A LIMITED ARITHMETIC ON SIMPLE CONTINUED FRACTIONS—III

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1. INTRODUCTION

The simple continued fraction expansions of rational multiples of quadratic surds of the form $[a, \dot{b}]$ and $[a, \dot{b}, \dot{c}]$ where the notation is that of Hardy and Wright [1, Ch. 10] were studied in some detail in the first two papers [2] and [3] in this series. Of course, for $a = b = c = 1$, the results concerned the golden ratio, $(1 + \sqrt{5})/2$, and the Fibonacci and Lucas numbers since, as is well known, $(1 + \sqrt{5})/2 = [\dot{1}]$ and the nth convergent to this fraction is F_{n+1}/F_n where F_n denotes the nth Fibonacci number.

In this paper, we consider the simple continued fraction expansions of powers of the surd $\xi = [\dot{a}]$ and of some related surds. We also consider the special case $(1 + \sqrt{5})/2 = [\dot{1}]$ since statements can be made about this surd that are not true in the more general case.

2. PRELIMINARY CONSIDERATIONS

Let a be a positive integer and let the integral sequences

$$\{f_n\}_{n\geq 0} \quad \text{and} \quad \{g_n\}_{n\geq 0}$$

be defined as follows:

$$f_0 = 0,\ f_1 = 1,\ f_n = af_{n-1} + f_{n-2},\ n \geq 2, \tag{1}$$

and

$$g_0 = 2,\ g_1 = a,\ g_n = ag_{n-1} + g_{n-2},\ n \geq 2. \tag{2}$$

These difference equations are easily solved to give

$$f_n = \frac{\xi^n - \overline{\xi}^n}{\sqrt{a^2 + 4}}, \quad n \geq 0, \tag{3}$$

and

$$g_n = \xi^n + \overline{\xi}^n, \quad n \geq 0, \tag{4}$$

where

$$\xi = (a + \sqrt{a^2 + 4})/2 \quad \text{and} \quad \overline{\xi} = (a - \sqrt{a^2 + 4})/2$$

are the two irrational roots of the equation

$$x^2 - ax - 1 = 0. \tag{5}$$

Of course, these results are entirely analogous to those for the Fibonacci and Lucas sequences, $\{F_n\}$ and $\{L_n\}$, and many of the Fibonacci and Lucas results translate immediately into corresponding results for $\{f_n\}$ and $\{g_n\}$. For example, if we solve (3) and (4) for f_n and g_n in terms of ξ^n and $\overline{\xi}^n$, we obtain

$$\xi^n = \frac{g_n + f_n\sqrt{a^2 + 4}}{2} \tag{6}$$

and

$$\overline{\xi}^n = \frac{g_n - f_n\sqrt{a^2 + 4}}{2}. \tag{7}$$

Also, since

$$\xi\overline{\xi} = \frac{a + \sqrt{a^2 + 4}}{2} \quad \frac{a - \sqrt{a^2 + 4}}{2} = \frac{a^2 - (a^2 + 4)}{4} = -1,$$

it follows that

$$(-1)^n = \xi^n\overline{\xi}^n = \frac{g_n^2 - (a^2 + 4)f_n^2}{4} \tag{8}$$

and also that

$$\bar{\xi}^n = \frac{(-1)^n}{\xi^n}. \tag{9}$$

We exhibit the first few terms of $\{f_n\}$ and $\{g_n\}$ in the following table and note that both sequences are strictly increasing for $n \geq 2$.

n	0	1	2	3	4	5
f_n	0	1	a	$a^2 + 1$	$a^3 + 2a$	$a^4 + 3a^2 + 1$
g_n	2	a	$a^2 + 2$	$a^3 + 3a$	$a^4 + 4a^2 + 2$	$a^5 + 5a^3 + 5a$

The following lemmas, of some interest in their own right, will prove useful in obtaining the main results.

Lemma 1: For $n > 1$,

(a) $[f_{2n}\sqrt{a^2 + 4}] = g_{2n} - 1$,

(b) $[f_{2n-1}\sqrt{a^2 + 4}] = g_{2n-1}$.

Proof of (a): By (8),

$$(a^2 + 4)f_{2n}^2 = g_{2n}^2 - 4 > g_{2n}^2 - 2g_{2n} + 1$$

since $2g_{2n} - 1 > 4$ for $n > 1$. Therefore,

$$f_{2n}\sqrt{a^2 + 4} > g_{2n} - 1 \tag{10}$$

for $n > 1$. On the other hand

$$g_{2n}^2 > g_{2n}^2 - 4 = (a^2 + 4)f_{2n}^2,$$

so that

$$g_{2n} > f_{2n}\sqrt{a^2 + 4} \tag{11}$$

for all n. But (10) and (11) together imply that

$$[f_{2n}\sqrt{a^2 + 4}] = g_{2n} - 1$$

for $n > 1$ as claimed.

