A COMPLETE CHARACTERIZATION OF THE DECIMAL FRACTIONS THAT CAN BE REPRESENTED AS $\Sigma 10^{-k(i+1)} F_{\alpha i}$, WHERE $F_{\alpha i}$ IS THE αiTH FIBONACCI NUMBER

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1. INTRODUCTION

In 1953 Fenton Stancliff [2] noted (without proof) that

$$\Sigma 10^{-(i+1)} F_i = \frac{1}{89}$$

where F_i denotes the ith Fibonacci number. Until recently this expansion was regarded as an anomalous numerical curiosity, possibly related to the fact that 89 is a Fibonacci number (see Remark in [2]), but not generalizing to other fractions in an obvious manner.

Recently, the second of us showed that the sums $\Sigma 10^{-(i+1)} F_{\alpha i}$ approximate 1/71, 2/59, and 3/31 for $\alpha = 2$, 3, and 4, respectively. Moreover, Winans showed that the sums $\Sigma 10^{-2(i+1)} F_{\alpha i}$ approximate 1/9899, 1/9701, 2/9599, and 3/9301 for $\alpha = 1$, 2, 3, and 4, respectively.

In this paper, we completely characterize all decimal fractions that can be approximated by sums of the type

$$\frac{1}{F_\alpha}\left(\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}\right), \quad \alpha \geq 1, \; k \geq 1.$$

In particular, all such fractions must be of the form

$$\frac{1}{10^{2k} - 10^k - 1 - 10^k\left(\sum_{j=1}^{(\alpha-1)/2} L_{2j}\right)} \tag{1.1}$$

when α is odd, and of the form

$$\frac{1}{10^{2k} - 3(10^k) + 1 - 10^k\left(\sum_{j=1}^{(\alpha+1)/2} L_{2j+1}\right)} \tag{1.2}$$

when α is even [L_j denotes the jth Lucas number and the denominators in (1.1) and (1.2) are assumed to be positive].

Recalling that the ith term of the Fibonacci sequence is given by

$$F_i = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i, \tag{1.3}$$

it is straightforward to prove that the sums $\frac{1}{F_\alpha}\left(\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}\right)$ converge to the

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During the writing of this article, this author was at Carleton University, Ottawa, Canada, and wishes to acknowledge with gratitude support under National Research Council of Canada Grant A-7233.

fractions indicated in (1.1) and (1.2) provided that $((1+\sqrt{5})/2)^{\alpha} < 10^k$. For example, we have $((1+\sqrt{5})/2)^2 = (3+\sqrt{5})/2$ and $(3+\sqrt{5})/2 < 10$. Hence, appealing to the formula for the sum of a convergent geometric series, we have

$$\sum_{i=1}^{\infty} \frac{F_{2i}}{10^{i+1}} = \frac{1}{10\sqrt{5}}\left(\frac{1}{1-(3+\sqrt{5})/20} - \frac{1}{1-(3-\sqrt{5})/20}\right) \tag{1.4}$$

$$= \frac{2\sqrt{5}}{5}\left(\frac{17+\sqrt{5}}{284} - \frac{17-\sqrt{5}}{284}\right) = \frac{1}{71}$$

The surprising fact, indeed the fact that motivates the writing of this paper, is that the fractions given by (1.1) and (1.2) are completely determined by values in the Lucas sequence, totally independent of any consideration regarding Fibonacci numbers. The manner in which this dependence on Lucas numbers arises seems to us thoroughly remarkable.

2. THE SUMS $\Sigma 10^{-k(i+1)}F_{\alpha i}$, $k = 1$

Case 1: $\alpha = 1$.

