PRIMES WHICH ARE FACTORS OF ALL FIBONACCI SEQUENCES

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In studying the Fibonacci and Lucas sequences, one of the striking differences observed is the fact that ALL primes are factors of some positive term of the Fibonacci sequence while for the Lucas sequence many primes are excluded as factors. This difference raises some interesting questions regarding Fibonacci sequences in general.

(1) For a given Fibonacci sequence, how do we find which primes are factors and which are non-factors of its terms?

(2) Are there certain primes which are factors of all Fibonacci sequences?

It is this latter question which will be given attention in this paper.

We are considering Fibonacci sequences in which there is a series of positive terms with successive terms relatively prime to each other. For any sequence we can find two consecutive terms $a \geq 0$, $b > 0$, $a < b$, and take these as

$$f_0 = a, \quad f_1 = b \quad ,$$

the defining relation for the sequence being

$$f_{n+1} = f_n + f_{n-1}, \quad (n \geq 2) \quad .$$

The particular sequence with $a = 0$ and $b = 1$ is known as <u>the</u> Fibonacci sequence and will have its terms designated by $F_0 = 0$, $F_1 = 1$, and so on.

<u>Theorem:</u> The only Fibonacci sequence having all primes as factors of some of its positive terms is the sequence with $a = 0$ and $b = 1$.

<u>Proof</u>: Since zero is an element of the sequence, the fact that all primes divide some positive terms of the sequence follows from the periodicity of the series relative to any given modulus.

To prove the converse, we note that each sequence is characterized by a quantity $D = b^2 - a(a + b)$. For if f_n is the n^{th} term of the sequence,

$$f_n = F_{n-1}b + F_{n-2}a \quad .$$

Then

$$f_n^2 - f_{n-1}f_{n+1} = (F_{n-1}b + F_{n-2}a)^2 - (F_{n-2}b + F_{n-3}a)(F_nb + F_{n-1}a)$$

which equals

$$b^2(F_{n-1}^2 - F_{n-2}F_n) + ab(F_{n-1}F_{n-2} - F_nF_{n-3}) + a^2(F_{n-2}^2 - F_{n-3}F_{n-1})$$

or

$$(-1)^n (b^2 - ab - a^2)$$

so that the values are successively $+D$ and $-D$.

Now D is equal to 1 in the case of the sequence $0, 1, 1, 2, 3, \cdots$ and in no other Fibonacci sequences. For if a is kept fixed, the quantity $b(b - a) - a^2$ increases with b. Therefore its minimum value is found for $b = a + 1$. But then $b(b - a) - a^2$ becomes $a + 1 - a^2$. Now if $a = 0, 1$ or 2, $|a + 1 - a^2| = 1$ and we have the Fibonacci sequence. If $a \geq 3$, $|a + 1 - a^2| \geq 5$.

Thus, apart from the Fibonacci sequence properly so-called, $D > 1$. Furthermore, D must be odd if a and b are relatively prime. Hence if $f_n \equiv 0$ modulo some prime factor p of D, we would then have

$$f_{n-1}f_{n+1} \equiv 0 \pmod{p}$$

from the relation

$$f_n^2 - f_{n-1}f_{n+1} = (-1)^n D$$

so that either f_{n-1} or $f_{n+1} \equiv 0 \pmod{p}$. Thus two successive terms of the series would be divisible by p and consequently all terms would be divisible by p which would lead to the conclusion that $p \mid (a, b)$, contrary to hypothesis.

Therefore, the only Fibonacci sequence having all primes as divisors one or the other of its terms is the one Fibonacci sequence with a zero element, namely: $0, 1, 1, 2, 3, 5, 8, 13, \cdots$.

CONGRUENTIAL FIBONACCI SEQUENCES

For a given prime modulus, such as eleven, there are eleven possible residues modulo 11: $0, 1, 2, 3, \cdots, 10$. These may be arranged in ordered pairs repetitions being allowed in 11^2 or 121 ways. Each such pair of residues can be made the starting point of a congruential Fibonacci sequence modulo 11, though of course various pairs will give rise to the same sequence. The one pair that needs to be excluded as trivial is 0 - 0 since all the terms of the sequence would then be 0 and we have assumed throughout that no two successive terms have a common factor. Hence there are 120 possible sequence pairs. A complete listing of these congruential sequences modulo 11 is displayed below.

(A)	1	1	2	3	5	8	2	10	1	0
(B)	2	2	4	6	10	5	4	9	2	0
(C)	3	3	6	9	4	2	6	8	3	0
(D)	4	4	8	1	9	10	8	7	4	0
(E)	5	5	10	4	3	7	10	6	5	0
(F)	6	6	1	7	8	4	1	5	6	0
(G)	7	7	3	10	2	1	3	4	7	0
(H)	8	8	5	2	7	9	5	3	8	0
(I)	9	9	7	5	1	6	7	2	9	0
(J)	10	10	9	8	6	3	9	1	10	0
(K)	1	8	9	6	4	10	3	2	5	7
(L)	1	4	5	9	3					
(M)	2	8	10	7	6					

That all possible sequence-pairs are covered is shown in the following table where the number in the column at the left is the first term of the pair and the number in the row at the top is the second.

