CONTINUED FRACTIONS OF FIBONACCI AND LUCAS RATIOS

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The purpose of this article is to lay the groundwork for continued fraction representations of Fibonacci and Lucas ratios. We assume the general theory of such fractions to be known and refer the unfamiliar or rusty reader to the very readable work of C. D. Olds [1]. This paper will deal with ratios in which the Fibonacci and Lucas numbers enter linearly since such results are the simplest and most fundamental, being necessary for more advanced developments.

1. THE RATIO F_n/F_{n-a}

Two cases may be distinguished depending on whether a is odd or even.

<u>Case 1. $a = 2k-1$</u>

$$F_n/F_{n-2k+1} = L_{2k-1} + F_{n-4k+2}/F_{n-2k+1}$$

This devolves from the relation:

$$F_n - F_{n-4k+2} = L_{2k-1}F_{n-2k+1}$$

The next partial quotient results from the reciprocal of the fraction F_{n-4k+2}/F_{n-2k+1} and hence is again L_{2k-1}. Thus for odd a, all the partial quotients are L_{2k-1}, the termination depending on the value of n modulo 2k-1.

Example. F_{54}/F_{47}. There will be six partial quotients $L_7(29)$ after which there will be a remainder F_5/F_{12}. This latter gives partial quotients 28, 1, 4. Thus

$$F_{54}/F_{47} = (29_6, 28, 1, 4),$$

where the subscript 6 adjacent to 29 indicates the number of times 29 appears as a partial quotient.

<u>Case 2. $a = 2k$</u>

It can be shown that

$$F_n/F_{n-2k} = L_{2k} - 1 + \frac{F_{n-2k} - F_{n-4k}}{F_{n-2k}}$$

Then

$$\frac{F_{n-2k}}{F_{n-2k} - F_{n-4k}} = 1 + \frac{F_{n-4k}}{F_{n-2k} - F_{n-4k}}$$

Next

$$\frac{F_{n-2k} - F_{n-4k}}{F_{n-4k}} = L_{2k} - 2 + \frac{F_{n-4k} - F_{n-6k}}{F_{n-4k}}$$

Thus, there is a repeating pattern. The first partial quotient is $L_{2k}-1$; this is followed by $(1, L_{2k}-2)$ as a repeated pattern, the remainder after r such partial quotient pairs being

$$\frac{F_{n-2(r+1)k} - F_{n-2(r+2)k}}{F_{n-2(r+1)k}}$$

Example. F_{40}/F_{32} has a first partial quotient of $L_8-1 = 46$ followed by three sets (1, 45) and a remainder

$$\frac{F_8 - F_0}{F_8} = 1$$

Thus

$$F_{40}/F_{32} = [46, (1, 45)_3, 1]$$

which could also be represented $[46, (1, 45)_2, 1, 46]$.

2. THE RATIO L_n/F_{n-a}

Case 1. a odd

$$L_n/F_{n-a} = 5F_a - 1 + \frac{F_{n-a} - L_{n-2a}}{F_{n-a}}$$

where the relation $5F_aF_{n-a} = L_n + L_{n-2a}$ has been used in arriving at this result.

Then

$$\frac{F_{n-a}}{F_{n-a} - L_{n-2a}} = 1 + \frac{L_{n-2a}}{F_{n-a} - L_{n-2a}}$$

Next

$$\frac{F_{n-a} - L_{n-2a}}{L_{n-2a}} = F_a - 2 + \frac{L_{n-2a} - F_{n-3a}}{L_{n-2a}}$$

where the relation $F_a L_{n-2a} = F_{n-a} + F_{n-3a}$ has been employed.

