2 \cdot 3^2 5 \cdot p \cdot q$. Now, $p < 103$, since $103 \cdot 121 > 11{,}111$ and $\sigma^*(q) | 16p$, so the only possibility is $p = 7$ and $q = 13$, and another unitary harmonic number is found:

$$H^*(2 \cdot 3^2 5 \cdot 7 \cdot 13) = 13.$$

If $3 \| n$, then $n = 2 \cdot 3 \cdot k$ with $k \leqslant 166{,}666$, $(2 \cdot 3, k) = 1$, $(\sigma^*(k), 3) = 1$ and $\omega(k) \geqslant 3$. But $\omega(k) \leqslant 4$, since

$$7 \cdot 13 \cdot 19 \cdot 25 \cdot 31 > 166{,}666.$$

If $\omega(k) = 4$, the smallest possible next prime power is 7, since

$$13 \cdot 19 \cdot 25 \cdot 31 > 166{,}666.$$

But if $3 \cdot 7 \| n$, then $H^*(n)$ has at least one excess 2 in its denominator. Therefore, $\omega(k) = 3$, so let $k = pqr$ with $p < q < r$. Now, $p < 49$, since $49 \cdot 61 \cdot 67 > 166{,}666$. We have the following possibilities:

$p = 43$ forces $11 | n$. But $11 \nmid n$, so $n > 2 \cdot 3 \cdot 7^2 11^2 43 > 10^6$.

$p = 37$ implies $19 | n$. But $19 \nmid n$, so $n > 2 \cdot 3 \cdot 19^2 37 \cdot 43 > 10^6$.

$p = 31$ leaves extra 2's in the denominator of $H^*(n)$.

$p = 25$ requires $13 | n$, but $13 \nmid n$. If $13^4 | n$, then $n > 10^6$, and the same is true if $13^3 \| n$, because then $157 | n$. Then $13^2 \| n$, so $17 | n$ and $17 \| n$, so $n > 2 \cdot 3 \cdot 5^2 13^2 17^2 > 10^6$.

$p = 19$ forces $5 | n$, but $5 \nmid n$. But $n > 10^6$ if $5^6 | n$ or $5^4 \| n$, and there are extra 3's in the denominator of $H^*(n)$ if $5^5 \| n$ or $5^3 \| n$. Therefore, $5^2 \| n$, so $13 | n$ and $13 \nmid n$, but $n > 10^6$ if $13^3 | n$, and hence $13^2 \| n$, whence $17 | n$ and $n > 10^6$.

$p = 13$ requires $7|n$, and $7\nparallel n$. If $7^5|n$ or $7^4\|n$ or $7^3\|n$, then $n > 10^6$. Thus, $7^2\|n$, so $5^2|n$. Then $n = 2 \cdot 3 \cdot 5^2 7^2 13 \cdot k$ with $k \leqslant 10$. The only value of k that checks out is $k = 1$:

$$H^*(2 \cdot 3 \cdot 5^2 7^2 13) = 14.$$

$p = 7$ leaves extra 2's in the denominator of $H^*(n)$.

Since $2\|n$, $3|n$ and the $3\nmid n$ subcase is eliminated. Thus, the $2\|n$ case of the theorem is proved.

4. LARGE INTEGRAL VALUES OF $H^*(n)$

It is not at all hard to construct n with $H^*(n)$ a large integer. For example, one may start with the fifth unitary perfect number [5],

$$2^{18}3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

and have $H^*(n) = 2^{11} = 2048$. However, substituting for various blocks of unitary divisors yields the related number

$$n = 2^{18}3^4 5^4 7^4 11^2 13^2 17 \cdot 19^2 31 \cdot 37 \cdot 41 \cdot 43 \cdot 61 \cdot 79 \cdot 109 \cdot 157 \cdot 181 \cdot 313 \cdot 601 \cdot 1201,$$

for which $H^*(n) = 2^{11} 3 \cdot 7 \cdot 19 = 817{,}152$.

The author conjectures that there are infinitely many unitary harmonic numbers, including infinitely many odd ones, but that there are only finitely many unitary harmonic numbers with $\omega(n)$ fixed.

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