RESTRICTED COMPOSITIONS

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1. INTRODUCTION

A *composition* of the integer n into k parts is defined [1, p. 107] as the number of ordered sets of non-negative integers $(a_1, a_2, \cdots, a_k)$ such that

(1.1) $$a_1 + a_2 + \cdots + a_k = n.$$

It is well known and easy to prove that the number of such compositions is equal to the binomial coefficient

$$\binom{n+k-1}{k-1} \quad .$$

If we require that the a_i be strictly positive then of course the number of solutions of (1.1) is equal to

$$\binom{n-1}{k-1}.$$

In the present paper we consider the problem of determining the number of solutions of (1.1) when we require that

(1.2) $$a_i \ne a_{i+1} \qquad (i = 1, 2, \cdots, k-1).$$

Let $c(n,k)$ denote the number of solutions of (1.1) and (1.2) in *positive* a_i and let $\bar{c}(n,k)$ denote the number of solutions of (1.1) and (1.2) in *non-negative* a_i. Then clearly

(1.3) $$c(n,k) = \bar{c}(n-k,k).$$

Also it is evident from the definition that

(1.4) $$\bar{c}(n,k) = 0 \qquad (k > 2n+1).$$

We shall show that

(1.5) $$\sum_{n,k=0}^{\infty} c(n,k)x^n z^k = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \dfrac{x^j z^j}{1-x^j}} \quad .$$

For $z = 1$, this reduces to

(1.6) $$\sum_{n=0}^{\infty} c(n)x^n = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \dfrac{x^j}{1-x^j}} \quad ,$$

where

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$$c(n) = \sum_{k=1}^{n} c(n,k), \qquad c(0) = 1. \tag{1.7}$$

Thus $c(n)$ is the number of solutions of (1.1) and (1.2) with $a_i > 0$ when the number of parts is unrestricted.

It follows from (1.3) and (1.6) that

$$1 + \sum_{n,k=1}^{\infty} \bar{c}(n,k)x^n z^k = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}}. \tag{1.8}$$

This is also proved independently.

The generating function for

$$\bar{c}(n) = \sum_{k} \bar{c}(n,k), \qquad \bar{c}(0) = 1 \tag{1.9}$$

is less immediate. It is proved that

$$\sum_{0}^{\infty} \bar{c}(n)x^n = \frac{1}{1-(1-x)\sum_{1}^{\infty} \frac{x^{2j-1}}{(1-x^{2j-1})(1-x^{2j})}}. \tag{1.10}$$

It is of some interest to determine the radius of convergence of the series

$$\sum_{0}^{\infty} c(n)x^n, \qquad \sum_{0}^{\infty} \bar{c}(n)x^n. \tag{1.11}$$

We show that the radius of convergence of the first is at least ½; the radius of convergence of the second is also probably > ½ but this is not proved.

2. GENERATING FUNCTIONS FOR $c(n,k)$ AND $c_a(n,k)$

It is convenient to define the following refinements of $c(n,k)$ and $\bar{c}(n,k)$. Let $c_a(n,k)$ denote the number of solutions of (1.1) and (1.2) in positive integers a_i with $a_i = a$; $\bar{c}_a(n,k)$ is defined as the corresponding number when the $a_i \geq 0$. Clearly

$$c(n,k) = \sum_{a=1}^{n} c_a(n,k), \qquad \bar{c}(n,k) = \sum_{a=0}^{n} \bar{c}_a(n,k). \tag{2.1}$$

The enumerant $c_a(n,k)$ satisfies the recurrence

$$c_a(n,k) = \sum_{b \neq a} c_b(n-a, k-1) \qquad (k > 1). \tag{2.2}$$

If we put, for $k \geq 1$,

$$F_a(x,k) = \sum_{n=1}^{\infty} c_a(n,k)x^n, \qquad \Phi_k(x,y) = \sum_{a=1}^{\infty} F_a(x,k)y^a,$$

it follows from (2.2) that

$$F_a(x,k) = x^a \sum_{b \neq a} F_b(x,k-1) \qquad (k > 1).$$

Then

$$\Phi_k(x,y) = \sum_{a=1}^{\infty} (xy)^a \sum_{b \neq a} F_b(x,k-1) = \sum_{b=1}^{\infty} F_b(x,k-1) \sum_{a \neq b} (xy)^a = \sum_{b=1}^{\infty} F_b(x,k-1) \left(\frac{xy}{1-xy} - (xy)^b \right),$$

so that

(2.3) $$\Phi_k(x,y) = \frac{xy}{1-xy} \Phi_{k-1}(x,1) - \Phi_{k-1}(x,xy) \qquad (k > 1).$$

