REFERENCES

1. M. Avriel. *Non-Linear Programming: Analysis and Methods*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1976.
2. M. Avriel & D. J. Wilde. "Optimality Proof for the Symmetric Fibonacci Search Technique." *The Fibonacci Quarterly* 4 (1966):265-69.
3. M. Avriel & D. J. Wilde. "Golden Block Search for the Maximum of Unimodal Functions." *Management Science* 14 (1968):307-19.
4. J. H. Beamer & D. J. Wilde. "Minimax Optimization of Unimodal Functions by Variable Block Search." *Management Science* 16 (1970):529-41.
5. M. J. Box, D. Davies, & W. H. Swann. *Non-Linear Optimization Techniques*. Edinburgh: Oliver & Boyd, 1969.
6. J. Kiefer. "Sequential Minimax Search for a Maximum." *Proc. Amer. Math. Soc.* 4 (1963):502-06.
7. L. T. Oliver & D. J. Wilde. "Symmetrical Sequential Minimax Search for a Maximum." *The Fibonacci Quarterly* 2 (1964):169-75.
8. H. H. Rosenbrock. "An Automatic Method for Finding the Greatest and Least Value of a Function." *Computer J.* 3 (1960):175-84.
9. D. J. Wilde. *Optimum Seeking Methods*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1964.
10. C. Witzgall. "Fibonacci Search with Arbitrary First Evaluation." *The Fibonacci Quarterly* 10 (1972):113-34.

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2,3 SEQUENCE AS BINARY MIXTURE

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The integer sequence formed by multiplying integral powers of the numbers 2 and 3 can be viewed as a binary sequence. The numbers 2 and 3 are the component factors of this binary. This paper explores the combination of these components to form the properties of the integers in the binary. Properties considered are: value, ordinality (position in the sequence), and exponents of the factors of each integer in the binary sequence.

Questions related to the properties of integer sequences with irregular nth differences are notoriously hard to answer [1]. The integers in the 2,3 sequence produce irregular nth differences. These integers can be related to the graphs constructed in the study of 2,3 trees [2, 3]. It is shown in this paper that the ordinality property of the integers in the 2,3 sequence can be derived from the irrational number log 3/log 2. This number also finds application in the derivation of a discontinuous spatial pattern found in the study of fractal dimension [4].

In Table 1, the first fifty-one numbers in the 2,3 sequence are listed according to their ordinality with respect to value. Since the 2,3 sequence consists of numbers which are integral multiples of the factors 2 and 3, it is convenient to plot the information in Table 1 in the form of a two-dimensional lattice, as shown in Figure 1. In this figure, the horizontal axis represents integral powers of 2 and the vertical axis represents integral powers of 3. The ordinality of each number is printed next to its corresponding lattice point. For example, the number $2592 = 2^5 3^4$ and $\text{Ord}(2^5 3^4) = 50$; therefore, at the coordinates 2^5, 3^4, the number "50" is printed.

TABLE 1. Value, Ordinality, and Factors of the First Fifty-one Numbers in the 2,3 Sequence

Value	Ordinality	Factors	Value	Ordinality	Factors
1	0	$2^0 3^0$	243	26	$2^0 3^5$
2	1	$2^1 3^0$	256	27	$2^8 3^0$
3	2	$2^0 3^1$	288	28	$2^5 3^2$
4	3	$2^2 3^0$	324	29	$2^2 3^4$
6	4	$2^1 3^1$	384	30	$2^7 3^1$
8	5	$2^3 3^0$	432	31	$2^4 3^3$
9	6	$2^0 3^2$	486	32	$2^1 3^5$
12	7	$2^2 3^1$	512	33	$2^9 3^0$
16	8	$2^4 3^0$	576	34	$2^6 3^2$
18	9	$2^1 3^2$	648	35	$2^3 3^4$
24	10	$2^3 3^1$	729	36	$2^0 3^6$
27	11	$2^0 3^3$	768	37	$2^8 3^1$
32	12	$2^5 3^0$	864	38	$2^5 3^3$
36	13	$2^2 3^2$	972	39	$2^2 3^5$
48	14	$2^4 3^1$	1024	40	$2^{10} 3^0$
54	15	$2^1 3^3$	1152	41	$2^7 3^2$
64	16	$2^6 3^0$	1296	42	$2^4 3^4$
72	17	$2^3 3^2$	1458	43	$2^1 3^6$
81	18	$2^0 3^4$	1536	44	$2^9 3^1$
96	19	$2^5 3^1$	1728	45	$2^6 3^3$
108	20	$2^2 3^3$	1944	46	$2^3 3^5$
128	21	$1^7 3^0$	2048	47	$2^{11} 3^0$
144	22	$2^4 3^2$	2187	48	$2^0 3^7$
162	23	$2^1 3^4$	2304	49	$2^8 3^2$
192	24	$2^6 3^1$	2592	50	$2^5 3^4$
216	25	$2^3 3^3$			