	0	1	2	3	4	5	6	7	8	9	10
0	X	A	B	C	D	E	F	G	H	I	J
1	A	A	A	G	L	F	I	F	K	D	J
2	B	G	B	A	B	K	C	H	M	I	A
3	C	L	K	C	G	A	C	E	H	J	G
4	D	F	C	E	D	L	B	G	D	B	K
5	E	I	H	H	B	E	F	K	A	L	E
6	F	F	M	J	K	E	F	I	C	C	B
7	G	K	I	G	D	I	M	G	F	H	E
8	H	D	A	C	F	H	J	D	H	K	M
9	I	J	B	L	C	H	K	I	J	I	D
10	J	A	G	K	E	B	E	M	D	J	J

We shall now consider various categories so as to cover all primes.

(A) $p = 2$

If either a or b is even, 2 is a factor of terms of the series; if both are odd, then $a + b \equiv 0 \pmod 2$. Thus, 2 is a factor of all Fibonacci sequences.

(B) $p = 5$

Since 5 is not a factor of terms of the Lucas series, it cannot be a divisor of all Fibonacci sequences.

(C) $p = 10x \pm 1$

For p of the form $10x \pm 1$, the period $h(p)$ for any Fibonacci sequence is a divisor of $p - 1$. Since there are $p^2 - 1$ sequence pairs of residues, the number of congruential sequences modulo p would have to be

$$\geq \frac{p^2 - 1}{p - 1} \quad \text{or} \quad p + 1 \quad .$$

But since there are only $p - 1$ residues other than zero, sequence triples a-0-a can only be $p - 1$ in number. Thus there cannot be one per sequence. Hence no prime of the form $10x \pm 1$ can be a divisor of all Fibonacci sequences.

(D) $p = 10x \pm 3$

For p of the form $10x \pm 3$, the situation is as follows:

(1) The period is a factor of $2p + 2$.

(2) $2p + 2$ is divisible by 4.

(3) The period contains all power of 2 found in $2p + 2$.

(4) The period is the same as the period of the Fibonacci sequence, F_n. [1]

Accordingly, if the period is less than $2p + 2$, it will also be less than $p - 1$ and hence as before there will not be enough sequence pairs with zeros to cover all the sequences. Thus a necessary condition is that the period be $2p + 2$ if a prime is to be found as a factor of all Fibonacci sequences.

Two cases may be distinguished: (a) The case in which the period $h(p) = 2^2 (2r + 1)$; (b) The case in which the period $h(p) = 2^m (2r + 1)$, $m \geq 3$.

(a) $h(p) = 2^2(2r + 1)$

In this instance, if a sequence has a zero at k, it will also have zeros at $k/4$, $k/2$, and $3k/4$ or four zeros per sequence. The number of sequences is

$$\frac{p^2 - 1}{2p + 2} = \frac{p - 1}{2} .$$

To provide 4 zeros per sequence there would have to be

$$\frac{4(p - 1)}{2} = 2(p - 1) \text{ zeros,}$$

whereas there are only $p - 1$.

(b) $k = 2^m(2r + 1),\ m \geq 3.$

For a period of this form, if there is a zero at k, there will also be a zero at $k/2$, but not at $k/4$ or $3k/4$. The number of zeros required for $(p - 1)/2$ sequences would be

$$2(p - 1)/2 = p - 1 ,$$

which is the exact number available. Thus the primes which divide all Fibonacci sequences are primes of the form $10x \pm 3$ for which $2p + 2$ is equal to $2^m(2r + 1)$, $m \geq 3$. In other words,

$$p \equiv \pm 3 \pmod{10}$$

$$p = 2^{m-1}(2r + 1) - 1 \text{ or } p \equiv -1 \pmod 4$$

These congruences lead to the solution $p \equiv 3, 7 \pmod{20}$.

CONCLUSION

The primes which are factors of all Fibonacci sequences are:

(1) The prime 2

(2) Primes of the form $20k + 3, 7$, having a period $2p + 2$.

LIST OF PRIMES WHICH DIVIDE
ALL FIBONACCI SEQUENCES ($p < 3000$)

2	383	787	1327	1783	2383
3	443	823	1367	1787	2423
7	463	827	1423	1847	2467
23	467	863	1447	1867	2503
43	487	883	1487	1907	2543
67	503	887	1543	1987	2647
83	523	907	1567	2003	2683
103	547	983	1583	2063	2707
127	587	1063	1607	2083	2767
163	607	1123	1627	2087	2803
167	643	1163	1663	2143	2843
223	647	1187	1667	2203	2887
227	683	1283	1723	2243	2903
283	727	1303	1747	2287	2927
367				2347	2963

REFERENCE

1. D. D. Wall, "Fibonacci Series Modulo m," The American Mathematical Monthly, June-July, 1960, p. 529.

SOME CORRECTIONS TO VOLUME 1, NO. 4

Pages 45–46: D = 31 should read (2,7), (3,8).

There was an omission in the Table of "D's" as follows:

D		D	
305	(1,18) (16,33)	361	(8,25) (9,26)
311	(5,21) (11,27)	379	(1,20) (18,37)
319	(2,19) (7,23) (9,25) (15,32)	389	(5,23) (13,31)
331	(3,20) (14,31)	395	(2,21) (17,36)
341	(1,19) (4,21) (13,30) (17,35)		
349	(5,22) (12,29)		
355	(6,23) (11,28)		
359	(7,24) (10,27)		