Iterating (2.3), we get

$$\Phi_k(x,y) = \frac{xy}{1-xy} \Phi_{k-1}(x,1) - \frac{x^2y}{1-x^2y} \Phi_{k-2}(x,1) + \Phi_{k-2}(x,x^2y) \qquad (k > 2)$$

and generally

$$\Phi_k(x,y) = \sum_{j=1}^{s} (-1)^{j-1} \frac{x^jy}{1-x^jy} \Phi_{k-j}(x,1) + (-1)^s \Phi_{k-s}(x,x^sy) \qquad (k > s).$$

In particular, for $s = k-1$, this becomes

(2.4) $$\Phi_k(x,y) = \sum_{j=1}^{k-1} (-1)^{j-1} \frac{x^jy}{1-x^jy} \Phi_{k-j}(x,1) + (-1)^{k-1} \Phi_1(x,x^{k-1},y) \qquad (k > 1).$$

Since

$$\Phi_1(x,y) = \sum_{a=1}^{\infty} (xy)^a = \frac{xy}{1-xy},$$

it is clear that (2.4) may be replaced by

(2.5) $$\Phi_k(x,y) = \sum_{j=1}^{k} (-1)^{j-1} \frac{x^jy}{1-x^jy} F_{k-1}(x,1) \qquad (k \geq 1),$$

where it is understood that

(2.6) $$\Phi_0(x,y) = 1.$$

For $y = 1$, (2.5) reduces to

(2.7) $$\Phi_k(x,1) + \sum_{j=1}^{k} (-1)^j \frac{x^j}{1-x^j} \Phi_{k-j}(x,1) = \delta_{k,1},$$

where $\delta_{k,1}$ is the Kronecker delta.

Using (2.6), this gives

$$\sum_{k=0}^{\infty} z^k \left\{ \Phi_k(x,1) + \sum_{j=1}^{k} (-1)^j \frac{x^j}{1-x^j} \Phi_{k-j}(x,1) \right\} = 1$$

and therefore

(2.8) $$\sum_{k=0}^{\infty} \Phi_k(x,1) z^k = \frac{1}{1 + \sum_{j=1}^{\infty} (-1)^j \frac{x^jz^j}{1-x^j}}.$$

In view of (2.1), (2.8) can be written in the more explicit form

$$\sum_{n,k=0}^{\infty} c(n,k)x^n z^k = \frac{1}{1+\sum_{j=1}^{\infty} (-1)^j \frac{x^j z^j}{1-x^j}} . \tag{2.9}$$

We now put

$$\frac{1}{1+\sum_{j=1}^{\infty} (-1)^j \frac{x^j z^j}{1-x^j}} = \sum_{k=0}^{\infty} \frac{P_k(x)}{(x)_k} z^k , \tag{2.10}$$

where

$$(x)_k = (1-x)(1-x^2)\cdots(1-x^k), \qquad (x)_0 = 1.$$

Clearly

$$\Phi_k(x,1) = \frac{P_k(x)}{(x)_k} . \tag{2.11}$$

The $P_k(x)$ are polynomials in x that satisfy

$$P_k(x) = \sum_{j=1}^{k} (-1)^{j-1} \begin{bmatrix} k \\ j \end{bmatrix} (x)_{j-1} x^j P_{k-j}(x) \qquad (k \geq 1), \tag{2.12}$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(x)_k}{(x)_j (x)_{k-j}} .$$

The first few values of $P_k(x)$ are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 2x^3, \quad P_3(x) = x^4 + x^5 + 4x^6 .$$

In the next place, by (2.5),

$$\sum_{k=1}^{\infty} \Phi_k(x,y)z^k = \sum_{k=1}^{\infty} z^k \sum_{j=1}^{k} (-1)^{j-1} \frac{x^j y}{1-x^j y} \Phi_{k-j}(x,1) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j y z^j}{1-x^j y} \sum_{k=0}^{\infty} \Phi_k(x,1)z^k .$$

Hence, by (2.8),

$$\sum_{k=1}^{\infty} \Phi_k(x,y)z^k = \frac{\sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j y z^j}{1-x^j y}}{1+\sum_{j=1}^{\infty} (-1)^j \frac{x^j z^j}{1-x^j}} . \tag{2.13}$$

This evidently reduces to (2.8) when $y = 1$.

