A COMBINATORIAL PROBLEM INVOLVING RECURSIVE SEQUENCES AND TRIDIAGONAL MATRICES

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In [2], C. A. Church, Jr., shows that the total number of k-combinations of the first n natural numbers such that no two elements i and $i+2$ appear together in the same selection is F^2_{m+2} if $n = 2m$ and $F_{m+2}F_{m+3}$ if $n = 2m+1$. Furthermore he shows, if the k-combinations are arranged in a circle, so that 1 and n are consecutive with no two elements i and $i+2$ appearing together in the same selection, then the number of k-combinations of the first n natural numbers is L^2_m if $n = 2m$ and $L_m L_{m+1}$ if $n = 2m+1$.

Letting $\{U_n\}_{n=0}^{\infty}$ be the sequence of k-combinations of the first n natural numbers such that no two elements i and $i+2$ appear together in the same selection, we have

$$U_0 = 1, \quad U_1 = 2, \quad U_2 = 4, \quad U_3 = 6, \quad U_4 = 9, \quad U_5 = 15, \quad U_6 = 25, \cdots.$$

By applying standard techniques, it is easy to show that the generating function for $\{U_n\}_{n=0}^{\infty}$ is

$$(1) \qquad \sum_{n=0}^{\infty} U_n x^n = \sum_{m=0}^{\infty} (F^2_{m+2} + F_{m+2}F_{m+3}x)x^{2m} = \frac{1+2x+2x^2+2x^3-x^4-x^5}{1-2x^2-2x^4+x^6} .$$

Although this rational function may be very interesting in its own right, it is also surprising to observe what happens if we replace m by $m-1$, multiply by x^2, and then start the summation from $m = 0$. Doing this we have

$$(2) \qquad \sum_{m=0}^{\infty} (F^2_{m+1} + F_{m+1}F_{m+2}x)x^{2m} = \frac{1+x-x^2}{1-2x^2-2x^4+x^6} = \frac{1}{(1-x-x^2)(1+x^2)} .$$

Incidentally, it can be shown that

$$(3) \qquad \sum_{m=0}^{\infty} (F^2_{m+1}x + F_m F_{m+1})x^{2m} = \frac{x}{(1-x-x^2)(1+x^2)} .$$

The results of (2) and (3) can be generalized in a very natural way to the sequence of Fibonacci polynomials defined recursively by

$$f_0(\lambda) = 0, \quad f_1(\lambda) = 1, \quad f_{n+1}(\lambda) = \lambda f_n(\lambda) + f_{n-1}(\lambda), \quad n \geqslant 1.$$

Using the well known fact that

$$f_n(\lambda) = \frac{\alpha^n - \beta^n}{\alpha - \beta} ,$$

where

$$\alpha = \frac{\lambda + \sqrt{\lambda^2+4}}{2} \quad \text{and} \quad \beta = \frac{\lambda - \sqrt{\lambda^2+4}}{2} ,$$

together with the techniques found in part VI of [6], we have

$$(4) \qquad \sum_{m=0}^{\infty} f^2_{m+2}(\lambda)x^{2m} = \frac{\lambda^2 + (\lambda^2+1)x^2 - x^4}{[1-(\lambda^2+2)x^2+x^4](1+x^2)}$$

and

$$\text{(5)} \qquad \sum_{m=0}^{\infty} f_{m+2}(\lambda)f_{m+3}(\lambda)x^{2m+1} = \frac{(\lambda^3+\lambda)x + (\lambda^3+\lambda)x^3 - \lambda x^5}{[1-(\lambda^2+2)x^2+x^4](1+x^2)} .$$

Adding (4) and (5), we obtain

$$\text{(6)} \qquad \sum_{m=0}^{\infty} [f_{m+2}^2(\lambda) + f_{m+2}(\lambda)f_{m+3}(\lambda)x]x^{2m} = \frac{\lambda^2 + (\lambda^3+\lambda)x + (\lambda^2+1)x^2 + (\lambda^3+\lambda)x^3 - x^4 - \lambda x^5}{[1-(\lambda^2+2)x^2+x^4](1+x^2)}$$