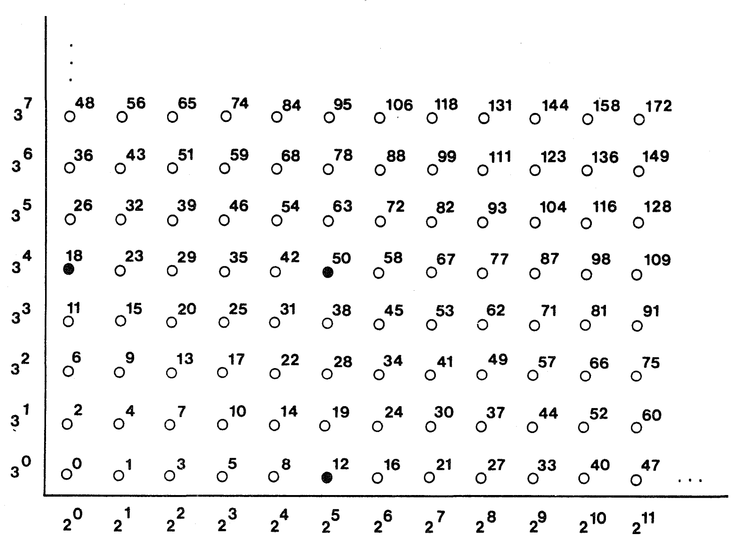


FIGURE 1

We shall now develop a theorem that will condense the information in Figure 1.

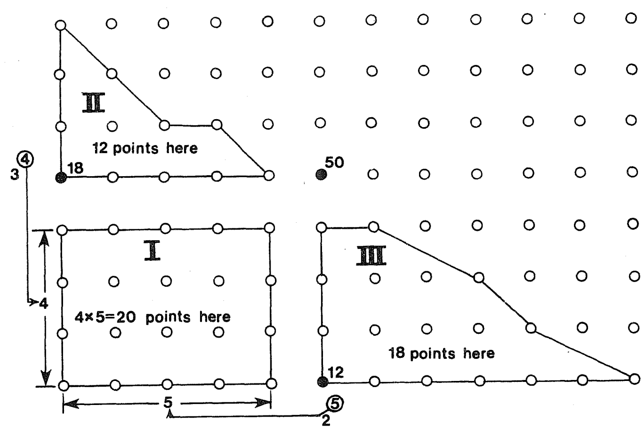
Theorem 1: $\mathrm{Ord}(2^a3^b) = ab + \mathrm{Ord}(2^a3^0) + \mathrm{Ord}(2^03^b)$.

This theorem states that the ordinality of any point in the 2,3 lattice can be determined from the exponents of the coordinates of the point, and a knowledge of the ordinalities of the projections of the point onto the horizontal and vertical baselines. For example, in the case of the number 2^53^4, this theorem takes the form

$$\begin{array}{ccccccc} \mathrm{Ord}(2^53^4) & = & (5)(4) & + & \mathrm{Ord}(2^53^0) & + & \mathrm{Ord}(2^03^4). \\ 50 & = & 20 & + & 12 & + & 18 \end{array}$$

Point 50 and its projections onto the horizontal and vertical baselines (i.e., points 12 and 18, respectively) can be seen in Figure 1 as the blacked-in points.