Note that the LHS of (2.13) is equal to

$$\sum_{n=1}^{\infty} \sum_{a,k} c_a(n,k) x^n y^a z^k . \tag{2.14}$$

3. GENERATING FUNCTION FOR $c(n)$ AND RELATED FUNCTIONS

For $z = 1$, (2.8) reduces to

(3.1)
$$\sum_{k=0}^{\infty} \Phi_k(x,1) = \frac{1}{1-\sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j}{1-x^j}} .$$

We have

$$\sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j}{1-x^j} = \sum_{j,k=1}^{\infty} (-1)^{j-1} x^{jk} = \sum_{n=1}^{\infty} x^n \sum_{j|n} (-1)^{j-1} .$$

Put

(3.2)
$$d'(n) = \sum_{j|n} (-1)^{j-1} ;$$

thus $d'(n)$ is the number of odd divisors of n less the number of even divisors.

For $n = 2^r m$, where m is odd, and $r \geq 0$,

$$d'(n) = \sum_{s=0}^{r} \sum_{j|m} (-1)^{2^s j-1} = (1-r) \sum_{j|m} 1,$$

so that

(3.3)
$$d'(n) = -(r-1)d(m),$$

where $d(n)$ is the number of divisors of n.

Thus we may replace (3.1) by

(3.4)
$$\sum_{k=0}^{\infty} \Phi_k(x,1) = \frac{1}{1-\sum_{1}^{\infty} d'(n)x^n} .$$

Since

$$\sum_{k=0}^{\infty} \Phi_k(x,1) = 1 + \sum_{n,k,a=1}^{\infty} c_a(n,k)x^n = 1 + \sum_{n=1}^{\infty} c(n)x^n ,$$

we have therefore

(3.5)
$$1 + \sum_{n=1}^{\infty} c(n)x^n = \frac{1}{1-\sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^j}{1-x^j}} = \frac{1}{1-\sum_{n=1}^{\infty} d'(n)x^n} .$$

It follows that $c(n)$ satisfies the recurrence

(3.6)
$$c(n) = \sum_{j=1}^{n} d'(j)c(n-j) \qquad (n \geq 1),$$

where $c(0) = 1$.

It is also of some interest to take $z = -1$ in (2.8). We get

$$\sum_{k=0}^{\infty} (-1)^k \Phi_k(x,1) = \frac{1}{1+\sum_{1}^{\infty} \frac{x^j}{1-x^j}} = \frac{1}{1+\sum_{1}^{\infty} d(n)x^n} .$$

Since

$$\sum_{k=0}^{\infty} (-1)^k \Phi_k(x,1) = 1 + \sum_{n,k,a=1}^{\infty} (-1)^k c_a(n,k)x^n = 1 + \sum_{n=1}^{\infty} c^*(n)x^n ,$$

where

$$c^*(n) = \sum_{k,a=1}^{n} (-1)^k c_a(n,k), \tag{3.7}$$

we get

$$1 + \sum_{1}^{\infty} c^*(n)x^n = \frac{1}{1 + \sum_{1}^{\infty} d(n)x^n} . \tag{3.8}$$

This yields the recurrence

$$c^*(n) + \sum_{j=1}^{n} d(j)c^*(n-j) = 0 \qquad (n \geqslant 1), \tag{3.9}$$

where $c^*(0) = 1$.

The first few values of $c^*(n)$ are

$$c^*(1) = -1, \quad c^*(2) = -1, \quad c^*(3) = 1, \quad c^*(4) = 0, \quad c^*(5) = 1, \quad c^*(6) = -2.$$

It is also of interest to take $y = -1$ in (2.13). For $y = -1$, $z = 1$ we get

$$\sum_{k=1}^{\infty} \Phi_k(x,-1) = \frac{\sum_{1}^{\infty} (-1)^j \dfrac{x^j}{1+x^j}}{1 + \sum_{1}^{\infty} (-1)^j \dfrac{x^j}{1-x^j}} ,$$

so that

$$\sum_{k=0}^{\infty} \Phi_k(x,-1) = \frac{1 + 2\sum_{1}^{\infty} (-1)^j \dfrac{x^j}{1-x^{2j}}}{1 + \sum_{1}^{\infty} (-1)^j \dfrac{x^j}{1-x^j}} . \tag{3.10}$$

