THE SEQUENCE: $1\ 5\ 16\ 45\ 121\ 320\ \cdots$ IN COMBINATORICS

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Extremely dedicated Fibonaccists might possibly recognize that this sequence can be derived by subtracting 2 from every other Lucas number. The purpose of this note is to describe how this rather bizarre sequence arises naturally in two quite disparate areas of combinatorics. For completeness, and to guarantee uniformity of notation, all basic definitions will be given.

A. FIBONACCI SEQUENCES

Any sequence $\{x_1, x_2, x_3, \cdots\}$ that satisfies $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$ will be called a Fibonacci sequence; such a sequence is completely determined by x_1 and x_2. The Fibonacci sequence $\{F_n\}$ with $F_1 = F_2 = 1$ is the sequence of Fibonacci numbers; the Fibonacci sequence $\{L_n\}$ with $L_1 = 1$, $L_2 = 3$ is the sequence of Lucas numbers. For reference, the first few numbers of these two sequences are given as follows:

n:	1	2	3	4	5	6	7	8	9	10	11	12
F_n:	1	1	2	3	5	8	13	21	34	55	89	144
L_n:	1	3	4	7	11	18	29	47	76	123	199	322

There are of course many identities involving these numbers; two which will be used here are:

$$F_{k+2} = 3F_k - F_{k-2} \qquad k \geq 3\,.$$

$$L_k = 3F_k - 2F_{k-2} \qquad k \geq 3\,.$$

Both of these identities can be verified by a straightforward induction argument.

B. THE FUNDAMENTAL MATRIX

In both of the combinatorial examples to be discussed, it will be important to evaluate the determinant of the $n \times n$ matrix A_n which is defined as:

$$A_n = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 3 \end{bmatrix}.$$

In words, A_n has 3's on the digaonal, -1's on the super- and sub-diagonals, -1's in the lower left and upper right-hand corners, and 0's elsewhere. This description explains why we set

$$A_1 = [1]\,, \qquad \text{and} \qquad A_2 = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}.$$

To facilitate the evaluation of $\det A_n$, define T_n to be the $n \times n$ continuant with 3's on the diagonal, -1's on the super- and sub-diagonals, and 0's elsewhere. That is:

$$T_n = \begin{bmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 3 \end{bmatrix}$$

Lemma. $\det T_n = F_{2n+2}$.

Proof. The lemma is certainly true for $n = 1$ and $n = 2$, since

$$T_1 = [3], \quad \text{and} \quad T_2 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Thus we will assume that the lemma is true for all $k < n$, and expand $\det T_n$ by the first row:

$$\det T_n = 3\det T_{n-1} - (-1)\det \left[\begin{array}{c|c} -1 & -1 \\ \hline & T_{n-2} \end{array}\right] = 3\det T_{n-1} - \det T_{n-2} = 3F_{2n} - F_{2n-2} = F_{2n+2}.$$

We are now able to verify that the sequence $\{\det A_1, \det A_2, \det A_3, \cdots\}$ is the sequence in the title.

Theorem. $\det A_n = L_{2n} - 2$.

Proof. The theorem is true for A_1 and A_2 as defined above; this can be easily verified. Now for $n > 2$, we can expand $\det A_n$ by its first row to obtain:

(1) $$\det A_n = 3\det T_{n-1} - (-1)\det R_{n-1} + (-1)^{n+1}(-1)\det S_{n-1},$$

where R_n and S_n are $n \times n$ matrices defined by:

$$R_n = \left[\begin{array}{c|c} -1 & -1 \\ \hline & T_{n-1} \\ -1 & \end{array}\right] \quad \text{and} \quad S_n = \left[\begin{array}{c|c} -1 & T_{n-1} \\ \hline -1 & -1 \end{array}\right].$$

Notice that T_{n-1} is symmetric, so we have

$$S_n^t = \left[\begin{array}{c|c} -1 & -1 \\ \hline T_{n-1} & \\ & -1 \end{array}\right].$$

Thus:

(2) $$\det S_n = \det S_n^t = (-1)^{n-1}\det R_n.$$

Now, expanding $\det R_n$ by the first column, we obtain:

$$\det R_n = (-1)\det T_{n-1} + (-1)^{n+1}(-1)\det \begin{bmatrix} -1 & & & \\ 3 & \ddots & & \\ -1 & \ddots & \ddots & \\ & -1 & 3 & -1 \end{bmatrix} = -\det T_{n-1} + (-1)^{n+2}(-1)^{n-1}.$$

Thus:

(3) $$\det R_n = -\det T_{n-1} - 1.$$

We can now substitute (2) and (3) into (1), and we obtain:

$$\det A_n = 3\det T_{n-1} + (-\det T_{n-2} - 1) + (-1)^{n+2}(-1)^{n-2}(-\det T_{n-2} - 1) = 3\det T_{n-1} - 2\det T_{n-2} - 2.$$

Then by using the Lemma and an identity mentioned earlier, we have:

$$\det A_n = 3F_{2n} - 2F_{2n-2} - 2 = L_{2n} - 2.$$

C. SPANNING TREES OF WHEELS

This section begins with some very basic definitions from graph theory. The reader uninitiated in this subject is urged to consult one of the many texts in this field (for example, [1] or [2]).