Since an ordinality of 50 means there are fifty points of lower value, and hence, lower ordinality in the lattice, it will be useful to examine in detail the locations of these points. In Figure 2, the three polygons enclose all the points with ordinalities less than 50.



$\mathrm{Ord}[2^a3^b] = ab + \mathrm{Ord}[2^a3^0] + \mathrm{Ord}[2^03^b]$

$\mathrm{Ord}[2^53^4] = 20 + 12 + 18 = 50$

FIGURE 2

Polygon:

I. Those points with $a < 5$ and $b < 4$ (since both a and b are smaller in these points than in point 50, the ordinalities of these points must be less than 50).

II. Those points with ordinalities less than 50, with $a < 5$ and $b \geq 4$.

III. Those points with ordinalities less than 50, with $b < 4$ and $a \geq 5$.

Since ordinality is determined with respect to value, the fifty points in polygons I, II, and III must represent numbers whose values are less than 2^53^4.

The reason that the ordinality of point 12 is exactly equal to the number of lattice points in polygon II can be seen from Figures 3 and 4, with the help of the following discussion. By the definition of "ordinality 12" and the fact that point 12 lies on the horizontal baseline, there must be twelve points of lower value west and northwest of point 12 (since there are no points south of the horizontal baseline, and all points north, northeast, or east are larger). But the relative values of all points in the 2,3 lattice are related to each other according to relative position. For example, take any lattice point, the point directly above it is three times greater in value, the point directly below it is one-third as great in value, the point directly to the right is twice as large in value, and the point directly to the left is half as large in value. If we normalize the *value* of point 12 to the relative value 1, the relative value of all points west and northwest that are lower in value can be seen in Figure 4. This relative value relationship holds for the points west and northwest of point 12 in exactly the same way that it holds for the points west and northwest of point 50, since the relative values of all points are related to each other according to their relative position to each other. Thus, the ordinality of point 12 is identical to the number of points in polygon II and the ordinality of point 18 is identical to the number of points in polygon III (this can be seen with the help of Figures 5 and 6).

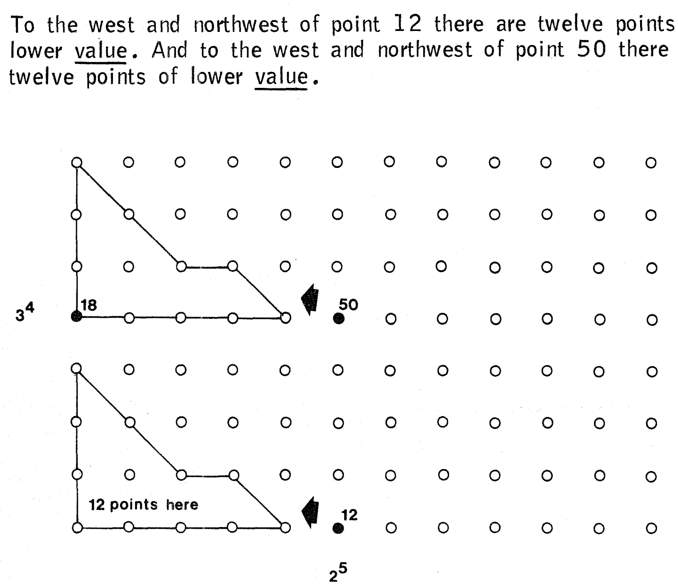


FIGURE 3

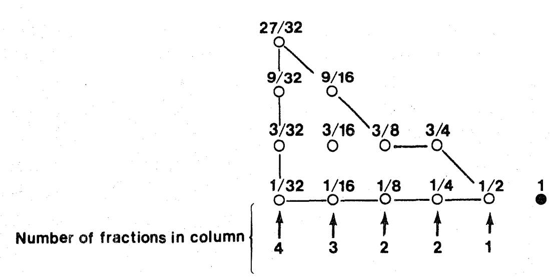


FIGURE 4

To the south and southeast of point 18 there are eighteen points of lower <u>value</u>. And to the south and southeast of point 50 there are eighteen points of lower <u>value</u>.