Proof of (b): Again by (8),

$$(a^2 + 4)f_{2n-1}^2 = g_{2n-1}^2 + 4$$

so that

$$f_{2n-1}\sqrt{a^2 + 4} = \sqrt{g_{2n-1}^2 + 4} > g_{2n-1}. \tag{12}$$

Also, for $n > 1$,

$$(g_{2n-1} + 1)^2 = g_{2n-1}^2 + 2g_{2n-1} + 1 > g_{2n-1}^2 + 4 = (a^2 + 4)f_{2n-1}^2$$

so that

$$g_{2n-1} + 1 > f_{2n-1}\sqrt{a^2 + 4}. \tag{13}$$

Thus, from (12) and (13),

$$[f_{2n-1}\sqrt{a^2 + 4}] = g_{2n-1}$$

and the proof is complete.

Lemma 2: For $n > 1$,

(a) $[g_{2n}\sqrt{a^2 + 4}] = (a^2 + 4)f_{2n}$,

(b) $[g_{2n-1}\sqrt{a^2 + 4}] = (a^2 + 4)f_{2n-1} - 1$.

Proof: The argument here is quite similar to that for Lemma 1 and is thus omitted.

3. THE GENERAL CASE

The first two theorems give the simple continued fraction expansions of ξ^n and $\bar{\xi}^n$.

Theorem 3: For $n \geq 1$,

(*a*) $\xi^{2n-1} = [\dot{g}_{2n-1}]$

(*b*) $\xi^{2n} = [g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$.

Proof: Since it is well known that $[\dot{g}_{2n-1}]$ converges, we may set

$$x = [\dot{g}_{2n-1}] = g_{2n-1} + \frac{1}{x}.$$

Thus,

$$x^2 - xg_{2n-1} - 1 = 0$$

and hence, using (8) and (6),

$$x = \frac{g_{2n-1} + \sqrt{g_{2n-1}^2 + 4}}{2} = \frac{g_{2n-1} + f_{2n-1}\sqrt{a^2 + 4}}{2} = \xi^{2n-1},$$

and this proves (*a*). Also, set

$$y = [\dot{1}, \dot{g}_{2n} - 2] = 1 + \frac{1}{g_{2n} - 2 + 1/y}$$

so that

$$y^2(g_{2n} - 2) - y(g_{2n} - 2) - 1 = 0.$$

Then,

$$y = \frac{g_{2n} - 2 + \sqrt{(g_{2n} - 2)^2 + 4(g_{2n} - 2)}}{2(g_{2n} - 2)} = \frac{g_{2n} - 2 + \sqrt{g_{2n}^2 - 4}}{2(g_{2n} - 2)}$$

and, again using (8) and (6),

$$[g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2] = g_{2n} - 1 + \frac{1}{y} = g_{2n} - 1 + \frac{2(g_{2n} - 2)}{g_{2n} - 2 + \sqrt{g_{2n}^2 - 4}}$$

$$= \frac{g_{2n} + \sqrt{g_{2n}^2 - 4}}{2} = \frac{g_{2n} + f_{2n}\sqrt{a^2 + 4}}{2} = \xi^{2n}$$

as claimed.

Theorem 4: For $n \geq 1$,

(*a*) $\bar{\xi}^{2n-1} = [-1, 1, g_{2n-1} - 1, \dot{g}_{2n-1}]$,

(*b*) $\bar{\xi}^{2n} = [0, g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$.

Proof: From (9) we have immediately that

$$\bar{\xi}^{2n} = \frac{1}{\xi^{2n}} \quad \text{and} \quad \bar{\xi}^{2n-1} = -\frac{1}{\xi^{2n-1}}.$$

Since $\xi^{2n} = [g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$ from the preceding theorem, it follows that $\bar{\xi}^{2n} = [0, g_{2n} - 1, \dot{1}, \dot{g}_{2n} - 2]$ as claimed. We also have from the preceding theorem that

$$\xi^{2n-1} = [\dot{g}_{2n-1}]$$

so that

$$\frac{1}{\xi^{2n-1}} = [0, \dot{g}_{2n-1}].$$

But it is well known that if α is real, $\alpha = [a_0, a_1, a_2, \ldots]$ and $a_1 > 1$, then $-\alpha = [-(a_0 + 1), 1, a_1 - 1, a_2, \ldots]$. Thus, it follows that

$$\bar{\xi}^{2n-1} = -\frac{1}{\xi^{2n-1}} = [-1, 1, g_{2n-1} - 1, \dot{g}_{2n-1}]$$

and the proof is complete.

Recall that two real numbers α and β are said to be equivalent if there exist integers A, B, C, and D such that $|AD - BC| = 1$ and

$$\alpha = \frac{A\beta + B}{C\beta + D}.$$

We indicate this equivalency by writing $\alpha \sim \beta$. Recall too that $\alpha \sim \beta$ if and only if the simple continued fraction expansions of α and β are identical from some point on. With this in mind we state the following corollary, which follows immediately from the two preceding theorems.