A graph on n vertices is a collection of n points (called vertices), some pairs of which are joined by lines (called edges).

A subgraph of a graph consists of a subset of the vertices, together with some (perhaps all or none) of the edges of the original graph that connect pairs of vertices in the chosen subset.

A subgraph containing all vertices of the original graph is called a spanning subgraph.

A graph is connected if every pair of vertices is joined by a sequence of edges.

A cycle is a sequence of three or more edges that goes from a vertex back to itself.

A tree is a connected graph containing no cycles. It is easy to verify that any tree with n vertices must have exactly $n-1$ edges.

A spanning tree of a graph is a spanning subgraph of the graph that is in fact a tree. Two spanning trees are considered distinct if there is at least one edge not common to them both.

Given a graph G, the complexity of the graph, denoted by $k(G)$, is the number of distinct spanning trees of the graph.

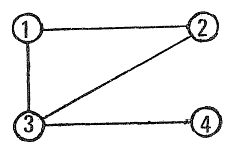
If a graph G has n vertices, number them $1, 2, \cdots, n$. The adjacency matrix of G, denoted by $A(G)$, is an $n \times n$ $(0,1)$ matrix with a 1 in the (i,j) position if and only if there is an edge joining vertex i to vertex j.

For any vertex i, the degree of i, denoted by $\deg i$, is the number of edges that are joined to i. Let $D(G)$ be the $n \times n$ diagonal matrix whose (i,i) entry is $\deg i$.

We are now able to state a quite remarkable theorem, attributed in [2] to Kirkhoff. For a proof of this theorem, see [1], page 159, or [2], page 152.

For any graph G, $k(G)$ is equal to the value of the determinant of any one of the n principal $(n-1)$-rowed minors of the matrix $D(G) - A(G)$.

As a simple example to illustrate this theorem, consider the graph G:

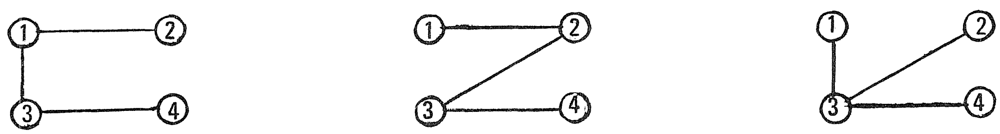


$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and thus

$$D(G) - A(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

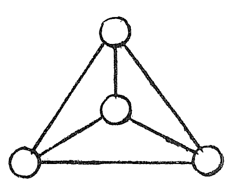
Each of the four principal 3-rowed minors of $D(G) - A(G)$ has determinant 3. The 3 spanning trees of G are:



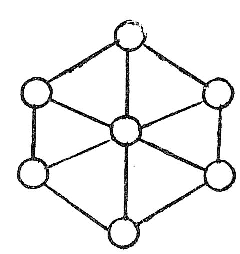
The relevance of these ideas to the title sequence will be established after making one more definition.

For $n \geq 3$, the n-wheel, denoted by W_n, is a graph with $n+1$ vertices; n of these vertices lie on a cycle (the rim) and the $(n+1)^{st}$ vertex (the hub) is connected to each rim vertex.

W_3:



W_6:



Theorem. $k(W_n) = L_{2n} - 2$.

Proof. Number the rim vertices $1, 2, \cdots, n$; the hub vertex is $n+1$. Each rim vertex i has degree 3; it is adjacent to vertices $i-1$ and $i+1 \pmod n$ and to vertex $n+1$. The hub vertex has degree n and is adjacent to all other vertices. Thus

$$D(W_n) - A(W_n) = \left[\begin{array}{ccc|c} & & & -1 \\ & A_n & & \vdots \\ & & & -1 \\ \hline -1 & -1 & \cdots -1 & n \end{array}\right].$$

To compute $k(W_n)$, any n-rowed principal minor will do. So delete row and column $n+1$. Then we have, by previous results:

$$k(W_n) = \det A_n = L_{2n} - 2.$$

This result can be found in [4] and in [7], but in neither instance is the number expressed explicitly in terms of the Lucas numbers. In [7], the formula for $k(W_n)$ is given by:

$$k(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$$

while in [4] the result is expressed:

$$k(W_n) = F_{2n+2} - F_{2n-2} - 2.$$

Readers familiar with Fibonacci identities will have no trouble verifying that both of these expressions are equivalent to the value given in the theorem.

D. GENERALIZING TOTAL UNIMODULARITY

A matrix M is said to be totally unimodular if every non-singular submatrix of M has determinant ± 1. Since the individual entries are 1×1 submatrices, they must necessarily be $0, \pm 1$. The following theorem, found in [3], provides sufficient conditions for total unimodularity:

Let M be a matrix satisfying the following four conditions:

(1) All entries of M are $0, \pm 1$.
(2) The rows of M are partitioned into two disjoint sets T_1 and T_2.
(3) If any column has two non-zero entries of the same sign, then one is in a row of T_1 and the other in a row of T_2.
(4) If any column has two non-zero entries of opposite sign, then they are both in rows of T_1 or both in rows of T_2.