Replacing m by $m-1$, multiplying by x^2, and then starting the summation from $m = 0$, we have

$$\text{(7)} \qquad \sum_{m=0}^{\infty} [f_{m+1}^2(\lambda) + f_{m+1}(\lambda)f_{m+2}(\lambda)x]x^{2m} = \frac{1+\lambda x - x^2}{[1-(\lambda^2+2)x^2+x^4](1+x^2)} = \frac{1}{(1-\lambda x - x^2)(1+x^2)} .$$

Minor manipulations of (4) and (5) will also yield

$$\text{(8)} \qquad \sum_{m=0}^{\infty} [f_{m+1}^2(\lambda)x + f_m(\lambda)f_{m+1}(\lambda)]x^{2m} = \frac{x}{(1-\lambda x - x^2)(1+x^2)} .$$

As should be the case, (7) and (8) are (2) and (3) when $\lambda = 1$.

Another generalization of (2) and (3) occurs when we examine the sequence of Pellian Polynomials defined recursively by

$$P_0(\lambda) = 0, \quad P_1(\lambda) = 1, \quad P_{n+1}(\lambda) = (1-\lambda)P_n(\lambda) - \lambda P_{n-1}(\lambda), \quad n \geqslant 2.$$

Since

$$P_n(\lambda) = \frac{a^n - \beta^n}{a - \beta},$$

where

$$a = \frac{(1-\lambda) + \sqrt{\lambda^2 - 6\lambda + 1}}{2} \quad \text{and} \quad \beta = \frac{(1-\lambda) - \sqrt{\lambda^2 - 6\lambda + 1}}{2} .$$

we can use the techniques found in part VI of [6] together with arguments used to develop (7) and (8) in order to show that

$$\text{(9)} \qquad \sum_{m=0}^{\infty} [P_{m+1}^2(\lambda) + P_{m+1}(\lambda)P_{m+2}(\lambda)x]x^{2m} = \frac{1}{[1-(1-\lambda)x+\lambda x^2[(1-\lambda x^2)}$$

and

$$\text{(10)} \qquad \sum_{m=0}^{\infty} [P_{m+1}^2(\lambda)x + P_m(\lambda)P_{m+1}(\lambda)]x^{2m} = \frac{x}{[1-(1-\lambda)x+\lambda x^2](1-\lambda x^2)} .$$

When $\lambda = -1$ we obtain the sequence of Pellian numbers.

Our final generalization of (2) and (3) is obtained by returning to subsets of a given set. Let $S_n = \{1, 2, 3, \cdots, n\}$ and $P(S_n)$ be the power set of S_n. Let T_n be the number of elements of $P(S_n)$ with no two elements congruent modulo two. The first nine terms of $\{T_n\}_{n=0}^{\infty}$ with $T_0 = 1$ are

$$1, 2, 4, 6, 9, 12, 16, 20, 25, \cdots.$$

To develop a formula for $\{T_n\}_{n=0}^{\infty}$ we first note that any element of $P(S_n)$ of order three or more is rejected. Furthermore there is one element of order zero and there are n elements of order one. The number of elements of order two is

$$\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)$$

if n is odd and $n^2/4$ if n is even, any even integer with any odd integer. Hence,

$$T_n = \frac{n^2-1}{4} + n + 1 = \frac{(n+1)(n+3)}{4}, \quad n \text{ odd}$$

and

$$T_n = \frac{n^2}{4} + n + 1 = \left(\frac{n+2}{2}\right)^2, \quad n \text{ even}.$$

The generating function for $\{T_n\}_{n=0}^{\infty}$ is

$$(11)\quad \sum_{n=0}^{\infty} T_n x^n = \sum_{m=0}^{\infty} [(m+1)^2 + (m+1)(m+2)x]x^{2m} = \frac{x^2+1}{(1-x^2)^3} + \frac{2x}{(1-x^2)^3} = \frac{1}{(1-x)^2(1-x^2)}$$

while

$$(12)\quad \sum_{m=0}^{\infty} [(m+1)^2 x + m(m+1)]x^{2m} = \frac{x}{(1-x)^2(1-x^2)}.$$