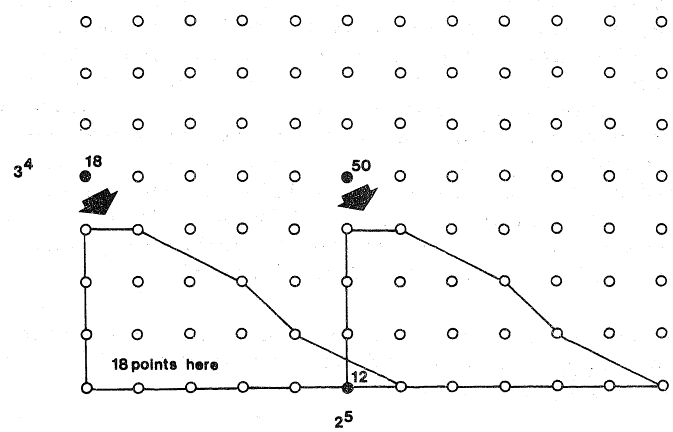


FIGURE 5

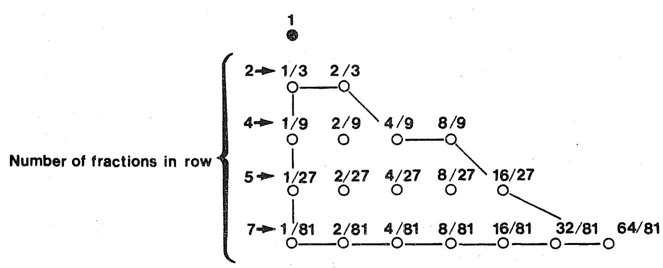


FIGURE 6

If the baseline ordinalities could be computed without recourse to any knowledge of non-baseline ordinalities, a considerable computational effort could be saved. A theorem that will allow us to compute baseline ordinalities directly will now be developed. However, before this new theorem is presented, it will be necessary to expand our nomenclature.

Up to this point, we have been concerned with only one sequence, the 2,3 sequence. All ordinalities were of 2,3 sequence numbers with respect to the 2,3 sequence. However, it is possible to conceive of ordinalities (with respect to the 2,3 sequence) of numbers that are not in this sequence. Take the number 5 as an example. In Table 1, we see that the 2,3 sequence skips from value 4 to value 6. The question "What is the ordinality of 5 with respect to the 2,3 sequence?" is written as: $\mathrm{Ord}(5)_{2,3} = ?$ Please note that the subscripts 2, 3 are written outside of the parentheses, whereas when we previously wrote $\mathrm{Ord}(2^5 3^4)$ there were no subscripts. We could have written $\mathrm{Ord}(2^5 3^4)_{2,3}$ but in order to make the notation more compact, the reference sequence will be specified only when it is different from the enclosed factors or when an ambiguity exists. The convention is also adopted that when the ordinality of a number that is not in a sequence is to be determined with respect to the sequence, the ordinality of the next highest number in the sequence (with respect to the number whose ordinality is to be determined) is the ordinality chosen. For example,

$\text{Ord}(5)_{2,3}$	=	$\text{Ord}(6)_{2,3}$	=	$\text{Ord}(2^1 3^1) = 4$
The ordinality of 5 with respect to the 2,3 sequence.		The next highest number in the 2,3 sequence is 6. That is, 5 "rounds up" to 6 in the 2,3 sequence.		And the ordinality of 6 in the 2,3 sequence is 4, as found in Table 1.

But, $\text{Ord}(4)_{2,3} = \text{Ord}(2^2 3^0) = 3$.

No round up, since the number 4 is found in the 2,3 sequence.