Corollary 5: If n is any positive integer, then $\xi^n \sim \bar{\xi}^n$.

Noting the form of the surds

$$\xi^n = \frac{g_n + f_n\sqrt{a^2 + 4}}{2} \quad \text{and} \quad \bar{\xi}^n = \frac{g_n - f_n\sqrt{a^2 + 4}}{2},$$

it seemed reasonable also to investigate the simple continued fraction expansions of surds of the form

$$\frac{ag_m \pm f_n\sqrt{a^2 + 4}}{2}, \frac{af_m \pm g_n\sqrt{a^2 + 4}}{2},$$

and so on. It turned out to be impossible to give explicit general expansions of these surds valid for all a, m, and n, but it was possible to obtain the following more modest results.

Theorem 6: Let a be as above and let m, n, and r be positive integers with $m \equiv r \equiv 0 \pmod 3$ or $mr \not\equiv 0 \pmod 3$ if a is odd. Also, let $\{u_n\}$ be either of the sequences $\{f_n\}$ or $\{g_n\}$ and similarly for $\{v_n\}$ and $\{w_n\}$. Then

$$\frac{au_m + w_n\sqrt{a^2 + 4}}{2} \sim \frac{av_r + w_n\sqrt{a^2 + 4}}{2}$$

and

$$\frac{au_m + w_n\sqrt{a^2 + 4}}{2} \sim \frac{av_r - w_n\sqrt{a^2 + 4}}{2}.$$

Proof: We first note that, if a is odd, $f_n \equiv g_n \equiv 0 \pmod 2$ if $n \equiv 0 \pmod 3$ and $f_n \equiv g_n \equiv 1 \pmod 2$ if $n \not\equiv 0 \pmod 3$. Thus $u_m \pm v_r \equiv 0 \pmod 2$ if and only if $m \equiv r \equiv 0 \pmod 3$ or $mr \not\equiv 0 \pmod 3$. To show the first equivalence, let $A = 1$, $B = a(u_m - v_r)/2$, $C = 0$, and $D = 1$. Then B is an integer, since either a or $u_m - v_r$ is divisible by 2 by the above. Moreover,

$$\frac{A \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + B}{C \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + D} = \frac{1 \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + \dfrac{a(u_m - v_r)}{2}}{0 \cdot \dfrac{av_r + w_n\sqrt{a^2 + 4}}{2} + 1}$$

$$= \frac{au_m + w_n\sqrt{a^2 + 4}}{2},$$

and this shows the first equivalence claimed, since $|AD - BC| = 1$. Since the proof of the second equivalence is the same, it is omitted here.

Corollary 7: If m and n are positive integers, then the surds in the following two sets are equivalent:

$$(a)\quad \frac{af_m + g_n\sqrt{a^2+4}}{2},\ \frac{af_m - g_n\sqrt{a^2+4}}{2},\ \frac{ag_m + g_n\sqrt{a^2+4}}{2},\ \frac{ag_m - g_n\sqrt{a^2+4}}{2},$$

and

$$(b)\quad \frac{ag_m + f_n\sqrt{a^2+4}}{2},\ \frac{ag_m - f_n\sqrt{a^2+4}}{2},\ \frac{af_m + f_n\sqrt{a^2+4}}{2},\ \frac{af_m - f_n\sqrt{a^2+4}}{2}.$$

Proof: The first of the above equivalences follows immediately from the second equivalence in Theorem 6 by setting $r = m$, $u_m = f_m$, and $w_n = g_n$ and the others are obtained similarly.

Theorem 8: Let a be as above and let $m > 0$ and $n > 2$ denote integers. Also, let $x = af_m + (a^2+4)f_n$ and $y = ag_m + (a^2+4)f_n$. Then

$$\frac{af_m + g_n\sqrt{a^2+4}}{2} = [a_0, \dot{a}_1, \ldots, \dot{a}_r] \text{ and } \frac{ag_m + g_n\sqrt{a^2+4}}{2} = [b_0, \dot{a}_1, \ldots, \dot{a}_r]$$

where the vector $(a_1, a_2, \ldots, a_{r-1})$ is symmetric and

$$a_r = 2a_0 - af_m = 2b_0 - ag_m.$$

Also

$$a_0 = \frac{af_m + (a^2+4)f_n - b}{2} = \frac{x-b}{2} \text{ and } b_0 = \frac{ag_m + (a^2+4)f_n - c}{2} = \frac{y-c}{2}$$

where

$$\begin{aligned}
b &= 0 \text{ if } n \equiv x \equiv 0 \pmod 2,\\
b &= 1 \text{ if } x \equiv 1 \pmod 2,\\
b &= 2 \text{ if } n - 1 \equiv x \equiv 0 \pmod 2,\\
c &= 0 \text{ if } n \equiv y \equiv 0 \pmod 2,\\
c &= 1 \text{ if } y \equiv 1 \pmod 2, \text{ and}\\
c &= 2 \text{ if } n - 1 \equiv y \equiv 0 \pmod 2.
\end{aligned}$$