If we take $y = z = -1$ in (2.13) we get

$$\sum_{k=1}^{\infty} (-1)^k \Phi_k(x,-1) = \frac{\sum_{1}^{\infty} \dfrac{x^j}{1+x^j}}{1 + \sum_{1}^{\infty} \dfrac{x^j}{1-x^j}} ,$$

so that

$$\sum_{k=0}^{\infty} (-1)^k \Phi_k(x,-1) = \frac{1 + 2\sum_{1}^{\infty} \dfrac{x^j}{1-x^{2j}}}{1 + \sum_{1}^{\infty} \dfrac{x^j}{1-x^j}} = \frac{1 + 2\sum_{1}^{\infty} d_0(n)x^n}{1 + \sum_{1}^{\infty} d(n)x^n} , \tag{3.11}$$

where $d_o(n)$ denotes the number of *odd* divisors of n. Note that the LHS of (3.11) is equal to

(3.12) $$1+\sum_{n=1}^{\infty} x^n \sum_{a,k} (-1)^{a+k} c_a(n,k).$$

4. GENERATING FUNCTION FOR $\bar{c}(n,k)$ AND $\bar{c}_a(n,k)$

While generating functions for $\bar{c}(n,k)$ and $\bar{c}_a(n,k)$ can be obtained from those for $c(n,k)$ and $c_a(n,k)$ by using (1.3), it is of some interest to derive them independently. Put

$$\bar{F}_a(x,k) = \sum_{n=0}^{\infty} \bar{c}_a(n,k)x^n, \qquad \bar{\Phi}_k(x,y) = \sum_{a=0}^{\infty} \bar{F}_a(x,k)y^a .$$

Then, exactly as in Section 2,

$$\bar{c}_a(n,k) = \sum_{b\neq a} c_b(n-a,k),$$

so that

$$\bar{F}_a(n,k) = x^a \sum_{b\neq a} \bar{F}_b(x,k-1)$$

and

$$\bar{\Phi}_k(x,y) = \sum_{a=0}^{\infty} (xy)^a \sum_{b\neq a} \bar{F}_b(x,k-1) = \sum_{b=0}^{\infty} \bar{F}_b(x,k-1)\left(\frac{1}{1-xy} - (xy)^b\right).$$

Thus

(4.1) $$\bar{\Phi}_k(x,y) = \frac{1}{1-xy}\Phi_{k-1}(x,1) - \bar{\Phi}_{k-1}(x,xy) \qquad (k>1).$$

As above, iteration yields

$$\bar{\Phi}_k(x,y) = \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{1-x^jy}\bar{\Phi}_{k-j}(x,1) + (-1)^{k-1}\bar{\Phi}_1(x,x^{k-1}y) \qquad (k>1).$$

Since

$$\bar{\Phi}_1(x,y) = \sum_{a=0}^{\infty} (xy)^a = \frac{1}{1-xy},$$

we get

(4.2) $$\bar{\Phi}_k(x,y) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{1-x^jy}\bar{\Phi}_{k-j}(x,1) \qquad (k\geqslant 1),$$

where

(4.3) $$\bar{\Phi}_0(x,y) = 1.$$

For $y=1$, (4.2) reduces to

(4.4) $$\bar{\Phi}_k(x,1) + \sum_{j=1}^{k} \frac{(-1)^j}{1-x^j}\bar{\Phi}_{k-j}(x,1) = \delta_{k,0}.$$

It follows that

(4.5)
$$\sum_{k=0}^{\infty} \bar{\Phi}_k(x,1)z^k = \frac{1}{1+\sum_{j=1}^{\infty}(-1)^j \dfrac{z^j}{1-x^j}} .$$

Now put

$$\frac{1}{1+\sum_{j=1}^{\infty}(-1)^j \dfrac{z^j}{1-x^j}} = \sum_{k=0}^{\infty} \frac{\bar{P}_k(x)}{(x)_k} z^k ,$$

so that

(4.6)
$$\bar{\Phi}_k(x,1) = \frac{\bar{P}_k(x)}{(x)_k} .$$

The $\bar{P}_k(x)$ are polynomials in x that satisfy the recurrence

(4.7)
$$\bar{P}_k(x) = \sum_{j=1}^{k} (-1)^{j-1} \begin{bmatrix} k \\ j \end{bmatrix} (x)_{j-1}\bar{P}_{k-j}(x) \qquad (k \geq 1);$$

also it is clear from the definition that

(4.8)
$$P_k(x) = x^k\bar{P}_k(x).$$

For $x = 1$, (4.7) reduces to

$$\bar{P}_k(1) = k\bar{P}_{k-1}(1),$$

so that

(4.9)
$$\bar{P}_k(1) = k!.$$

Also it is easy to show by induction that

$$\deg \bar{P}_j(x) \leq \tfrac{1}{2}j(j-1).$$

Indeed, assuming that this holds for $j < k$, it follows that the degree of the j^{th} term on the right of (4.7)

$$\leq j(k-j) + \tfrac{1}{2}j(j-1) + \tfrac{1}{2}(k-j)(k-j-1) = \tfrac{1}{2}k(k-1).$$