The authors also found the generating function for the sequence of k-combinations of the first n natural numbers arranged in a circle, so that 1 and n are consecutive, with no two elements i and $i+2$ appearing together in the same selection. Letting $\{V_n\}_{n=1}^{\infty}$ be the stated sequence, we see that

$$V_1 = 1, \quad V_2 = 3, \quad V_3 = 9, \quad V_4 = 12, \quad V_5 = 16, \quad V_6 = 28, \quad V_7 = 49, \quad \cdots,$$

and

$$(13)\quad \sum_{n=1}^{\infty} V_n x^n = \sum_{m=1}^{\infty} (L_m^2 + L_m L_{m+1} x)x^{2m} = \frac{(1+7x^2-4x^4)x^2}{1-2x^2-2x^4+x^6} + \frac{(3+6x^2-2x^4)x^3}{1-2x^2-2x^4+x^6}$$
$$= \frac{(1+3x+7x^2+6x^3-4x^4-2x^5)x^2}{1-2x^2-2x^4+x^6}.$$

Replacing m by $m+1$ and summing from $m = 0$, in order to obtain the same form as (2), we have

$$(14)\quad \sum_{m=0}^{\infty} (L_{m+1}^2 + L_{m+1}L_{m+2}x)x^{2m} = \frac{1+3x+7x^2+6x^3-4x^4-2x^5}{1-2x^2-2x^4+x^6}$$

which does not simplify and is therefore not as appealing as the result in (2).

If we replace m by $m-1$ in (13), multiply by x^2, and then sum from $m = 0$ we have

$$(15)\quad \sum_{m=0}^{\infty} (L_{m-1}^2 + L_{m-1}L_m x)x^{2m} = \frac{1-2x+2x^2+6x^3-9x^4+3x^5}{1-2x^2-2x^4+x^6} = \frac{1-3x+6x^2-3x^3}{(1-x-x^2)(1+x^2)}$$

which does not simplify further and is not as appealing as equation (2).

The authors tried several other substitutions and manipulations of (13) in order to obtain a rational function whose numerator is a one or an x. However, they were not successful.

We now turn to the major result of this article which is the establishment of a relationship between (2), (7), (9), (11) and a sequence of determinants of tridiagonal matrices defined by the rule $P_n(a,b,c) = (a_{ij})$, where

$$a_{ij} = a \text{ if } i = j, \quad a_{ij} = b \text{ if } i = j-2, \quad a_{ij} = c \text{ if } i = j+2, \text{ and } a_{ij} = 0 \text{ otherwise.}$$

The first eleven values of $P_n(a,b,c)$ with $P_0(a,b,c)$ defined to be one are

$$P_0(a,b,c) = 1$$
$$P_1(a,b,c) = a$$
$$P_2(a,b,c) = a^2$$
$$P_3(a,b,c) = a^3 - abc$$
$$P_4(a,b,c) = (a^2-bc)^2$$
$$P_5(a,b,c) = a^5 - 3a^3bc + 2ab^2c^2$$
$$P_6(a,b,c) = (a^3-2abc)^2$$

$$P_7(a,b,c) = a^7 - 5a^5bc + 7a^3b^2c^2 - 2ab^3c^3$$
$$P_8(a,b,c) = (a^4 - 3a^2bc + b^2c^2)^2$$
$$P_9(a,b,c) = a^9 - 7a^7bc + 16a^5b^2c^2 - 13a^3b^3c^3 + 3ab^4c^4$$
$$P_{10}(a,b,c) = (a^5 - 4a^3bc + 3ab^2c^2)^2 .$$

It would seem at first that there is no order, except for the perfect squares, to the sequence $\{P_n(a,b,c)\}_{n=0}^{\infty}$. However if one were to actually evaluate the determinants he would see a nice pattern developing in the way he finds those values. In fact it can be shown by induction that

(16) $$P_n(a,b,c) = aP_{n-1}(a,b,c) - abcP_{n-3}(a,b,c) + b^2c^2P_{n-4} .$$

The generating function for $\{P_n(a,b,c)\}_{n=0}^{\infty}$ is found to be

(17) $$\sum_{n=0}^{\infty} P_n(a,b,c)x^n = \frac{1}{(1-bcx^2)(1-ax+bcx^2)} .$$

When $bc = -1$ and $a = 1$, (17) becomes (2). When $bc = -1$ and $a = \lambda$, (17) becomes (7). When $bc = \lambda$ and $a = (1-\lambda)$, (17) becomes (9). When $bc = 1$ and $a = 2$, (17) becomes (11).