Using Table 1 (see Section 6 below), we have

$$\sum_{i=1}^{60} 10^{-(i+1)}F_i \tag{2.1}$$

$$= .0112359550561797752808988764044943820224719\underline{10112}3296681836230.$$

It is easily verified that 1/89 repeats with period 44 and that

$$\frac{1}{89} = .0112359550561797752808988764044943820224719\underline{1011235}\ldots \tag{2.2}$$

The approximation $\sum_{i=1}^{60} 10^{-(i+1)}F_i \approx \frac{1}{89}$ is accurate *only* to 49 places, solely because we have used only the first 60 Fibonacci numbers. A good ballpark estimate of the accuracy of the approximation $\sum_{i=1}^{s} 10^{-k(i+1)}F_{\alpha i} \approx \frac{p}{q}$ may be obtained by looking at the number of zeros preceding the first nonzero entry in the expansion

$$\frac{F_{\alpha s}}{10^{k(s+1)}} = .000\ldots\ a_n \cdot a_{n+1} \cdots a_{\ell} \tag{2.3}$$

a_n is the first nonzero entry and $\ell = k(s+1)$.

Thus, e.g.,

$$\frac{F_{60}}{10^{61}} = .000\ldots1548008755920 \tag{2.4}$$

The number of zeros preceding a_n above is 48, so that the 49-place accuracy found is to be expected.

Case 2: $\alpha = 2$.

Look at every second Fibonacci number; then, using Table 1, we have

$$\sum_{i=1}^{25} 10^{-(i+1)}F_{2i} = .01408450704225347648922085 \tag{2.5}$$

Now,

(2.6) $$\frac{1}{71} = .0140845070422535\ldots$$

Note that

(2.7) $$\frac{F_{50}}{10^{26}} = .000\ldots12586269025$$

where the number of zeros preceding $a_n = 1$ is 15.

Case 3: $\alpha = 3$.

Looking at every third Fibonacci number, we have

(2.8) $$\sum_{i=1}^{16} 10^{-(i+1)}F_{3i} = .03389826975294276$$

Moreover,

(2.9) $$\frac{2}{59} = .0338983\ldots$$

The six place accuracy is to be expected in light of the fact that

(2.10) $$\frac{F_{48}}{10^{17}} = .00000004807526976$$

Case 4: $\alpha = 4$.

Looking at every fourth Fibonacci number up to F_{100}, we have

(2.11) $$\sum_{i=1}^{25} 10^{-(i+1)}F_{4i} = .09676657589472715467557065$$

Now

(2.12) $$\frac{3}{31} = .096774\ldots$$

The convergence of (2.11) is very slow, as can be seen by the fact that $\dfrac{F_{100}}{10^{26}}$ has only five zeros preceding its first nonzero entry:

(2.13) $$\frac{F_{100}}{10^{26}} = .00000354224638179261842845$$

Case 5: $\alpha \geq 5$.

Consider $\Sigma 10^{-(i+1)}F_{5i}$. The sum is of the form

(2.14) $$\begin{array}{r} .05 \\ +\ .055 \\ +\ .0610 \\ +\ .06765 \\ \underline{+\ \ \ \ldots} \end{array}$$

Clearly this sum does not converge at all and, *a fortiori*, $\Sigma 10^{-(i+1)}F_{\alpha i}$ does not converge for any $\alpha \geq 5$.

Summary of Section 2:

(2.15) $$\sum_{i=1}^{n} 10^{-(i+1)}F_i \approx \frac{1}{89} \qquad \sum_{i=1}^{n} 10^{-(i+1)}F_{2i} \approx \frac{1}{71}$$

(2.16) $$\sum_{i=1}^{n} 10^{-(i+1)} F_{3i} \approx \frac{2}{59} \qquad \sum_{i=1}^{n} 10^{-(i+1)} F_{4i} \approx \frac{3}{31}$$

(2.17) $$\sum_{i=1}^{n} 10^{-(i+1)} F_{\alpha i} \to \infty \text{ as } n \to \infty \text{ if } \alpha \geq 5$$