Let γ_k denote the coefficient of $x^{\frac{1}{2}k(k-1)}$ in $\bar{P}_k(x)$. Then we have

$$\gamma_k = \sum_{j=1}^{k} \gamma_{k-j} = \sum_{j=0}^{k-1} \gamma_j \qquad (k \geq 1).$$

This gives

$$\sum_{k=0}^{\infty} \gamma_k x^k \left(1 - \sum_{j=1}^{\infty} x^j \right) = 1,$$

so that

$$\sum_{k=0}^{\infty} \gamma_k x^k = \frac{1-x}{1-2x} .$$

Thus $\gamma_k = 2^{k-1}$, $k \geq 1$, and so

(4.10)
$$\deg \bar{P}_k(x) = \tfrac{1}{2}k(k-1).$$

Since, by (2.4),

$$\bar{c}(n,k) = 0 \qquad (k > 2n+1),$$

it follows that $\bar{P}_k(x)$ begins with a term in $x^{[k/2]}$; moreover the coefficient of this term is 1 for k odd and 2 for k even and positive.

It is clear from the recurrence (4.7) that all the coefficients are integers. It would be interesting to know if they are positive.

If we put

$$\bar{P}_k(x) = \sum_j \gamma(k,j)x^j \quad \text{and} \quad \frac{1}{(x)_k} = \sum_{n=0}^{\infty} p(n,k)x^n,$$

so that $p(n,k)$ is the number of partitions (in the usual sense) of n into parts $\leqslant k$, it follows from (4.6) that

$$\bar{c}(n,k) = \sum_j p(n-j,\, k)\gamma(k,j). \tag{4.11}$$

Returning to (4.2), we have

$$\sum_{k=1}^{\infty} \bar{\Phi}_k(x,y)z^k = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{1-x^jy} \sum_{k=0}^{\infty} \bar{\Phi}_k(x,1)z^k .$$

This gives

$$\sum_{k=1}^{\infty} \bar{\Phi}_k(x,y)z^k = \frac{\displaystyle\sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{1-x^jy}}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}} . \tag{4.12}$$

We may rewrite (4.5) and (4.12) as

$$1+\sum_{n,k=1}^{\infty} \bar{c}(n,k)x^nz^k = \frac{1}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}}, \tag{4.13}$$

$$1+\sum_{n=1}^{\infty}\sum_{a,k} \bar{c}_a(n,k)x^ny^az^k = \frac{\displaystyle\sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{1-x^jy}}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{z^j}{1-x^j}} . \tag{4.14}$$

By (1.3) we have

$$\bar{c}(n,k) = c(n+k,\, k). \tag{4.15}$$

Hence, replacing z by xz in (4.5), we have

$$1+\sum_{n,k=1}^{\infty} c(n+k,\, k)x^{n+k}z^k = \frac{1}{\displaystyle 1+\sum_{j=1}^{\infty} (-1)^j \frac{x^jz^j}{1-x^j}} . \tag{4.16}$$

This is of course equivalent to (2.9).

Since

$$\bar{c}_a(n,k) = c_{a+1}(n+k,\, k) \qquad (k > 0),$$

the equivalence of (4.16) and (2.9) follows easily.

Note that it follows from (4.6) and (4.12) that

$$\bar{\Phi}_k(x,y) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{1-x^jy} \frac{\bar{P}_{k-j}(x)}{(x)_{k-j}} . \tag{4.17}$$

In addition to (4.15) another relation expressing $\bar{c}(n,k)$ in terms of $c(n,k)$ can be obtained by considering the possible location of zero elements. There may be a zero on the extreme left or the extreme right; also there may be one or more zeros on the inside. Thus we get relations such as the following.