Instead of rounding up in the binary 2,3 sequence, as the example above illustrates, we shall be concerned with rounding up between the two unary sequences: the 2 sequence and the 3 sequence. Thus, from Table 2, we learn that

$\text{Ord}(2^0)_3 = 0$, $\text{Ord}(2^1)_3 = 1$, $\text{Ord}(2^2)_3 = 2$, $\text{Ord}(2^3)_3 = 2$, $\text{Ord}(2^4)_3 = 3$, $\text{Ord}(2^5)_3 = 4$, $\text{Ord}(3^0)_2 = 0$, $\text{Ord}(3^1)_2 = 2$, $\text{Ord}(3^2)_2 = 4$, $\text{Ord}(3^3)_2 = 5$, $\text{Ord}(3^4)_2 = 7$.

TABLE 2

2 Sequence			3 Sequence		
Value	Ordinality	Factors	Value	Ordinality	Factors
1	0	2^0	1	0	3^0
2	1	2^1			
			3	1	3^1
4	2	2^2			
8	3	2^3			
			9	2	3^2
16	4	2^4			
			27	3	3^3
32	5	2^5			
64	6	2^6			
			81	4	3^4
128	7	2^7			

With this nomenclature in mind, we can proceed to the next theorem.

Theorem 2: $\text{Ord}(2^a 3^0) = \sum_{k=0}^{a} \text{Ord}(2^k)_3$.

This theorem states that the ordinality of any point on the horizontal baseline of the 2,3 lattice can be determined from a knowledge of the ordinality of terms in the 3 sequence. And since the ordinality of any term in the 3 sequence is simply its exponent (as can be seen from Table 2), the determination of baseline ordinalities is straightforward.

For example, in the case of the number $2^5 3^0$, this theorem takes the form

$$\text{Ord}(2^5 3^0) = \sum_{k=0}^{5} \text{Ord}(2^k)_3.$$

$$12 = 0 + 1 + 2 + 2 + 3 + 4$$

Table 3 should help clarify this result.

TABLE 3

$k =$	$2^k =$		"Rounded Up" to the Next Highest Number in the 3 Sequence	$\text{Ord}(2^k)_3 =$
0	1	→	$1 = 3^0$	0
1	2	→	$3 = 3^1$	1
2	4	→	$9 = 3^2$	2
3	8	→	$9 = 3^2$	2
4	16	→	$27 = 3^3$	3
5	32	→	$81 = 3^4$	4
			Total =	12

The origin of this result can also be seen in Figure 4. If we list the number of fractions in each *column* to the left of the blacked-in point, we obtain (going right to left), 1, 2, 2, 3, 4. Since each fraction in these columns is less than one and consists of a numerator that is a power of 3 and a denominator that is a power of 2, the question "What is the highest power of 3 in the numerator, for a given power of 2 in the denominator, consistent with a fraction less than one?" can be seen to be related to the question

$$\text{Ord}(2^k)_3 = ?$$

For example, let $k = 5$, then, as previously developed, $\text{Ord}(2^5)_3 = 4$. But the highest power of 3 in the numerator consistent with 32 in the denominator, and a fraction whose overall value is less than one, is 3. That is,

$$3^4/2^5 > 1 > 3^3/2^5, \text{ or } 3^4 > 2 > 3^3.$$

Counting 27/32 and the three fractions beneath it in the leftmost column of Figure 4 gives

$$1 + 3 = 4 \text{ fractions: } 27/32,\ 9/32,\ 3/32,\ 1/32.$$

Thus we see that a numerator power of 3 gives four fractions, since the fraction with the numerator 3^0 must be counted. Therefore, "rounding up" counts this zero exponent term.

The next theorem applies to the vertical baseline.

Theorem 3: $\text{Ord}(2^0 3^b) = \sum_{k=0}^{b} \text{Ord}(3^k)_2.$

This theorem states that the ordinality of any point on the vertical baseline of the 2,3 lattice can be determined from a knowledge of the ordinality of terms in the 2 sequence. And since the ordinality of any term in the 2 sequence is simply its exponent (as can be seen from Table 2), the determination of these ordinalities is straightforward.

For example, in the case of the number $2^0 3^4$, this theorem takes the form

$$\text{Ord}(2^0 3^4) = \sum_{k=0}^{4} \text{Ord}(3^k)_2.$$

$$18 = 0 + 2 + 4 + 5 + 7$$

Table 4 should help clarify this result.