THE SUMS $\Sigma 10^{-k(i+1)} F_{\alpha i}$, $k = 2$

If $\alpha = 10$, the sum $\Sigma 10^{-2(i+1)} F_{\alpha i}$ is of the form

(3.1)
```
  .0055
+ .006765
+ .00832040
+ .0102334155
+      ...
```

and this clearly does not converge. There are, consequently, exactly nine fractions with four-digit denominators that are approximated by sums of the type

$$\sum_{i=1}^{n} 10^{-2(i+1)} F_{\alpha i}.$$

Henceforth, for brevity, we denote $\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}$ by $S_{\alpha i}(k)$. Then, for $\alpha = 1$, $2, \ldots, 9$, we have, respectively, $S_{\alpha i}(2) \approx$ 1/9899, 1/9701, 2/9599, 3/9301, 5/8899, 8/8201, 13/7099, 21/5301, and 34/2399.

We indicate the computation for $S_{4i}(2)$, leaving the reader to check the remaining values. To compute $\sum_{i=1}^{12} 10^{-2(i+1)} F_{4i}$, we must perform the addition:

(3.2)
```
.0003
.000021
.00000144
           987
             6765
              46368
                317811
                 2178309
                   14930352
                    102334155
                       701408733
                        4807526976
.000322545962799695419502 76
```

Now

(3.3) $$\frac{3}{9301} = .000322545962799698\ldots$$

Notice that the approximation is considerably more accurate for small n than the analogous approximation given by (2.11). Of course, this is because, from the point of rapidity of convergence (or lack thereof), $S_{4i}(1)$ is more closely analogous to $S_{8i}(2)$—each represents the largest value of α for which convergence is possible for the respective value of k.

The reader may well wonder how we arrived at fractions such as 21/5301 and 34/2399, since $S_{8i}(2)$ and $S_{9i}(2)$ converge so slowly that it is not obvious what fractions they are approximating. The values for $S_{\alpha i}(2)$, $\alpha = 1, \ldots, 6$, were obtained from empirical evidence. The pattern for the numerators is obvious. After looking at the denominators for some time, the first of us noted (with some astonishment) the following pattern governing the first two digits of the denominators:

$$\begin{aligned} 98 - 95 &= 3 \\ 97 - 93 &= 4 \\ 95 - 88 &= 7 \\ 93 - 82 &= 11 \\ 88 - 70 &= 18 \end{aligned} \tag{3.4}$$

Subsequent empirical evidence revealed what poetic justice required, namely that the eighth and ninth denominators must be 5301 and 2301, for

$$82 - 53 = 29 \quad \text{and} \quad 70 - 23 = 47 \tag{3.5}$$

The indicated differences are, of course, precisely the Lucas numbers beginning with $L_2 = 3$. Notice that entirely apart from any numerical values for the Fibonacci numbers, the existence of a value for $S_{10i}(2)$ is outlawed by the above pattern. For the first two digits of the denominator of such a fraction would be (on the basis of the pattern) $53 - 76 < 0$, presumably an absurdity.

Naturally, the real value of recognizing the pattern is that values can easily be given for $S_{\alpha i}(k)$ for every k and every α for which it is possible that these sums converge. Moreover, values of α for which convergence is an obvious impossibility (because terms in the sum are increasing), and the denominators of the fractions which these sums approximate for the remaining α, may be determined by consideration of the Lucas numbers alone.

We may proceed at once to the general case, but for the sake of illustration we briefly sketch the case $k = 3$ employing the newly discovered pattern.

4. THE SUMS $\Sigma 10^{-k(i+3)}F_{\alpha i}$, $k = 3$

In analogy to the earlier cases it is not difficult to obtain and empirically check that 1/998999 and 1/997001 are fractions that are approximated by $S_i(3)$ and $S_{2i}(3)$, respectively.

