REFERENCES

1. D. E. Knuth. *The Art of Computer Programming I: Fundamental Algorithms*. Reading, Mass.: Addison-Wesley, 1968.
2. D. E. Knuth. *The Art of Computer Programming III: Sorting and Searching*. Reading, Mass.: Addison-Wesley, 1973.
3. R. E. Miller, N. Pippenger, A. L. Rosenberg, & L. Snyder, "Optimal 2,3-Trees." *SIAM J. Comput.* 8 (1979):42-59.
4. A. L. Rosenberg & L. Snyder. "Minimal-Comparison 2,3 Trees." *SIAM J. Comput.* 7 (1978):465-480.

*****

A STUDY OF THE MAXIMAL VALUES IN PASCAL'S QUADRINOMIAL TRIANGLE

CLAUDIA SMITH and VERNER E. HOGGATT, JR.
San Jose State University, San Jose, CA 95112

1. INTRODUCTION

In this paper we search for the generating function of the maximal values in Pascal's quadrinomial triangle. We challenge the reader to find this function as well as a general formula for obtaining all generating functions of the $(H - L)/k$ sequences obtained from partition sums in Pascal's quadrinomial triangle.

Generalized Pascal triangles arise from the multinomial coefficients obtained by the expansion of

$$(1 + x + x^2 + \cdots + x^{j-1})^n,\ j \geq 2,\ n \geq 0,$$

where n denotes the row in each triangle. For $j = 4$, the quadrinomial coefficients produce the following triangle:

1									
1	1	1	1						
1	2	3	4	3	2	1			
1	3	6	10	12	12	10	6	3	1

The partition sums are defined by

$$S(n,j,k,r) = \sum_{i=0}^{M} \left[\!\left[\begin{matrix} n \\ r + ik \end{matrix} \right]\!\right]_j;\ 0 \leq r \leq k - 1,$$

where

$$M = \left[\frac{(j-1)n - r}{k}\right];$$

the brackets [ ] denote the greatest integer function. To clarify, we give a numerical example. Consider $S(3,4,5,1)$. This denotes the partition sums in the third row of the quadrinomial triangle, in which every fifth element is added, beginning with the first column. Thus,

$$S(3,4,5,1) = 3 + 10 = 13.$$

(Conventionally, the column of 1's at the far left is the 0th column and the top row is the 0th row.)

In the nth row of the j-nomial triangle, the sum of the elements is j^n. This is expressed by

$$S(n,j,k,0) + S(n,j,k,1) + \cdots + S(n,j,k,k-1) = j^n.$$

Let

$$S(n,j,k,0) = (j^n + A_n)/k$$
$$S(n,j,k,1) = (j^n + B_n)/k \ldots$$
$$S(n,j,k,k-1) = (j^n + Z_n)/k.$$

Since $S(0,j,k,0) = 1$,

$$S(0,j,k,1) = 0 \ldots S(0,j,k,k-1) = 0,$$

we can solve for A_0, B_0, ..., Z_0 to get $A_0 = k - 1$, $B_0 = -1$, ..., $Z_0 = -1$.

Now a departure table can be formed with A_0, B_0, ..., Z_0 as the 0th row. Pascal's rule of addition is the simplest method for finding the successive rows in each departure table. The departure table for six partitions in the quadrinomial triangle appears below.

TABLE 1. SUMS OF SIX PARTITIONS IN THE QUADRINOMIAL TRIANGLE

5	-1	-1	-1	-1	-1
2	2	2	2	-4	-4
-4	-4	2	8	2	-4
2	-10	-10	2	8	8
20	8	-10	-16	-10	8

In particular, the $(H - L)/k$ sequences defined as the difference of the maximum and minimum value sequences in a departure table, divided by k partitions will be of prime importance. Table 2 is a table of the $(H - L)/k$ sequences for $k = 5$ to $k = 15$ partitions.

TABLE 2. $(H - L)/k$ SEQUENCES FOR $k = 5$ TO $k = 15$

$k = 5$	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
1	3	3	4	4	4	4	4	4	4	4
1	6	6	8	10	11	12	12	12	12	12
1	9	14	24	30	36	39	42	43	44	44
1	18	31	56	85	105	125	135	145	149	153
1	27	70	160	246	340	404	468	503	538	553

The primary purpose of this paper is to share the progress that has been made toward finding a generating function for the maximal values in Pascal's quadrinomial triangle. The generating functions for the maximal values in the binomial and trinomial triangles are known. In the February 1979 issue of *The Fibonacci Quarterly*, we showed that the limit of the generating functions for the $(H - L)/k$ sequences was precisely the generating function for the maximal values in the rows of the binomial and trinomial triangles. We would like to establish this for the quadrinomial triangle as well.

2. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN THE QUADRINOMIAL TRIANGLE

As k increases, one sees the $(H - L)/k$ sequences obtain more of the values of the sequence of central (maximal) values in the quadrinomial triangle. For $k = 14$, we observe from Table 2 that the $(H - L)/14$ sequence contains the first five values. We examined all even values of k, up to $k = 52$. The $(H - L)/50$ sequence has its first 17 values coinciding with the central values in the quadrinomial triangle. The $(H - L)/50$ sequence is

1, 1, 4, 12, 44, 155, 580, 2128, 8092, 30276, 116304, 440484, 1703636, 6506786, 25288120, 97181760, 379061020, 1463609338... .