Then

$$\frac{L_{n-2a}}{L_{n-2a} - F_{n-3a}} = 1 + \frac{F_{n-3a}}{L_{n-2a} - F_{n-3a}}$$

Finally

$$\frac{L_{n-2a} - F_{n-3a}}{F_{n-3a}} = 5F_a - 2 + \frac{F_{n-3a} - L_{n-4a}}{F_{n-3a}}$$

The form of the remainder is the same as that of the first remainder so that a cycle has been completed. In summary, the first term is $5F_a - 1$; the cycle that is repeated is $1,\ F_a - 2,\ 1,\ 5F_a - 2$; the remainder after r cycles is:

$$\frac{F_{n-(2r+1)a} - L_{n-(2r+2)a}}{F_{n-(2r+1)a}}$$

Example.

$$L_{86}/F_{79} = [64,\ (1, 11, 1, 63)_5,\ 1, 10, 3]$$

The verification of this development is shown below.

	0	1
	1	0
64	64	1
1	65	1
11	779	12
1	844	13
63	53951	831
1	54795	844
11	6 56696	10115
1	7 11491	10959
63	454 80629	7 00532
1	461 92120	7 11491
11	5535 93949	85 26933
1	5997 86069	92 38424
63	3 83401 16296	5905 47645
1	3 89399 02365	5997 86069
11	46 66790 42311	71881 94404
1	50 56189 44676	77879 80473
63	3232 06725 56899	49 78309 64203
1	3282 62915 01575	50 56189 44676
11	39340 98790 74224	605 96393 55639
1	42623 61705 75799	656 52583 00315
63	27 24628 86253 49561	41967 09122 75484
1	27 67252 47959 25360	42623 61705 75799
10	303 97153 65846 03161	4 68203 26180 33474
3	939 58713 45497 34843	14 47233 40246 76221
	L_{86}	F_{79}

Case 2. a even

$$L_n/F_{n-a} = 5F_a + \frac{L_{n-2a}}{F_{n-a}}$$

where the relation $5F_aF_{n-a} = L_n - L_{n-2a}$ has been used in the transformation. Then

$$\frac{F_{n-a}}{L_{n-2a}} = F_a + \frac{F_{n-3a}}{L_{n-2a}}$$

by virtue of the relation $F_aL_{n-2a} = F_{n-a} - F_{n-3a}$. Thus the pattern is

$$(5F_a, F_a)_r$$

with a remainder after r periods of

$$\frac{F_{n-(2r+1)a}}{L_{n-2ra}} .$$

Example. $L_{79}/F_{71} = (5F_8,\ F_8)_4$ with a remainder of F_7/L_{15}. Thus

$$L_{79}/F_{71} = \left[(105, 21)_4,\ 104, 1, 12\right]$$

3. THE RATIO F_n/L_{n-a}

The algebra is quite similar to that in the case of L_n/F_{n-a} so that only the final results will be given. If a is even, the partial quotients are given by

$$(F_a, 5F_a)_r$$

with a remainder of

$$\frac{L_{n-(2r+1)a}}{F_{n-2ra}} \quad .$$

If a is odd, there is a first partial quotient of $F_a - 1$ followed by cycles

$$(1,\ 5F_a - 2,\ 1,\ F_a - 2)_r$$

with a remainder of

$$\frac{L_{n-(2r+1)a} - F_{n-(2r+2)a}}{L_{n-(2r+1)a}}$$

4. THE RATIO L_n/L_{n-a}

Case I. a even

$$L_n/L_{n-2k} = L_{2k} - 1 + \frac{L_{n-2k} - L_{n-4k}}{L_{n-2k}}$$

the relation $L_n - L_{2k}L_{n-2k} = -L_{n-4k}$ being used in the transformation.

Then

$$\frac{L_{n-2k}}{L_{n-2k} - L_{n-4k}} = 1 + \frac{L_{n-4k}}{L_{n-2k} - L_{n-4k}}$$

and

$$\frac{L_{n-2k} - L_{n-4k}}{L_{n-4k}} = L_{2k} - 2 + \frac{L_{n-4k} - L_{n-6k}}{L_{n-4k}} \quad .$$