TABLE 4

k =	3^k =		"Rounded Up" to the Next Highest Number in the 2 Sequence	$\text{Ord}(3^k)_2$ =
0	1	→	$1 = 2^0$	0
1	3	→	$4 = 2^2$	2
2	9	→	$16 = 2^4$	4
3	27	→	$32 = 2^5$	5
4	81	→	$128 = 2^7$	7
				Total = 18

The origin of this result can be seen in Figure 6. If we list the number of fractions in each *row* beneath the blacked-in point in Figure 6, we obtain (from top to bottom) 2, 4, 5, 7. Since each fraction in these rows is less than one and consists of a numerator that is a power of 2 and a denominator that is a power of 3, the question "What is the highest power of 2 in the numerator, for a given power of 3 in the denominator, consistent with a fraction less than one?" can be seen to be related to the question

$$\text{Ord}(3^k)_2 = ?$$

For example, let $k = 4$, then, as previously developed, $\text{Ord}(3^4)_2 = 7$. But the highest power of 2 in the numerator consistent with 81 in the denominator, and a fraction whose overall value is less than one, is 6. That is,

$$2^7/3^4 > 1 > 2^6/3^4, \text{ or } 2^7 > 3^4 > 2^6.$$

Counting 64/81 and the six fractions to its left, in the southmost row of Figure 6 gives

$1 + 6 = 7$ fractions: 64/81, 32/81, 16/81, 8/81, 4/81, 2/81, 1/81.

Thus we see that a numerator power of 6 gives seven fractions, since the fraction with numerator 2^0 must be counted. Therefore "rounding up" counts this zero exponent term.

The combination of Theorems 1-3 gives Theorem 4.

<u>*Theorem 4*</u>: $\text{Ord}(2^a 3^b) = ab + \sum_{k=0}^{a} \text{Ord}(2^k)_3 + \sum_{k=0}^{b} \text{Ord}(3^k)_2.$

This is the mathematical equivalent of describing a binary mixture in terms of its pure components.

Evaluating $\text{Ord}(2^5)_3$ has been shown to be equivalent to finding the integral power of 3 (i.e., 3^k) such that

$$3^{k+1} > 2^5 > 3^k.$$

The ordinality was then shown to be one more than k (i.e., ordinality $= k + 1$), since the fraction with zero power in the numerator had to be counted. This problem can be simplified to a linear problem if the logarithms of the terms involved are used. For example, take the above problem. If $2^5 > 3^k$, then

$$5 \log 2 > k \log 3 \quad \text{or} \quad k < 5 \log 2/\log 3.$$

The term on the right of the last inequality must have an integral and a non-integral part (since log 2 and log 3 are independent irrationals). To five places, $5 \log 2/\log 3 = 3.15465$. Since 3^{k+1} was constrained to be greater than 2^5, we can write

$$(k + 1)\log 3 > 5 \log 2.$$

Also, since k was specified to be an integer, we evaluate k as the integral part of $5 \log 2/\log 3$. Therefore, 1 + integral part of $5 \log 2/\log 3$ is the same as the round up of $5 \log 2/\log 3$ to the next positive integer. Since this is also $k + 1$, and $k + 1$ is equal to the ordinality, we can write Theorem 5.

Theorem 5: $\text{Ord}(2^k)_3 = \text{Ord}(k \log 2/\log 3)_1$.

The subscript 1 in Theorem 5 represents a round up process that rounds up to the next highest integer (i.e., We call the sequence of positive integers the 1 sequence. In this sequence, the ordinality of an integer is defined to be its value).