Proof: Let $\nu = (af_m + g_n\sqrt{a^2+4})/2$. Then, by Lemma 2,

$$a_0 = [\nu] = \left[\frac{af_m + g_n\sqrt{a^2+4}}{2}\right] = \left[\frac{af_m + [g_n\sqrt{a^2+4}]}{2}\right]$$

$$= \begin{cases} \left[\dfrac{af_m + (a^2+4)f_n}{2}\right], & n \text{ even}, n > 2\\[2ex] \left[\dfrac{af_m + (a^2+4)f_n - 1}{2}\right], & n \text{ odd}, n > 2 \end{cases}$$

$$= \frac{af_m + (a^2+4)f_n - b}{2},$$

where it is clear that

$$b = 0 \text{ if } n \equiv x \equiv 0 \pmod 2,$$
$$b = 1 \text{ if } x \equiv 1 \pmod 2, \text{ and}$$
$$b = 2 \text{ if } n - 1 \equiv x \equiv 0 \pmod 2.$$

Thus a_0 is as claimed. Moreover, $0 < \nu - a_0 < 1$, so if we set $\nu_1 = 1/(\nu - a_0)$, it follows that

$$\nu_1 > 1. \tag{14}$$

Taking conjugates, we have that

$$\overline{\nu}_1 = \frac{1}{\dfrac{af_m - g_n\sqrt{a^2 + 4}}{2} - \dfrac{af_m + (a^2 + 4)f_n - b}{2}} = \frac{-2}{(a^2 + 4)f_n - b + g_n\sqrt{a^2 + 4}} \tag{15}$$

and it is clear that

$$-1 < \overline{\nu}_1 < 0, \tag{16}$$

since a and n are both positive. But (14) and (16) together show that ν_1 is reduced and so, by [4, p. 101], for example, has a purely periodic simple continued fraction expansion $[\dot{a}_1, a_2, \ldots, \dot{a}_r]$. Thus

$$\nu = \frac{af_m + g_n\sqrt{a^2 + 4}}{2} = [a_0, \nu_1] = [a_0, \dot{a}_1, a_2, \ldots, \dot{a}_r]. \tag{17}$$

On the other hand, again by [4, p. 93],

$$-\frac{1}{\overline{\nu}_1} = [\dot{a}_r, a_{r-1}, \ldots, \dot{a}_1]. \tag{18}$$

But then

$$-\frac{1}{\overline{\nu}_1} = \frac{(a^2 + 4)f_n - b + g_n\sqrt{a^2 + 4}}{2}$$
$$= \frac{af_m + g_n\sqrt{a^2 + 4}}{2} + \frac{af_m + f_n(a^2 + 4) - b}{2} - \frac{2af_m}{2}$$
$$= \nu + a_0 - af_m = [2a_0 - af_m, \dot{a}_1, a_2, \ldots, \dot{a}_r].$$

Comparing (18) and (19), we immediately have that $2a_0 - af_m = a_r$, $a_1 = a_{r-1}$, $a_2 = a_{r-2}, \ldots, a_{r-1} = a_1$. This completes the proof for ν. The proof for $\mu = (ag_m + g_n\sqrt{a^2 + 4})/2$ is similar and is omitted.

The following theorem is similar to Theorem 8 and is stated without proof.

Theorem 9: Let a be as above and let $m > 0$ and $n > 2$ denote integers. Also, let $x = af_m + g_n$ and $y = ag_m + g_n$. Then

$$\frac{af_m + f_n\sqrt{a^2 + 4}}{2} = [c_0, \dot{c}_1, \ldots, \dot{c}_r] \text{ and } \frac{ag_m + f_n\sqrt{a^2 + 4}}{2} = [d_0, \dot{c}_1, \ldots, \dot{c}_r]$$

where the vector $(c_1, c_2, \ldots, c_{r-1})$ is symmetric and

$$c_r = 2c_0 - af_m = 2d_0 - ag_m.$$

Also

$$c_0 = \frac{af_m + g_n - b}{2} = \frac{x - b}{2} \text{ and } d_0 = \frac{ag_m + g_n - c}{2} = \frac{y - c}{2}$$

where

$$b = 0 \text{ if } n - 1 \equiv x \equiv 0 \pmod 2,$$
$$b = 1 \text{ if } x \equiv 1 \pmod 2,$$
$$b = 2 \text{ if } n \equiv x \equiv 0 \pmod 2,$$

$$c = 0 \text{ if } n - 1 \equiv y \equiv 0 \pmod 2,$$
$$c = 1 \text{ if } y \equiv 1 \pmod 2, \text{ and}$$
$$c = 2 \text{ if } n \equiv y \equiv 0 \pmod 2.$$