We observed that $k = 3m + 2$ has $m + 1$ of the central values in the quadrinomial triangle.

In an attempt to discover a pattern for predicting all recurrence relations of the $(H - L)/k$ sequences, we examined the recurrence relations for the even partitions up to $k = 48$. These equations are displayed in Table 3.

As the reader can see, the size (both degree and coefficient) of the equations grows rapidly. For example, in finding the recurrence equation in the case with 48 partitions, we used the first 30 elements in the $(H - L)/48$ sequence. The last element has 17 digits, too large for accuracy in most computers and calculators; thus, much computation was done by hand using the pivotal element method. Even after examining so many cases, we were unable to derive a general formula for predicting successive recurrence equations. However, we discovered several patterns that enabled us to make accurate conjectures about most of the coefficients and the degree of the recurrence equation.

We predict that for $N = 8m$, $8m + 2$, or $8m + 4$, the degree of the recurrence equation is $4m$. For $N = 8m + 6$, the degree is $4m + 2$.

The first coefficient is 1.

The second coefficient we predict to have the form $-N$ for $N \geq 8$. A difference of 2 is observed between successive elements in the sequence of all second coefficients for even partitions.

The third coefficient we predict to have the form $\frac{1}{2}N(N - 11)$ for $N \geq 14$. A second difference of 4 is observed between successive elements in the sequence of third coefficients for even partitions. We show this below:

21	40	63	90	121	156	195	238	285	336	391	450 ...
19	12	23	31	35	39	43	47	51	55	59 ...	
4	4	4	4	4	4	4	4	4	4 ...		

The fourth coefficient can be found by making a table of third differences between successive elements in the sequence of fourth coefficients for even partitions. This appears to be valid for $N \geq 20$. The third difference is 8. We show this below:

TABLE 3. RECURRENCE EQUATIONS FOR $(H - L)/k$ SEQUENCES FOR EVEN PARTITIONS IN THE QUADRINOMIAL TRIANGLE

N	Recurrence Equation
4	$x - 1 = 0$
6	$x^2 - 3 = 0$
8	$x^4 - 8x^2 + 8 = 0$
10	$x^4 - 10x^2 + 5 = 0$
12	$x^4 - 12x^2 + 9 = 0$
14	$x^6 - 14x^4 + 21x^2 - 7 = 0$
16	$x^8 - 16x^6 + 40x^4 - 32x^2 + 8 = 0$
18	$x^8 - 18x^6 + 63x^4 - 57x^2 + 9 = 0$
20	$x^8 - 20x^6 + 90x^4 - 100x^2 + 25 = 0$
22	$x^{10} - 22x^8 + 121x^6 - 176x^4 + 88x^2 - 11 = 0$
24	$x^{12} - 24x^{10} + 156x^8 - 296x^6 + 225x^4 - 72x^2 + 8 = 0$
26	$x^{12} - 26x^{10} + 195x^8 - 468x^6 + 455x^4 - 169x^2 + 13 = 0$
28	$x^{12} - 28x^{10} + 238x^8 - 700x^6 + 833x^4 - 392x^2 + 49 = 0$
30	$x^{14} - 30x^{12} + 285x - 1000x + 1440x - 903x + 230x^2 - 15 = 0$
32	$x^{16} - 32x^{14} + 336x^{12} - 1376x^{10} + 2376x^8 - 1920x^6 + 736x^4 - 128x^2$ $+ 8 = 0$
34	$x^{16} - 34x^{14} + 391x^{12} - 1836x^{10} + 3757x^8 - 3740x^6 + 1819x^4 - 374x^2$ $+ 17 = 0$
36	$x^{16} - 36x^{14} + 450x^{12} - 2388x^{10} + 5715x^8 - 6804x^6 + 4059x^4 - 1080x^2$ $+ 81 = 0$
38	$x^{18} - 38x^{16} + 613x^{14} - 3040x^{12} + 8398x^{10} - 11742x^8 + 8512x^6 - 3059x^4$ $+ 475x^2 - 19 = 0$
40	$x^{20} - 40x^{18} + 580x^{16} - 3800x^{14} + 11970x^{12} - 19408x^{10} + 16860x^8$ $- 7800x^6 + 1825x^4 - 200x^2 + 8 = 0$
42	$x^{20} - 42x^{18} + 651x^{16} - 4676x^{14} + 16611x^{12} - 30912x^{10} + 31647x^8$ $- 17937x^6 + 5334x^4 - 700x^2 + 21 = 0$
44	$x^{20} - 44x^{18} + 726x^{16} - 5676x^{14} + 22517x^{12} - 47652x^{10} + 56628x^8$ $- 38236x^6 + 14036x^4 - 2420x^2 + 121 = 0$
46	$x^{22} - 46x^{20} + 805x^{18} - 6808x^{16} + 29900x^{14} - 71346x^{12} + 97198x^{10}$ $- 76912x^8 + 34500x^6 - 8119x^4 + 851x^2 - 23 = 0$
48	$x^{24} - 48x^{22} + 888x^{20} - 8080x^{18} + 38988x^{16} - 104064x^{14} + 160888x^{12}$ $- 147360x^{10} + 79329x^8 - 24080x^6 + 3816x^4 - 288x^2 + 8 = 0$

100	176	296	468	700	1000	1376	1836	2388	...
76	120	172	232	300	376	460	552	...	
44	52	60	68	76	84	92	...		
8	8	8	8	8	8	...			

