ORESME NUMBERS

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1. INTRODUCTION

The purpose of this article is to make known some properties of an interesting sequence of numbers which I believe has not received much (if any) attention.

In the mid-fourteenth century, the scholar and cleric, Nicole Oresme, found the sum of the sequence of rational numbers

$$\frac{1}{2},\ \frac{2}{4},\ \frac{3}{8},\ \frac{4}{16},\ \frac{5}{32},\ \frac{6}{64},\ \frac{7}{128},\ \frac{8}{256},\ \cdots \quad . \tag{1}$$

Unfortunately, Oresme's original calculations were not published.

Such a sequence is of considerable biological interest. As Hogben [3] remarks: "...what is of importance to the biologist is an answer to the question: if we know the first two terms, *i.e.,* the proportion of grandparents and parents of different genotypes, how do we calculate the proportions in any later generations?"

2. ORESME NUMBERS

The sequence (1) of Oresme can be extended "to the left" to include negative numbers if we see the pattern of the sequence, which is easily discernible. More is gained by recognizing the sequence (1) as a special case of a general sequence discussed by Horadam [4], [5] and [6].

This general sequence $\{w_n(a,b;p,q)\}$ is defined by

$$w_{n+2} = pw_{n+1} - qw_n\ , \tag{2}$$

where

$$w_0 = a, \quad w_1 = b \tag{3}$$

and p,q are arbitrary integers at our disposal. To achieve our purpose, we now extend the values of p,q to be arbitrary rational numbers.

Taking $a = 0$, $b = 1$, $p = 1$, $q = ¼$, and denoting a term of the special sequence by O_n $(n = \cdots, -2, -1, 0, 1, 2, \cdots)$, we write the sequence $\{O_n\} \equiv \{w_n(0, ½, 1, ¼)\}$ as

$$\begin{array}{cccccccccccccccc} \cdots O_{-7} & O_{-6} & O_{-5} & O_{-4} & O_{-3} & O_{-2} & O_{-1} & O_0 & O_1 & O_2 & O_3 & O_4 & O_5 & O_6 & O_7 & \cdots \\ \cdots -896 & -384 & -160 & -64 & -24 & -8 & -2 & 0 & \frac{1}{2} & \frac{2}{4} & \frac{3}{8} & \frac{4}{16} & \frac{5}{32} & \frac{6}{64} & \frac{7}{128} & \cdots \end{array} \tag{4}$$

The extension (4) of the original sequence (1) studied by Oresme we will call the *Oresme sequence.* Terms of this sequence are called *Oresme numbers.* Thus, Oresme numbers are, by (2), (3), (4), given by the second-order relation

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n \tag{5}$$

with

$$O_0 = 0\ ,\ O_1 = O_2 = \frac{1}{2}\ . \tag{6}$$

An interesting feature of the Oresme sequence is that it is a degenerate case of $\{w_n\}$ occurring when $p^2 - 4q = 0$ $(i.e., 1^2 - 4 \times ¼ = 0)$. Further comments will be made on this aspect later in §6.

A number which characterizes special cases of $\{w_n(a,b;p,q)\}$ is $e = pab - qa^2 - b^2$ which depends on the initial values a, b and on p, q. For the Oresme sequence,

(7) $$e = -\frac{1}{4}.$$

Immediate observations from (4) include these facts:

(8) $$O_n = n2^{-n} \quad (n \geqslant 0)$$

(9) $$O_{-n} = -n2^n \quad (n \leqslant 0)$$

i.e.,

$$O_m = m2^{-m} \quad (m \text{ integer})$$

whence

(10) $$O_{-n}O_n = -n^2$$

(11) $$\frac{O_{-n}}{O_n} = -2^{2n}$$

and

(12) $$\lim_{n\to\infty} O_n \to 0, \quad \lim_{n\to\infty} O_n \to -\infty$$

(13) $$\lim_{m\to\infty} \frac{O_m}{O_{m-1}} \to ½\,.$$

Two well-known sequences, associated with the researches of Lucas [9] are:

(14) $$\{U_n\} \equiv \{w_n(1,p;p,q)\}$$

(15) $$\{V_n\} \equiv \{w_n(2,p;p,q)\}.$$

When $p = q = -1$, (14) gives the ordinary Fibonacci sequence and (15) the ordinary Lucas sequence.