Theorem 10: Let m, n, and a denote positive integers and let $\{u_n\}$ and $\{v_n\}$ be as in Theorem 6. Also, let

$$\frac{au_m + v_n\sqrt{a^2 + 4}}{2} = [a_0, \dot{a}_1, \ldots, \dot{a}_r].$$

(a) If $a_1 > 1$, then

$$\frac{au_m - v_n\sqrt{a^2 + 4}}{2} = [-a_0 + au_m - 1, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1].$$

(b) If $a_1 = 1$, then

$$\frac{au_m - v_n\sqrt{a^2 + 4}}{2} = [-a_0 + au_m - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_2, \dot{a}_1].$$

Proof of (a): Let $\eta = (au_m + v_n\sqrt{a^2 + 4})/2$. Then by hypothesis,

$$\eta = [a_0, \dot{a}_1, \ldots, \dot{a}_r]$$

and

$$\frac{1}{\dfrac{1}{\eta - a_0} - a_1} = [\dot{a}_2, \ldots, a_r, \dot{a}_1].$$

But then

$$\begin{aligned}
&[-a_0 + au_m - 1, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1]\\
&= -a_0 + au_m - 1 + \cfrac{1}{1 + \cfrac{1}{a_1 - 1 + \cfrac{1}{\cfrac{1}{\cfrac{1}{\eta - a_0} - a_1}}}}\\
&= au_m - \eta\\
&= \frac{au_m - u_n\sqrt{a^2 + 4}}{2}
\end{aligned}$$

as claimed.

Proof of (b): If $a_1 = 1$, the above analysis still holds except that $a_1 - 1 = 0$, so that we no longer have a simple continued fraction. But then, we immediately have that

$$\begin{aligned}
\frac{au_m - v_n\sqrt{a^2 + 4}}{2} &= [-a_0 + au_m - 1, 1, 0, \dot{a}_2, \ldots, a_r, \dot{a}_1]\\
&= [-a_0 + au_m - 1, 1, 0, a_2, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2]\\
&= [-a_0 + au_m - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2]
\end{aligned}$$

and the proof is complete.

Interestingly, it appears that the integer r in the above results is always even but we have not been able to show this. Also, while it first seemed that r was bounded for all a, m, and n, this now appears not to be the case. For example, if $a = 4$ and we consider the related surd, $f_m + g_n\sqrt{5}$, r is sometimes quite large and appears to grow with n without bound. On the other hand, if $a = 2$, and we consider the related surds, $f_m + g_n\sqrt{2}$ and $g_m + g_n\sqrt{2}$, it can no doubt be shown that r equals 2 or 4 according as n is even or odd, and that for $f_m + f_n\sqrt{2}$ and $g_m + f_n\sqrt{2}$, r equals 1 or 2 as n is odd or even.

4. SPECIAL RESULTS WHEN $a = 1$

Of course, all the preceding theorems hold when $a = 1$, in which case

$$\xi = (1 + \sqrt{5})/2,\ f_n = F_n,\ \text{and } g_n = L_n$$

for all n. On the other hand, in this special case, far more specific results can be obtained as the following theorems show. Note especially that throughout the remainder of the paper we use m and k to denote a positive integer and a nonnegative integer, respectively.

Theorem 11: If $3\nmid m$ and $n = 2 + 6k$ or $4 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5F_n}{2}, \dot{F}_n, 5\dot{F}_n\right]$$

and

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n}{2}, \dot{F}_n, 5\dot{F}_n\right].$$

Proof: It is immediate from the hypotheses and Theorem 8 that

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5F_n}{2}, \dot{a}_1, \ldots, \dot{a}_r\right]$$

and that

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n}{2}, \dot{a}_1, \ldots, \dot{a}_r\right].$$

Let

$$x = \cfrac{1}{F_n + \cfrac{1}{5F_n + x}}.$$

Then

$$x^2 + 5F_nx - 5 = 0,$$

and, since x is clearly positive and $5F_n^2 + 4 = L_n^2$ is a special case of (8),

$$x = \frac{-5F_n + \sqrt{25F_n^2 + 20}}{2} = \frac{-5F_n + L_n\sqrt{5}}{2}.$$

But then, $$\left[\frac{F_m + 5F_n}{2}, \dot{F}_n, 5\dot{F}_n\right] = \frac{F_m + 5F_n}{2} + \frac{-5F + L_n\sqrt{5}}{2} = \frac{F_m + L_n\sqrt{5}}{2},$$

and similarly,

$$\left[\frac{L_m + 5F_n}{2}, \dot{F}_n, 5\dot{F}_n\right] = \frac{L_m + L_n\sqrt{5}}{2}$$

as claimed.