The authors were unsuccessful in trying to find a sequence of determinants whose generating function was related to (15). Similarly we had no success in trying to find such a sequence of determinants for the last two examples which we shall now discuss.

Our first example deals with a generalization of the problem of C. A. Church which can be found in [1]. Using S_n and $P(S_n)$ as previously defined, we wish to determine the number of subsets of S_n for which $3n$, $3n+3$ or $3n+1$, $3n+4$ or $3n+2$, $3n+5$ are not in the same subset. Letting U_n be the number of acceptable subsets for a given n, it is easy to illustrate that

$U_0 = 1,\ U_1 = 2,\ U_2 = 4,\ U_3 = 8,\ U_4 = 12,\ U_5 = 18,\ U_6 = 27,\ U_7 = 45,\ U_8 = 75,\ U_9 = 125, \cdots$

By applying the results of [3], it can be shown that

(18) $$U_n = \begin{cases} F_{k+2}^3 , & \text{if } n = 3k \\ F_{k+2}^2F_{k+3}, & \text{if } n = 3k+1 \\ F_{k+2}F_{k+3}^2, & \text{if } n = 3k+2 , \end{cases}$$

where F_k is the k^{th} Fibonacci number. Hence, the generating function for $\{U_n\}_{n=0}^{\infty}$ is

(19) $$\sum_{n=0}^{\infty} U_nx^n = \sum_{m=1}^{\infty} [F_{m+1}^3 + F_{m+1}^2F_{m+2}x + F_{m+1}F_{m+2}^2x^2]x^{3m-3}$$
$$= \frac{1+2x+4x^2+5x^3+6x^4+6x^5-3x^6-3x^7-3x^8-x^9-x^{10}-x^{11}}{(x^6-x^3-1)(x^6+4x^3-1)} .$$

Summing (19) from $m = 0$ and multiplying both sides by x^3 we have

(20) $$\sum_{m=0}^{\infty} [F_{m+1}^3 + F_{m+1}^2F_{m+2}x + F_{m+1}F_{m+2}^2x^2]x^{3m} = \frac{1+x+x^2-2x^3-x^4+x^5-x^6}{(x^6-x^3-1)(x^6+4x^3-1)}$$
$$= \frac{(1-x^2)(1+x+2x^2-x^3+x^4)}{(x^6-x^3-1)(x^6+4x^3-1)} .$$

Our final example deals with counting the number of elements of $P(S_n)$ which have no two members of the same subset congruent modulo three. Denoting the sequence by $\{V_n\}_{n=0}^{\infty}$, it is easy to illustrate that

$V_0 = 1,\ V_1 = 2,\ V_2 = 4,\ V_3 = 8,\ V_4 = 12,\ V_5 = 18,\ V_6 = 27,\ V_7 = 36,\ V_8 = 48,\ V_9 = 64, \cdots .$

In order to determine a formula for V_n, we first note that all elements of $P(S_n)$ with four or more members are rejected in the counting process. Furthermore there is always one element of $P(S_n)$ with no members and there are n elements of $P(S_n)$ with one member. Let us now assume $n = 3k + 1$ and arrange the numbers from 1 to n as follows

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ \cdots & \cdots & \cdots \\ 3k-2 & 3k-1 & 3k \\ 3k+1 & & \end{array}$$

An acceptable element of $P(S_n)$ of order two is found by taking any element of the first column with any element in columns two or three and any element of the second column with any element of the third column. Hence, the number of valid elements of $P(S_n)$ of order two is

$$2k(k+1) + k^2 = 3k^2 + 2k$$

provided $n = 3k + 1$. When $n = 3k + 2$ there are $3k^2 + 4k + 1$ allowable sets of order two while the number of such sets if $n = 3k$ is $3k^2$.