Hence the pattern is: $L_{2k}-1,\ (1,\ L_{2k}-2)_r$ with a remainder

$$\frac{L_{n-2(r+1)k} - L_{n-2(r+2)k}}{L_{n-2(r+1)k}}$$

<u>Case 2. a odd</u>

$$\frac{L_n}{L_{n-2k+1}} = L_{2k-1} + \frac{L_{n-4k+2}}{L_{n-2k+1}} \quad .$$

Thus the process is a repeating one, the remainder after r partial quotients being

$$\frac{L_{n-(r+1)(2k-1)}}{L_{n-r(2k-1)}}$$

5. GENERAL FIBONACCI SEQUENCE

Let the sequence be taken in the standard form [2] in which

$$f_1 = a, \quad f_2 = b, \quad a < b/2$$

Then

$$f_n = F_{n-1}b + F_{n-2}a$$

so that

$$\frac{f_n}{f_{n-k}} = \frac{F_{n-1}b + F_{n-2}a}{F_{n-1-k}b + F_{n-2-k}a}$$

If k is odd,

$$F_n/F_{n-k} = L_k + F_{n-2k}/F_{n-k}$$

so that

$$\frac{f_n}{f_{n-k}} = L_k + \frac{(F_{n-1} - L_k F_{n-1-k})b + (F_{n-2} - F_{n-2-k}L_k)a}{b\,F_{n-1-k} + a\,F_{n-2-k}}$$

$$= L_k + \frac{f_{n-2k}}{f_{n-k}} \quad .$$

Hence, there is a series of partial quotients $(L_k)_r$ with a remainder

$$\frac{f_{n-(r+1)k}}{f_{n-rk}}$$

Example. Using the series (1, 4),

$$f_{62}/f_{55} = (L_7)_7 \text{ with a remainder } f_6/f_{13} = 23/665$$

Thus

$$f_{62}/f_{55} = \left[(29)_7,\ 28,\ 1, 10\right]$$

If k is even,

$$f_n/f_{n-k} = L_k - 1 + \frac{f_{n-k} - f_{n-2k}}{f_{n-k}}$$

Then

$$\frac{f_{n-k}}{f_{n-k} - f_{n-2k}} = 1 + \frac{f_{n-2k}}{f_{n-k} - f_{n-2k}}$$

$$\frac{f_{n-k} - f_{n-2k}}{f_{n-2k}} = L_k - 2 + \frac{f_{n-2k} - f_{n-3k}}{f_{n-2k}}$$

so that the pattern is

$$L_k - 1,\ (1,\ L_k - 2)_r$$

with a remainder

$$\frac{f_{n-(r+1)k} - f_{n-(r+2)k}}{f_{n-(r+1)k}}$$

Example. f_{93}/f_{83} in the (1, 4) series.

$$f_{93}/f_{83} = \left[122,\ (1, 121)_7\right]$$

with a remainder

$$\frac{f_{13} - f_3}{f_{13}}$$

the latter yielding partial quotients 1, 132. Thus

$$f_{93}/f_{83} = [122, (1, 121)_7, 1, 132]$$

The verification of this expansion is shown below.

	0	1
	1	0
122	122	1
1	123	1
121	15005	122
1	15128	123
121	18 45493	15005
1	18 60621	15128
121	2269 80634	18 45493
1	2288 41255	18 60621
121	2 79167 72489	2269 80634
1	2 81456 13744	2288 41255
121	343 35360 35513	2 79167 72489
1	346 16816 49257	2 81456 13744
121	42229 70155 95610	343 35360 35513
1	42575 86972 44867	346 16816 49257
121	51 93909 93822 24517	42229 70155 95610
1	52 36485 80794 69384	42575 86972 44867
132	696410036 58721 83205	56 62244 50519 18054

Since f_{93} and f_{83} both have a factor of 5, these final quantities differ from them by this factor.

CONCLUSION

The continued fraction developments of the Fibonacci and Lucas ratios featured in this article are not only of interest in themselves by their mathematical patterns. They provide a ready means of recognizing Fibonacci and Lucas ratios that arise in attempting to formulate laws for the continued fraction developments of non-linear relations. This wider field offers many a challenge to the searcher after additional relations characterizing the Fibonacci sequences.

REFERENCES

1. C. D. Olds, "Continued Fractions," Random House, 1963.
2. Brother U. Alfred, "On the Order of the Fibonacci Sequence," Fibonacci Quarterly, Dec. 1963, pp. 43-46.

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