Now, using Table 2 (see Section 6 below),

$$\begin{aligned} &998 - 3 = 995,\ 997 - 4 = 993,\ 995 - 7 = 988,\ 993 - 11 = 982, \\ &988 - 18 = 970,\ 982 - 29 = 953,\ 970 - 47 = 923,\ 953 - 76 = 877, \\ &923 - 123 = 800,\ 877 - 199 = 678,\ 800 - 322 = 478, \\ &678 - 521 = 157, \text{ and } 478 - 843 < 0 \end{aligned} \tag{4.1}$$

Therefore, we expect that $S_{\alpha i}(3)$ is meaningful if $\alpha \leq 14$ and if the fourteen fractions corresponding to these α's are precisely:

$$\begin{aligned} &\frac{1}{998999}, \frac{1}{997001}, \frac{2}{995999}, \frac{3}{993001}, \frac{5}{988999}, \\ &\frac{8}{982001}, \frac{13}{970999}, \frac{21}{953001}, \frac{34}{923999}, \frac{55}{877001}, \\ &\frac{89}{800999}, \frac{144}{678001}, \frac{233}{478999}, \frac{377}{157001} \end{aligned} \tag{4.2}$$

We leave for the reader the aesthetic satisfaction of checking that $\alpha = 15$ is, indeed, the smallest value of α such that the terms of $S_{\alpha i}(3)$ are not decreasing.

Example:

Consider $\sum_{i=1}^{7} 10^{-3(i+1)}F_{9i}$

This sums as follows:

(4.3)

```
.000034
.000002584
.000000196418
        14930352
         1134903170
           86267571272
            6557470319842
-------------------------
.000036796576080211591842
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On the other hand, the ninth fraction in (4.2) is

$$\frac{34}{923999} = .00003679657\ldots \tag{4.4}$$

5. THE GENERAL CASE

All that has gone before can be summarized succinctly as follows. The totality of decimal fractions that can be approximated by sums of the form

$$\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}, \quad \alpha \geq 1,\ k \geq 1,$$

are given by

$$\frac{F_\alpha}{10^{2k} - 10^k - 1 - 10^k\left(\sum_{j=1}^{(\alpha-1)/2} L_{2j}\right)} \tag{5.1}$$

when α is odd and the denominator is positive, and by

$$\frac{F_\alpha}{10^{2k} - 3(10^k) + 1 - 10^k\left(\sum_{j=1}^{(\alpha-2)/2} L_{2j+1}\right)} \tag{5.2}$$

when α is even and the denominator is positive.

Remark: The appearance of F_α in the numerator of the above fractions is not essential to the analysis. One can just as well look at sums of the form

$$\frac{1}{F_\alpha}\sum_{i=1}^{n} 10^{-k(i+1)} F_{\alpha i}$$

These approximate fractions identical with those in (5.1) and (5.2), except that their numerators are always 1. These fractions are determined, then, only by Lucas numbers with no reference at all to the Fibonacci sequence.

Example 1: Let $k = 4$. The smallest positive value of the denominators in (5.1), (5.2) is

$$10^8 - 10^4 - 1 - 10^4\left(\sum_{j=1}^{(19-1)/2} L_{2j}\right) = 6509999.$$

This means that there are exactly nineteen fractions arising in the case $k = 4$ and

$$S_{19i}(4) \approx \frac{4184}{6509999}, \tag{5.3}$$

although it will be necessary to sum a large number of terms to get a good approximation (or even to get an approximation that remotely resembles 4184/6509999). However, if one looks at the nineteenth fraction arising when $k = 5$, one obtains

$$\frac{4184}{9065099999} = .0000004612\ldots \tag{5.4}$$

On the other hand, $\sum_{i=1}^{5} 10^{-5(i+1)} F_{19i}$ equals

$$\begin{array}{rl} & .0000004181 \\ + & .000000039088169 \\ & 365435296162 \\ & 3416454622906707 \\ & 31940414634990093395 \\ \hline & .000000461216107838545660793395 \end{array} \tag{5.5}$$

which restores one's faith in (5.3) with much less pain than employing direct computation.

<u>*Example 2:*</u> Let $k = 8$ and let $\alpha = 32$ so that (5.2) must be used. From Table 1, we have

$$\sum_{i=1}^{3} 10^{-8(i+1)} F_{32i}$$

$$\begin{array}{rl} & .0000000002178309 \\ + & .000000000010610209857723 \\ & .00000000000051680678854858312532 \\ \hline & .00000000022895791664627158312532 \end{array} \tag{5.6}$$

On the other hand, from (5.2) and Tables 1 and 2 we have that the thirty-second fraction arising when $k = 8$ is:

$$\frac{2178309}{10^{16} - 3(10^8) + 1 - 10^8\left(\sum_{j=1}^{15} L_{2j+1}\right)} = \frac{2178309}{9512915300000001} = .0000000002289\ldots \tag{5.7}$$

a good approximation considering that only three Fibonacci numbers (F_{32}, F_{64}, and F_{96}) are used in (5.6).

6. TABLES OF FIBONACCI AND LUCAS NUMBERS

TABLE 1

F_1	1	F_{14}	377	F_{27}	196418	F_{40}	102334155
F_2	1	F_{15}	610	F_{28}	317811	F_{41}	165580141
F_3	2	F_{16}	987	F_{29}	514229	F_{42}	267914296
F_4	3	F_{17}	1597	F_{30}	832040	F_{43}	433494437
F_5	5	F_{18}	2584	F_{31}	1346269	F_{44}	701408733
F_6	8	F_{19}	4184	F_{32}	2178309	F_{45}	1134903170
F_7	13	F_{20}	6765	F_{33}	3524578	F_{46}	1836311903
F_8	21	F_{21}	10946	F_{34}	5702889	F_{47}	2971215073
F_9	34	F_{22}	17711	F_{35}	9227465	F_{48}	4807526976
F_{10}	55	F_{23}	28657	F_{36}	14930352	F_{49}	7778742049
F_{11}	89	F_{24}	46368	F_{37}	24157817	F_{50}	12586269025
F_{12}	144	F_{25}	75025	F_{38}	39088169	F_{51}	20365011074
F_{13}	233	F_{26}	121393	F_{39}	63245986	F_{52}	32951280099

TABLE 1 (continued)

F_{53}	53316291173	F_{77}	5527939700884757
F_{54}	86267571272	F_{78}	8944394323791464
F_{55}	139583862445	F_{79}	14472334024676221
F_{56}	225851433717	F_{80}	23416728348467685
F_{57}	365435296162		
F_{58}	591286729879		
F_{59}	956722026041		
F_{60}	548008755920		
F_{61}	2504730781961		
F_{62}	4052739537881		
F_{63}	6557470319842		
F_{64}	10610209857723		
F_{65}	17167680177565		
F_{66}	27777890035288		
F_{67}	44945570212853		
F_{68}	72723460248141		
F_{69}	117669030460994		
F_{70}	190392490709135		
F_{71}	308061521170129		
F_{72}	498454011879264		
F_{73}	806515533049393		
F_{74}	1304969454928657		
F_{75}	2111485077978050		
F_{76}	3416454622906707		

TABLE 2

L_1	1	L_{11}	199	L_{21}	24476	L_{31}	3010349
L_2	3	L_{12}	322	L_{22}	39603	L_{32}	4870847
L_3	4	L_{13}	521	L_{23}	64079	L_{33}	7881196
L_4	7	L_{14}	843	L_{24}	103682	L_{34}	12752043
L_5	11	L_{15}	1364	L_{25}	167761	L_{35}	20633239
L_6	18	L_{16}	2207	L_{26}	271443	L_{36}	33385282
L_7	29	L_{17}	3571	L_{27}	439204	L_{37}	54068521
L_8	47	L_{18}	5778	L_{28}	710647	L_{38}	87483803
L_9	76	L_{19}	9349	L_{29}	1149851	L_{39}	141552324
L_{10}	123	L_{20}	15127	L_{30}	1860498	L_{40}	228826127

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