It is a ready consequence of (4) and (14) that

(16) $$O_n = \frac{1}{2}U_{n-1},$$

where, for this $\{U_n\}$, $p = 1$, $q = ¼$.

That is, the Oresme sequence turns out to be a special case of the sequence $\{U_n\}$ after division by 2.

3. LINEAR RELATIONS FOR ORESME NUMBERS

Two simple expressions derived readily from (5) are:

(17) $$O_{n+2} - \frac{3}{4}O_n + \frac{1}{4}O_{n-1} = 0$$

(18) $$O_{n+2} - \frac{3}{4}O_{n+1} + \frac{1}{16}O_{n-1} = 0\,.$$

Sums of interest are:

(19) $$\sum_{j=0}^{n-1} O_j = 4(½ - O_{n+1})$$

(20) $$\sum_{j=0}^{\infty} O_j = 2$$

(21) $$\sum_{j=0}^{n-1} (-1)^j O_j = \frac{4}{9}[-½ + (-1)^n(O_{n+1} - 2O_n)]$$

(22) $$\sum_{j=0}^{n-1} O_{2j} = \frac{4}{9}[2 + O_{2n-1} - 5O_{2n}]$$

(23) $$\sum_{j=0}^{n-1} O_{2j+1} = \frac{1}{9}(10 + 5O_{2n-1} - 16O_{2n}).$$

Also,

(24) $$O_{n+r} = O_r U_n - \frac{1}{4} O_{r-1} U_{n-1} = O_n U_r - \frac{1}{4} O_{n-1} U_{r-1}$$

(25) $$O_{n+r} = O_{r-j} U_{n+j} - \frac{1}{4} O_{r-j-1} U_{n+j-1} = O_{n+j} U_{r-j} - \frac{1}{4} O_{n+j-1} U_{r-j-1}$$

(26) $$\frac{O_{n+r} + 4^{-r} O_{n-r}}{O_n} = V_r \text{ (independent of } n\text{)}$$

(27) $$\frac{O_{n+r} - 4^{-r} O_{n-r}}{O_{n+s} - 4^{-s} O_{n-s}} = \frac{U_{r-1}}{U_{s-1}}$$

(28) $$O_{2n} = (-¼)^n \sum_{j=0}^{n} \binom{n}{j} (-4)^{n-j} O_{n-j} .$$

4. NON-LINEAR PROPERTIES OF ORESME NUMBERS

A basic quadratic expression, corresponding to Simson's result for Fibonacci numbers, is

(29) $$O_{n+1} O_{n-1} - O_n^2 = -(¼)^n .$$

This result is the basis of a geometric paradox of which the general expression is given in Horadam [5].

A specially interesting result is the "Pythagorean" theorem of which the generalization is discussed in Horadam [5]:

(30) $$(O_{n+2}^2 - O_{n+1}^2)^2 + (2O_{n+2} O_{n+1})^2 = (O_{n+2}^2 + O_{n+1}^2)^2$$

For instance, $n = 3$ leads to the Pythagorean triple 39, 80, 89 after we have ignored a common denominator $(= 1024)$; $n = 4$ leads to the Pythagorean triple 8, 6, 10 after simplification (and division by 64, which we ignore).

Some other quadratic properties are:

(31) $$\frac{1}{2} O_{m+n-1} = O_m O_n - \frac{1}{4} O_{m-1} O_{n-1}$$

(32) $$\frac{1}{2} O_{2n-1} = O_n^2 - \frac{1}{4} O_{n-1}^2 = O_{n+1} O_{n-1} - \frac{1}{4} O_n O_{n-2}$$

(33) $$O_{n+r} O_{n-r} - O_n^2 = -(¼)^{n-r+1} U_{r-1}^2$$

(an extension of (29))

(34) $$O_{n+1}^2 - (¼)^2 O_{n-1}^2 = \frac{1}{2} O_{2n+1} + \frac{1}{8} O_{2n-1}$$

(35) $$O_{n-r} O_{n+r+t} - O_n O_{n+t} = -(¼)^{n-r+1} U_{r-1} U_{r+t-1}$$

(an extension of (33)).

Many other results can be obtained, if we use the fact that $\{O_n\}$ is a special case of $\{w_n\}$. Rather than produce numerous identities here, we suggest (as we did in [7] with Pell identities) that the reader may entertain himself by discovering them. Recent articles by Zeitlin [11], [12] and [13] give many properties of $\{w_n\}$ which may be of assistance.