Evaluating $\text{Ord}(3^4)_2$ has been shown to be equivalent to finding the integral power of 2 (i.e., 2^k) such that

$$2^{k+1} > 3^4 > 2^k.$$

The ordinality was then shown to be one more than k (i.e., ordinality $= k + 1$), since the fraction with zero power in the numerator had to be counted. This problem can be simplified to a linear problem if the logarithms of the terms involved are used. For example, take the above problem. If $3^4 > 2^k$, then

$$4 \log 3 > k \log 2 \quad \text{or} \quad k < 4 \log 3/\log 2.$$

The term on the right of the last inequality must have an integral and non-integral part (since log 2 and log 3 are independent irrationals). To five places, $4 \log 3/\log 2 = 6.33985$. Since 2^{k+1} was constrained to be greater than 3^4, we can write

$$(k + 1)\log 2 > 4 \log 3.$$

Also, since k was specified to be an integer, we evaluate k as the integral part of $4 \log 3/\log 2$. Therefore, 1 + integral part of $4 \log 3/\log 2$ is the same as the round up of $4 \log 3/\log 2$ to the next positive integer. Since this is also $k + 1$, and $k + 1$ is equal to the ordinality, we can write Theorem 6.

Theorem 6: $\text{Ord}(3^k)_2 = \text{Ord}(k \log 3/\log 2)_1$.

The combination of Theorems 4-6 gives Theorem 7.

Theorem 7: $$\text{Ord}(2^a3^b) = ab + \sum_{k=0}^{a} \text{Ord}\left(\frac{k}{\log 3/\log 2}\right)_1 + \sum_{k=0}^{b} \text{Ord}\left(\frac{k}{\log 2/\log 3}\right)_1.$$

The lattice for the 2,3 sequence is not unique to numbers of the form 2^a3^b, a, b integers, $a \geq 0$, $b \geq 0$. Instead, it represents the ordinality sequence of all numbers of the form

$$(2^x)^a(3^x)^b,\ a,\ b \text{ integers},\ a \geq 0,\ b \geq 0,\ x > 0.$$

2^a3^b is seen as the special case in which $x = 1$. However, the right side of

Theorem 7 applies to the ordinality of any number in the $2^x, 3^x$ sequence, since

$$\operatorname{Ord}((2^x)^a(3^x)^b)_{2^x,\,3^x} = ab + \sum_{k=0}^{a} \operatorname{Ord}\left(\frac{k}{x \log 3/x \log 2}\right)_1 + \sum_{k=0}^{b} \operatorname{Ord}\left(\frac{k}{x \log 2/x \log 3}\right)_1.$$

And since the x's cancel, we obtain the terms on the right side of the equals sign in Theorem 7.

Therefore, all sequences with component terms of the form $(2^x)^a(3^x)^b$ have in common the fact that their lattice representations are identical. If a lattice does not uniquely specify a sequence, is there anything that it does specify uniquely? The answer lies in Theorem 7. From this theorem, we see that the number log 3/log 2 (and its reciprocal) are uniquely specified by the lattice representation of the $2^x, 3^x$ sequence. Therefore, to generate the lattice associated with any real number N, we generalize the results of Theorem 7, to give

Theorem 8: $\operatorname{Ord}(a, b) = ab + \sum_{k=0}^{a} \operatorname{Ord}\left(\frac{k}{N}\right)_1 + \sum_{k=0}^{b} \operatorname{Ord}\left(\frac{k}{1/N}\right)_1.$

In Theorem 8, $\operatorname{Ord}(a, b)$ is defined as the ordinality of the point at coordinates a, b. Since Theorem 8 is derived from Theorem 1, we can combine the two theorems to obtain

Theorem 9: $\operatorname{Ord}(a, b) = ab + \operatorname{Ord}(a, 0) + \operatorname{Ord}(0, b)$.

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REFERENCES

1. S. Ulam. "Combinatorial Analysis in Infinite Sets and Some Physical Theories." *SIAM Review* 6 (1964):343-55.
2. R. E. Miller, N. Pippenger, A. L. Rosenberg, & L. Snyder. "Optimal 2,3-Trees." *SIAM J. Comput.* 8 (1979):42-59.
3. E. M. Reingold. "A Note on 3-2 Trees." *The Fibonacci Quarterly* 17 (1979): 151-57.
4. B. B. Mandelbrot. *Fractals: Form, Chance, and Dimension.* San Francisco: W. H. Freeman and Co., 1977, p. 56.

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