Theorem 12: If $3\nmid m$ and $n = 5 + 6k$ or $7 + 6k$, or if $3|m$ and $n = 3 + 6k$, then

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5L_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F}_n - 2\right]$$

and

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F}_n - 2\right].$$

Proof: Again it is immediate from the hypotheses and Theorem 8 that

$$\frac{F_m + L_n\sqrt{5}}{2} = \left[\frac{F_m + 5F_n - 2}{2}, \dot{a}_1, \ldots, \dot{a}_r\right]$$

and that

$$\frac{L_m + L_n\sqrt{5}}{2} = \left[\frac{L_m + 5F_n - 2}{2}, \dot{a}_1, \ldots, \dot{a}_r\right].$$

Then, since n is odd, we have from Theorem 3 of [2] that

$$x = [\dot{1}, F_n - 2, 1, 5\dot{F}_n - 2] = \frac{L_n + L_n\sqrt{5}}{2} - L_{n+1} + 1.$$

Thus,

$$\left[\frac{F_m + 5F_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F}_n - 2\right] = \frac{F_m + 5F_n - 2}{2} + x$$
$$= \frac{F_m + 5F_n - 2}{2} + \frac{L_n + L_n\sqrt{5} - 2L_{n+1} + 2}{2}$$
$$= \frac{F_m + 5F_n - 2}{2} + \frac{-5F_n + L_n\sqrt{5} + 2}{2}$$
$$= \frac{F_m + L_n\sqrt{5}}{2}.$$

Similarly,

$$\left[\frac{L_m + 5F_n - 2}{2}, \dot{1}, F_n - 2, 1, 5\dot{F}_n - 2\right] = \frac{L_m + L_n\sqrt{5}}{2}$$

and the proof is complete.

Theorem 13: If $3\nmid m$ and $n = 6 + 6k$ or $9 + 6k$, or if $3|m$ and $n = 4 + 6k$, $5 + 6k$, $7 + 6k$, or $8 + 6k$, then

$$\frac{F_m + L_n\sqrt{5}}{2} = [a_0, \dot{a}_1, \ldots, \dot{a}_r] \quad \text{and} \quad \frac{L_m + L_n\sqrt{5}}{2} = [b_0, \dot{a}_1, \ldots, \dot{a}_r]$$

with $a_0 = (F_m + 5F_n - 1)/2$, $b_0 = (L_m + 5F_n - 1)/2$, $a_r = 5F_n - 1$, and where the vector $(a_1, \ldots, a_{r-1})$ is symmetric.

Proof: This is an immediate consequence of Theorem 8.

The only surds of the form $(F_m + L_n\sqrt{5})/2$ and $(L_m + L_n\sqrt{5})/2$ not treated by the above theorems are when $3\nmid m$ and $n = 1$ or 3, and when $3|m$ and $n = 1$ or 2. For these cases, the results are as follows.

Theorem 14:

(*a*) If $3\nmid m$, then

$$\frac{F_m + L_1\sqrt{5}}{2} = \left[\frac{F_m + 1}{2}, \dot{1}\right],$$
$$\frac{L_m + L_1\sqrt{5}}{2} = \left[\frac{L_m + 1}{2}, \dot{1}\right],$$
$$\frac{F_m + L_3\sqrt{5}}{2} = \left[\frac{F_m + 7}{2}, \dot{1}, 34. 1, \dot{7}\right],$$

and

$$\frac{L_m + L_3\sqrt{5}}{2} = \left[\frac{L_m + 7}{2}, \dot{1}, 34, 1, \dot{7}\right].$$

(*b*) If $3|m$, then

$$\frac{F_m + L_1\sqrt{5}}{2} = \left[\frac{F_m + 2}{2}, \dot{8}, \dot{2}\right],$$

$$\frac{L_m + L_1\sqrt{5}}{2} = \left[\frac{L_m + 2}{2}, \dot{8}, \dot{2}\right],$$

$$\frac{F_m + L_2\sqrt{5}}{2} = \left[\frac{F_m + 6}{2}, \dot{2}, 1, 4, 1, 2, \dot{6}\right],$$

and

$$\frac{L_m + L_2\sqrt{5}}{2} = \left[\frac{L_m + 6}{2}, \dot{2}, 1, 4, 1, 2, \dot{6}\right].$$

Theorem 15:

(*a*) If $3\nmid m$ and $n = 4 + 6k$ or $n = 8 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - F_n - 2}{2}, 1, F_n - 1, 5\dot{F}_n, \dot{F}_n\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - F_n - 2}{2}, 1, F_n - 1, 5\dot{F}_n, \dot{F}_n\right].$$

(*b*) If $3\nmid m$ and $n = 5 + 6k$ or $7 + 6k$, or if $3|m$ and $n = 9 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - 5F_n}{2}, F_n - 1, \dot{1}, 5F_n - 2, 1, F_n \overset{\cdot}{-} 2\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - 5F_n}{2}, F_n - 1, \dot{1}, 5F_n - 2, 1, F_n \overset{\cdot}{-} 2\right].$$

(*c*) Let $(F_m + L_n\sqrt{5})/2 = [a_0, \dot{a}_1, \ldots, \dot{a}_r]$ as is always the case from Theorem 8. If $3\nmid m$ and $n = 6 + 6k$, or if $3|m$ and $n = 4 + 6k$ or $8 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - 5F_n - 1}{2}, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - 5F_n - 1}{2}, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2\right].$$

And if $3\nmid m$ and $n = 9 + 6k$, or if $3|m$ and $n = 5 + 6k$ or $7 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = \left[\frac{F_m - 5F_n - 1}{2}, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1\right],$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = \left[\frac{L_m - 5F_n - 1}{2}, 1, a_1 - 1, \dot{a}_2, \ldots, a_r, \dot{a}_1\right].$$

The preceding theorem omits the cases when $n = 1, 2,$ or 3. These cases are treated in the following result, which is also stated without proof.