Some of the distinguishing features of $\{O_n\}$ arise from the fact that it is a degenerate case of (2), occurring when $p^2 - 4q = 0$.

5. GENERATING FUNCTION

A generating function for the Oresme numbers O_n $(n \geqslant 1)$ is

$$\sum_{n=1}^{\infty} O_n x^n = \frac{\frac{1}{2}x}{1 - x + \frac{1}{4}x^2} \tag{36}$$

This may be obtained from the general result for w_n in Horadam [6], by the appropriate specialization.

6. COMMENTS ON THE DEGENERACY PROPERTY

Since the general term of $\{w_n\}$ is

$$w_n = A\alpha^n + B\beta^n, \tag{37}$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad \beta = \frac{p - \sqrt{p - 4q}}{2} \tag{38}$$

are the roots of $x^2 - px + q = 0$, and

$$A = \frac{b - a\beta}{\alpha - \beta}, \qquad B = \frac{a\alpha - \beta}{\alpha - \beta}, \qquad (\alpha - \beta = \sqrt{p^2 - 4q}), \tag{39}$$

it follows that in the degenerate case, O_n cannot be expressed in the form (36), as we have seen earlier in (8) and (9). An interesting derivation from Eq. (4.6) of Horadam [4] is the relationship $O_n^2 - \frac{1}{4}U_{n-1}^2 = 0$, leading back to (16).

Carlitz [2], acknowledging the work of Riordan, established an interesting relationship between the sum of k^{th} powers of terms of the degenerate sequence $\{U_n\}$ (for which $q = p^2/4$) and the Eulerian polynomial $A_k(x)$ which satisfies the differential equation

$$A_{n+1}(x) = (1 + nx)A_n(x) + x(1 - x)\frac{d}{dx}A_n(x), \tag{40}$$

where

$$A_0(x) = A_1(x) = 1, \quad A_2(x) = 1 + x, \quad A_3(x) = 1 + 4x + x^2. \tag{41}$$

This result specializes to the Oresme case where $p = 1$.

7. HISTORICAL

It is thought that Nicole Oresme was born in 1323 in the small village of Allemagne, about two miles from Caen, in Normandy. Records show that in 1348 he was a theology student at the College of Navarre—of which he became principal during the period 1356–1361—and that he attended Paris University.

His star in the Church rose quickly. Successively he became archdeacon of Bayeux (1361), then caron (1362), and later dean (1364) of Rouen Cathedral. In this period, he journeyed to Avignon with a party of royal emissaries and preached a sermon at the papal court of Urban V. While dean of Rouen, Oresme translated several of Aristotle's works, at the request of Charles V.

Thanks to his imperial patron (Charles V), Oresme was made bishop of Lisieux in 1377, being enthroned in Rouen Cathedral the following year. In 1382, Oresme died at Lisieux and was buried in his cathedral church.

Mathematically, Oresme is important for at least three reasons. Firstly, he expounded a graphic representation of of qualities and velocities, though there is no mention of the (functional) dependence of one quality upon another, as found in Descartes. Secondly, he was the first person to conceive the notion of fractional powers (afterwards rediscovered by Stevin), and suggested a notation.

In Oresme's notation, $4^{1\frac{1}{2}}$ is written as

$$\boxed{1p \cdot \frac{1}{2}}\ 4 \text{ or } \boxed{\frac{p \cdot 1}{1 \cdot 2}}\ 4 .$$

Thirdly, in an unpublished manuscript, Oresme found the sum of the series derived from the sequence (1). Such recurrent infinite series did not generally appear again until the eighteenth century.

In all, Oresme was one of the chief medieval theological scholars and mathematical innovators. It is the writer's hope that something of Oresme's intellectual capacity has been appreciated by the reader. With this in mind, we honor his name by associating him with the extended recurrence sequence (4), of which he had a glimpse so long ago.

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INCREDIBLE IDENTITIES

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Consider the algebraic numbers

$$A = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$$

$$B = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}$$

To 25 decimals they both equal

$$7.38117\ 59408\ 95657\ 97098\ 72669 .$$

Either this is an incredible coincidence or

(1) $$A = B$$

is an incredible identity, since A and B do not appear to lie in the same algebraic field. But they do. One has

(2) $$A = B = 4X - 1,$$

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