The fifth coefficient can be found by making a table of fourth differences between successive elements in the sequence of fifth coefficients for even partitions. This appears to be valid for $N \geq 26$. The fourth difference is 16. See Table 4.

TABLE 4. PREDICTING 5th, 6th, AND 7th COEFFICIENTS

455	833	1440	2376	3757	5715	8398	11970	16611	22517	29900	...
378	607	936	1381	1958	2683	3572	4641	5906	7383	...	
229	329	445	577	725	889	1069	1265	1477	...		
100	116	132	148	164	180	196	212	...			
16	16	16	16	16	16	16	...				
1920	3740	6804	11742	19408	30912	47652	71346	104064	...		
1820	3064	4938	7666	11504	16740	23964	32718	...			
1244	1874	2728	3838	5236	6954	9024	...				
630	854	1110	1398	1718	2070	...					
224	256	288	320	352	...						
32	32	32	32	...							
8512	16860	31647	56628	97198	160888	257925	...				
8348	14787	24981	40570	63690	97037	...					
6439	10194	15589	23120	33347	...						
3755	5395	7531	10227	...							
1640	2136	2696	...								
496	560	...									
64	...										

3. GENERATING FUNCTIONS OF THE $(H - L)/k$ SEQUENCES IN A MULTINOMIAL TRIANGLE

We challenge the reader to finish this problem: to find the generating functions of the $(H - L)/k$ sequences for all k in the quadrinomial triangle. Then perhaps we could find the generating function of maximal values in Pascal's quadrinomial triangle, and show that the limits of the $(H - L)/k$ generating functions are precisely the generating function of maximal values.

This problem can be extended to examine the maximal values in Pascal's pentanomial triangle and larger multinomial triangles. Again, the pursuer of such an adventure will encounter numbers with up to 20 digits, in which the accuracy of each digit matters in order to find recurrence equations.

REFERENCES

1. John L. Brown, Jr., & V. E. Hoggatt, Jr. "A Primer for the Fibonacci Numbers, Part XVI: The Central Column Sequence." *The Fibonacci Quarterly* 16, No. 1 (1978):41.

2. Michel Y. Rondeau. "The Generating Functions for the Vertical Columns of $(N + 1)$-Nomial Triangles." Master's thesis, San Jose State University, San Jose, California, December 1973.
3. Claudia Smith & V. E. Hoggatt, Jr. "Generating Functions of Central Values in Generalized Pascal Triangles." *The Fibonacci Quarterly* 17, No. 1 (1979):58.
4. Claudia R. Smith. "Sums of Partition Sets in the Rows of Generalized Pascal's Triangles." Master's thesis, San Jose State University, San Jose, California, May 1978.

*****

THE STUDY OF POSITIVE INTEGERS (a,b) SUCH THAT $ab + 1$ IS A SQUARE

PETER HEICHELHEIM
Peat Marwick and Partners, Toronto, Ontario, Canada

1. INTRODUCTION

A P-set will be defined as a set of positive integers such that if a and b are two distinct elements of this set, $ab + 1$ is a square.

There are many examples of P-sets such as [2, 12] or [1, 3, 8, 120] and even formulas such as

$$[n - 1,\ n + 1,\ 4n,\ 4n(4n^2 - 1)]$$

or

$$[m,\ n^2 - 1 + (m - 1)(n - 1)^2,\ n(mn + 2),\ 4m(mn^2 - mn + 2n - 1)^2 + 4(mn^2 - mn + 2n - 1)].$$

(See Cross [1].) However, none of these formulas are general.

More recently, there has been considerable work on P-sets with polynomials (by Jones [2, 3]) and in connection with Fibonacci numbers (by Hoggatt and Bergum [4]).

It is of interest to find out how much these sets can be extended by adding new positive integers to the set; for example [2, 12] can be extended to [2, 12, 420]. A P-set which cannot be extended will be called nonextendible. One purpose of this article is to show that a nonextendible set must have at least four members. Then it will be demonstrated that the number of members of a P-set is finite. Finally, it will be shown that certain types of five-member P-sets will be impossible.

2. EXTENDING P-SETS TO FOUR ELEMENTS

The proof that sets of one or two elements are extendible is very simple, for $[N]$ can always be extended to $[N,\ N + 2]$ and $[a,\ b]$ can be extended to $[a,\ b,\ a + b + 2x]$ where $x^2 = ab + 1$. (See Euler [5].)

Let $[a,\ b,\ N]$ be members of a P-set. Then,

(1) $$ab + 1 = x^2,$$

(2) $$aN + 1 = y^2,$$

(3) $$bN + 1 = z^2.$$