Theorem 16:

(*a*) If $3\nmid m$, then

$$\frac{F_m - L_1\sqrt{5}}{2} = \left[\frac{F_m - 3}{2},\ 2,\ \dot{1}\right],$$

$$\frac{L_m - L_1\sqrt{5}}{2} = \left[\frac{L_m - 3}{2},\ 2,\ \dot{1}\right],$$

$$\frac{F_m - L_2\sqrt{5}}{2} = \left[\frac{F_m - 7}{2},\ 6,\ \dot{1},\ \dot{5}\right],$$

$$\frac{L_m - L_2\sqrt{5}}{2} = \left[\frac{L_m - 7}{2},\ 6,\ \dot{1},\ \dot{5}\right],$$

$$\frac{F_m - L_3\sqrt{5}}{2} = \left[\frac{F_m - 9}{2},\ 35,\ \dot{1},\ 7,\ 1,\ \dot{34}\right],$$

and

$$\frac{L_m - L_3\sqrt{5}}{2} = \left[\frac{L_m - 9}{2},\ 35,\ \dot{1},\ 7,\ 1,\ \dot{34}\right].$$

(*b*) If $3|m$, then

$$\frac{F_m - L_1\sqrt{5}}{2} = \left[\frac{F_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right],$$

$$\frac{L_m - L_1\sqrt{5}}{2} = \left[\frac{L_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right],$$

$$\frac{F_m - L_2\sqrt{5}}{2} = \left[\frac{F_m - 8}{2},\ 1,\ 1,\ \dot{1},\ 4,\ 1,\ 2,\ 6,\ \dot{2}\right],$$

$$\frac{L_m - L_2\sqrt{5}}{2} = \left[\frac{L_m - 8}{2},\ 1,\ 1,\ \dot{1},\ 4,\ 1,\ 2,\ 6,\ \dot{2}\right],$$

$$\frac{F_m - L_3\sqrt{5}}{2} = \left[\frac{F_m - 10}{2},\ 1,\ 1,\ \dot{8},\ \dot{2}\right],$$

and

$$\frac{L_m - L_3\sqrt{5}}{2} = \left[\frac{L_m - 10}{2},\ 1,\ 1,\ \dot{8},\ \dot{2}\right].$$

We close with two theorems which give the expansions for $(F_m \pm F_n\sqrt{5})/2$ and $(L_m \pm F_n\sqrt{5})/2$ for all positive integers m and n. Again, these theorems are stated without proof.

Theorem 17

(*a*) If $3\nmid m$ and $n = 1 + 6k$ or $5 + 6k$, or if $3|m$ and $n = 3 + 6k$, then

$$\frac{F_m + F_n\sqrt{5}}{2} = \left[\frac{F_m + L_n}{2},\ \dot{L}_n\right]$$

and

$$\frac{L_m + F_n\sqrt{5}}{2} = \left[\frac{L_m + L_n}{2},\ \dot{L}_n\right].$$

(*b*) If $3\nmid m$ and $n = 2 + 6k$ or $4 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m + F_n\sqrt{5}}{2} = \left[\frac{F_m + L_n - 2}{2},\ \dot{1},\ \dot{L}_n - 2\right]$$

and

$$\frac{L_m + F_n\sqrt{5}}{2} = \left[\frac{L_m + L_n - 2}{2}, \dot{1}, \dot{L_n} - 2\right].$$

(c) Let $(F_m + F_n\sqrt{5})/2 = [a_0, a_1, \ldots, a\ ]$. If $3\nmid m$ and $n = 3 + 6k$ or $6 + 6k$, or if $3|m$ and $n = 2 + 6k$, $4 + 6k$, $5 + 6k$, or $7 + 6k$, then

$$\frac{F_m + F_n\sqrt{5}}{2} = \left[\frac{F_m + L_n - 1}{2}, \dot{a}_1, \ldots, a_{r-1}, \dot{L_n} - 1\right]$$

and

$$\frac{L_m + F_n\sqrt{5}}{2} = \left[\frac{L_m + L_n - 1}{2}, \dot{a}_1, \ldots, a_{r-1}, \dot{L_n} - 1\right]$$

and the vector $(a_1, \ldots, a_{r-1})$ is symmetric.