Then M is totally unimodular.

This result usually includes the additional condition that there be at most two non-zero entries per column; this, however, is actually a consequence of conditions (3) and (4).

We are thus motivated to consider the class $\mathcal{M}$ of matrices which satisfy conditions (1), (2), and (3), but not (4). If $M \in \mathcal{M}$, then as an immediate consequence of (3), we see that there are at most four non-zero entries in any column of M; at most two non-zero entries (with opposite sign) in rows of T_1, and at most two non-zero entries (with opposite sign) in rows of T_2.

It is then natural to define the subclasses: $\mathcal{M}'' \subset \mathcal{M}' \subset \mathcal{M}$, where any matrix in $\mathcal{M}'$ satisfies conditions (1), (2), and (3) and has at most three non-zero entries per column; any matrix in $\mathcal{M}''$ satisfies (1), (2), and (3), and has at most two non-zero entries per column. An obvious problem is to find the maximum determinantal value of an $n \times n$ matrix in any one of these three classes. This problem is completely solved only for the class $\mathcal{M}''$; the following theorem appears in [6]:

If M is any $n \times n$ matrix in the class $\mathcal{M}''$, then $\det M \leqslant 2^{[n/2]}$. Moreover, for each $n \geqslant 1$, there is an $n \times n$ matrix in $\mathcal{M}''$ whose determinant achieves this upper bound.

The title sequence is relevant in considering the class $\mathcal{M}'$. For any $k \geqslant 1$, let I_k be the $k \times k$ identity matrix, and define J_k to be the $k \times k$ matrix with 1's on the diagonal, -1's on the super-diagonal, and a -1 in the lower left-hand corner. That is,

$$J_k = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \\ & & & -1 \\ -1 & & & 1 \end{bmatrix}.$$

Then for n even, say $n = 2k$, we can define the $n \times n$ matrices H_n and G_n as follows:

$$H_n = \begin{bmatrix} I_k & -J_k^t \\ J_k & I_k \end{bmatrix} \qquad G_n = \begin{bmatrix} I_k & 0 \\ -J_k & I_k \end{bmatrix} .$$

Notice first that $H_n \in \mathcal{M}'$. Now since $det\, G_n = 1$, we have:

$$det\, H_n = det\,(H_n G_n) = det \begin{bmatrix} I_k + J_k^t J_k & -J_k^t \\ 0 & I_k \end{bmatrix} = det\,(I_k + J_k^t J_k) .$$

But the (i,j) entry of $J_k^t J_k$ is simply the inner product of the i^{th} and j^{th} columns of J_k. It is thus not difficult to verify that

$$I_k + J_k^t J_k = A_k ,$$

where A_k is the fundamental matrix of this paper. We have thus verified the following result:

For n even, there is an $n \times n$ matrix in $\mathcal{M}'$ with determinant $L_n - 2$. A comparable result for odd n is proved in [5].

For n odd, there is an $n \times n$ matrix in $\mathcal{M}'$ with determinant $2F_n - 2$. It is my present conjecture that, for any given n, these determinantal values are the maximum possible for an $n \times n$ matrix in the class $\mathcal{M}'$, or in the class $\mathcal{M}$.

Finally, it should be noted that totally unimodular matrices occur naturally in the formulation of a problem in optimization theory known as the transportation problem. In [6], it is shown that matrices from class $\mathcal{M}$ arise in a discussion of the two-commodity transportation problem.

REFERENCES

1. C. Berge, *The Theory of Graphs and its Applications,* Methuen, London, 1962.
2. F. Harary, *Graph Theory,* Addison-Wesley, Reading, Massachusetts, 1969.
3. A.J. Hoffman and J.B. Kruskal, "Integral Boundary Points of Convex Polyhedra," *Linear Inequalities and Related Systems,* pp. 223–246, Annals of Mathematics, No. 38, Princeton University Press, Princeton, 1956.
4. B.R. Myers (proposer) and P.M. Gibson (solver), "The Spanning Trees of an *n*-Wheel," Advanced Problem 5795, *American Math. Montly,* Vol. 79, No. 8 (October, 1972), pp. 914–915.
5. K.R. Rebman, *Non-Unimodular Network Programming,* Ph.D. Dissertation, University of Michigan, Ann Arbor, 1969.
6. K.R. Rebman, "Total Unimodularity and the Transportation Problem: A Generalization," *Linear Algebra and its Applications,* Vol. 8, No. 1 (1974), pp. 11–24.
7. J. Sedlacek, "On the Sekeletons of a Graph or Digraph," *Proceedings of the Calgary International Conference of Combinatorial Structures and their Applications,* Gordon and Breach, New York, 1970, pp. 387–391.

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