$$\bar{c}(0,0) = \bar{c}(0,1) = 1, \qquad \bar{c}(n,1) = c(n,1) = 1 \quad (n \geqslant 1),$$

$$\bar{c}(n,2) = c(n,2) + 2c(n,1) \qquad (n \geqslant 2),$$

$$\bar{c}(n,3) = c(n,3) + 2c(n,2) + x(n,1) + \sum_{n_1+n_2=n} c(n_1, 1)c(n_2, 1),$$

$$\bar{c}(n,4) = c(n,4) + 2c(n,3) + c(n,2) + 2\sum_{n_1+n_2=n} c(n_1, 1)c(n_2, 1) + 2\sum_{n_1+n_2=n} c(n_1, 1)c(n_2, 2).$$

It follows that

$$\sum_0^\infty \bar{\Phi}_k(x,1)z^k = 1 + z + (1+z)^2 \sum_1^\infty \Phi_k(x,1)z^k + (1+z)^2 z\left\{\sum_1^\infty \Phi_k(x,1)z^k\right\}^2 + (1+z)^3 z\left\{\sum_1^\infty \Phi_k(x,1)z^k\right\}^3 + \cdots$$

$$= 1 + z + \frac{(1+z)^2 \sum_1^\infty \Phi_k(x,1)z^k}{1 - z\sum_1^\infty \Phi_k(x,1)z^k}.$$

It is easily verified that this is in agreement with (2.8) and (4.5).

5. GENERATING FUNCTIONS FOR $c(n)$ AND $\bar{c}(n)$

We may not put $z = 1$ in (4.5) since the right-hand side then becomes meaningless. We can get around this difficulty in the following way.

To begin with, we shall get crude upper bounds for $c(n)$ and $\bar{c}(n)$. Let $\nu(n,k)$ denote the number of solutions in positive integers of

$$n = a_1 + a_2 + \cdots + a_k$$

and let $\bar{\nu}(n,k)$ denote the number of solutions in non-negative integers. Then

$$\nu(n,k) = \binom{n-1}{k-1}, \qquad \bar{\nu}(n,k) = \binom{n+k-1}{k-1}.$$

Clearly

$$c(n,k) \leqslant \nu(n,k), \qquad \bar{c}(n,k) \leqslant \bar{\nu}(n,k).$$

It follows that

$$c(n) \leqslant 2^{n-1} \qquad (n \geqslant 1), \tag{5.1}$$

so that the radius of convergence of

$$\sum_0^\infty c(n)x^n \tag{5.2}$$

is at least ½.

As for $\bar{c}(n)$, since

$$\bar{c}(n,k) = 0 \qquad (k > 2n+1),$$

we get

$$\bar{c}(n) \leqslant \sum_{k=1}^{2n+1} \binom{n+k-1}{k-1} = \sum_{k=0}^{2n} \binom{n+k}{k} \leqslant \sum_{k=0}^{2n} \binom{3n}{k},$$

so that

(5.3)
$$\bar{c}(n) \leqslant 2^{3n}.$$

Hence the radius of convergence of

(5.4)
$$\sum_{0}^{\infty} \bar{c}(n)x^n$$

is at least 1/8;

Presumably these bounds are by no means best possible. It seems likely that the radius of convergence of (5.4) is about ½.

Next consider

$$\sum_{j=1}^{2k} (-1)^{j-1} \frac{z^j}{1-x^j} = \sum_{j=1}^{k} \left(\frac{z^{2j-1}}{1-x^{2j-1}} - \frac{z^{2j}}{1-x^{2j}} \right) = \sum_{j=1}^{\infty} \frac{1-z+x^{2j-1}(z-x)}{(1-x^{2j-1})(1-x^{2j})} z^{2j-1}.$$

Thus (4.5) becomes

(5.5)
$$\sum_{0}^{\infty} \Phi_k(x,1)z^k = \frac{1}{1-\sum_{1}^{\infty} \frac{1-z+x^{2j-1}(z-x)}{(1-x^{2j-1})(1-x^{2j})}}.$$

It is now permissible to let $z \to 1$. We get

(5.6)
$$\sum_{0}^{\infty} \bar{c}(n)x^n = \frac{1}{1-(1-x)\sum_{1}^{\infty} \frac{x^{2j-1}}{(1-x^{2j-1})(1-x^{2j})}}.$$

For $x = ½$ we get

$$\frac{1/2}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)} + \frac{1/8}{\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)} + \frac{1/32}{\left(1-\frac{1}{32}\right)\left(1-\frac{1}{64}\right)} = \frac{4}{3} + \frac{16}{105} + \frac{32}{31.63} < 1.$$

Thus the radius of convergence of (5.4) is probably somewhat greater than ½.

REFERENCE

1. John Riordan, *An Introduction to Combinatorial Analysis,* Wiley, New York, 1958.

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