The number of subsets of S_n of order three is

$$\binom{3k+1}{3}$$

provided $n = 3k + 1$. A subset of S_n of order three is not counted if it contains two elements of one column and one element from either of the two remaining columns or if it contains three elements from the same column. Hence the number of valid sets of order three if $n = 3k + 1$ is

$$\binom{3k+1}{3} - 2k\binom{k+1}{2} - (2k+1)\binom{k}{2} - (2k+1)\binom{k}{2} - \binom{k+1}{3} - 2\binom{k}{3} = k^3 + k^2.$$

When $n = 3k + 2$, the number of valid sets of order three is

$$\binom{3k+2}{3} - 2\binom{k+1}{2}(2k+1) - 2(k+1)\binom{k}{2} - 2\binom{k+1}{3} - \binom{k}{3} = k(k+1)^2.$$

When $n = 3k$, the number of valid sets of order three is

$$\binom{3k}{3} - 6k\binom{k}{2} - 3\binom{k}{3} = k^3.$$

Combining the results above, we conclude that

$$V_n = \begin{cases} k^3 + 3k^2 + 3k + 1 = (k+1)^3, & n = 3k \\ k^3 + 4k^2 + 3k + 2 = (k+1)^2(k+2), & n = 3k+1 \\ k^3 + 5k^2 + 8k + 4 = (k+1)(k+2)^2, & n = 3k+2. \end{cases} \tag{21}$$

Hence, the generating function for $\{V_n\}_{n=0}^{\infty}$ is

$$\sum_{n=0}^{\infty} V_n x^n = \sum_{m=0}^{\infty} [(m+1)^3 + (m+1)^2(m+2)x + (m+1)(m+2)^2x^2]x^{3m} \tag{22}$$

$$= \frac{x^2+1}{(x^2+x+1)^2(x-1)^4}.$$

REFERENCES

1. C.A. Church, Jr., "Advanced Problems and Solutions," H-70, *The Fibonacci Quarterly*, Vol. 3, No. 4 (Dec. 1965), p. 299.
2. C.A. Church, Jr., "Advanced Problems and Solutions," H-70, *The Fibonacci Quarterly*, Vol. 5, No. 3 (Oct. 1967), pp. 253–255.
3. J.L. Brown, Jr., "A Combinatorial Problem Involving Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 6, No. 1 (Feb. 1968), pp. 34–35.

4. J. L. Brown, Jr., "Advanced Problems and Solutions," H-137, *The Fibonacci Quarterly,* Vol. 6, No. 4 (Oct. 1968), p. 250.
5. J. L. Brown, Jr., "Advanced Problems and Solutions," H-137, *The Fibonacci Quarterly,* Vol. 8, No. 1 (Feb. 1970), pp. 75–76.
6. V. E. Hoggatt, Jr., and Marjorie Bicknell, "A Primer for the Fibonacci Numbers," The Fibonacci Association, 1973, pp. 52–64.

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ON THE FORMULA $\pi = 2\Sigma$ arcot f_{2k+1}

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While questing the $n + 1^{st}$ digit of π,
With series by Taylor, MacLauren, et al;
I tried the arcotan of integers high,
While old Leonardo de Pisa did call.

Old friends are a joy and, at times, a surprise
When they serependiciously drop by to chat.
They lighten our labors and open our eyes.
"Eureka!" quoth I. "Now, how about that!"

For what to my wondering eyes should appear,
Intermix't with the spurious inverse contans,
Were eight Fibonacci terms standing right here,
Waiting and patiently holding their hands.

The even term's arcotangent's easily seen
to equal the sum of the next pair in line.
Now start back with π, and keep your eyes keen
It makes 4 arcotan the unit sublime.

Note: 1 is the first and the second old friend.
So rewrite: π equals twice this plus twice that.
"This" is the arcot of the first term of Len.
"That," which we'll split, is from the second old hat.

From 2 we get 3, 4; from 4, 5 and 6.
The evens keep splitting; the odds hang behind.
Forming convergent series: sum twice arcot f
Sub $2k + 1$ which is π, I remind.

We don't know the digit half-million and one.
Guiness, keep stout! There'll be other tries.
I've got half my friends in a pretty new sum.
Well worth the labor to open my eyes.