(d) If $3|m$, then

$$\frac{F_m + F_1\sqrt{5}}{2} = \left[\frac{F_m + 2}{2}, \dot{8}, \dot{2}\right]$$

and

$$\frac{L_m + F_1\sqrt{5}}{2} = \left[\frac{L_m + 2}{2}, \dot{8}, \dot{2}\right].$$

Theorem 18

(a) If $3\nmid m$ and $n = 5 + 6k$ or $7 + 6k$, or if $3|m$ and $n = 3 + 6k$, then

$$\frac{F_m - F_n\sqrt{5}}{2} = \left[\frac{F_m - L_n - 2}{2}, 1, L_n - 1, \dot{L}_n\right]$$

and

$$\frac{L_m - F_n\sqrt{5}}{2} = \left[\frac{L_m - L_n - 2}{2}, 1, L_n - 1, \dot{L}_n\right].$$

(b) If $3\nmid m$ and $n = 2 + 6k$ or $4 + 6k$, or if $3|m$ and $n = 6 + 6k$, then

$$\frac{F_m - F_n\sqrt{5}}{2} = \left[\frac{F_m - L_n}{2}, L_n - 1, \dot{1}, \dot{L_n} - 2\right]$$

and

$$\frac{L_m - F_n\sqrt{5}}{2} = \left[\frac{L_m - L_n}{2}, L_n - 1, \dot{1}, \dot{L_n} - 2\right].$$

(c) Let

$$(F_m + F_n\sqrt{5})/2 = [a_0, \dot{a}_1, \ldots, \dot{a}_r]$$

and let

$$(L_m + L_n\sqrt{5})/2 = [b_0, \dot{a}_1, \ldots, \dot{a}_r].$$

If $3\nmid m$ and $n = 3 + 6k$, or if $3|m$ and $n = 5 + 6k$ or $7 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = [a_0 - a_r - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2]$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = [b_0 - a_r - 1, a_2 + 1, \dot{a}_3, \ldots, a_r, a_1, \dot{a}_2].$$

If $3\nmid m$ and $n = 6 + 6k$ or if $3|m$ and $n = 4 + 6k$ or $8 + 6k$, then

$$\frac{F_m - L_n\sqrt{5}}{2} = [a_0 - a_r - 1, 1, 1, \dot{a}_2, \ldots, a_r, \dot{a}_1]$$

and

$$\frac{L_m - L_n\sqrt{5}}{2} = [b_0 - a_r - 1,\ 1,\ 1,\ \dot{a}_2,\ \ldots,\ a_r,\ \dot{a}_1].$$

(*d*) If $3\nmid m$, then

$$\frac{F_m - F_1\sqrt{5}}{2} = \left[\frac{F_m - 3}{2},\ 2,\ \dot{1}\right]$$

and

$$\frac{L_m - F_1\sqrt{5}}{2} = \left[\frac{L_m - 3}{2},\ 2,\ \dot{1}\right].$$

If $3|m$, then

$$\frac{F_m - F_1\sqrt{5}}{2} = \frac{F_m - F_2\sqrt{5}}{2} = \left[\frac{F_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right]$$

and

$$\frac{L_m - F_1\sqrt{5}}{2} = \frac{F_m - F_2\sqrt{5}}{2} = \left[\frac{L_m - 4}{2},\ 1,\ 7,\ \dot{2},\ \dot{8}\right].$$

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BENFORD'S LAW FOR FIBONACCI AND LUCAS NUMBERS

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Benford's law states that the probability that a random decimal begins (on the left) with the digit p is $\log_{10}(p+1)/p$. Recent computations by J. Wlodarski [3] and W. G. Brady [1] show that the Fibonacci and Lucas numbers tend to obey both this law and its natural extension: the probability that a random decimal in base b begins with p is $\log_b(p+1)/p$. By using the fact that the terms of the Fibonacci and Lucas sequences have exponential growth, we prove the following result.

Theorem: The Fibonacci and Lucas numbers obey the extended Benford's law. More precisely, let $b \geq 2$ and let p satisfy $1 \leq p \leq b - 1$. Let $A_p(N)$ be the number of Fibonacci (or Lucas) numbers F_n (or L_n) with $n \leq N$ and whose first digit in base b is p. Then

$$\lim_{N\to\infty} \frac{1}{N} A_p(N) = \log_b\left(\frac{p+1}{p}\right).$$

Proof: We give the proof for the Fibonacci sequence. The proof for the Lucas sequence is similar.

Throughout the proof, log will mean $\log_b$. Also, $\langle x\rangle = x - [x]$ will denote the fractional part of x.

Let $\alpha = \frac{1}{2}(1+\sqrt{5})$, so $F_n = (\alpha^n - (-\alpha)^{-n})/\sqrt{5}$. We first need the following:

Lemma: The sequence $\{\langle n \log \alpha\rangle\}_{n=1}^{\infty}$ is uniformly distributed mod 1.