SOME ARITHMETIC FUNCTIONS RELATED TO FIBONACCI NUMBERS

L. CARLITZ, RICHARD SCOVILLE, and THERESA VAUGHAN
Duke University, Durham, North Carolina

1. INTRODUCTION

As is customary, we define the Fibonacci and Lucas numbers by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \qquad (n \geq 2)$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \qquad (n \geq 2),$$

respectively. It is well known that a positive integer N has the unique representation

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where $r = r(N)$ and the k_j satisfy

$$k_1 \geq 2, \quad k_j - k_{j-1} \geq 2 \qquad (j = 2, 3, \cdots, r). \tag{1.2}$$

The representation (1.1) is called the canonical or Zeckendorf representation of N.

It is proved in [1] that the set A_t of integers $\{N\}$ with $k_1 = t$ can be described in the following way:

$$\begin{cases} A_{2t} = \{ab^{t-1}a(n) \mid n = 1, 2, 3, \cdots\} \\ A_{2t+1} = \{b^t a(n) \mid n = 1, 2, 3, \cdots\} \end{cases} \qquad (t = 1, 2, 3, \cdots), \tag{1.3}$$

where juxtaposition of functions denotes composition and

$$a(n) = [\alpha n], \quad b(n) = [\alpha^2 n], \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}). \tag{1.4}$$

For the Lucas numbers it is known that every positive integer is uniquely representable in either the form

$$N = L_0 + L_{k_1} + L_{k_2} + \cdots + L_{k_r}, \tag{1.5}$$

where

$$k_1 \geq 3, \quad k_j - k_{j-1} \geq 2 \qquad (j = 2, 3, \cdots, r) \tag{1.6}$$

or in the form

---

* Supported in part by NSF Grant GP-17031.

(1.7) $$N = L_{k_1} + L_{k_2} + \cdots + L_{k_r} ,$$

where now

(1.8) $$k_1 \geq 1, \quad k_j - k_{j-1} \geq 2 \qquad (j = 2, 3, \cdots, r) ;$$

but not in both (1.5) and (1.7).

Let B_0 denote the set of positive integers representable in the form (1.5) and let B_t denote the set of positive integers representable in the form (1.7) with $k_1 = t$. Then it is proved in [2] that

(1.9) $$\begin{cases} B_0 = \{a^2(n) + n \mid n = 1, 2, 3, \cdots\} \\ B_1 = \{a^2(n) + n - 1 \mid n = 1, 2, 3, \cdots\} \end{cases},$$

and

(1.10) $$\begin{cases} B_{2t} = \{b^{t-1}a(n) + b^t a(n) \mid n = 1, 2, 3, \cdots\} \\ B_{2t+1} = \{ab^{t-1}a(n) + ab^t a(n) \mid n = 1, 2, 3, \cdots\} \end{cases} \qquad (t = 1, 2, 3, \cdots) .$$

The functions $a(n)$, $b(n)$ satisfy numerous relations that are consequences of the following.

(1.11) $$b(n) = a(n) + n = a^2(n) + 1$$

and

(1.12) $$ab(n) = a(n) + b(n) = ba(n) + 1.$$

Moreover if the function $e(n)$ is defined by

(1.13) $$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_r-1} ,$$

where N is defined by (1.1), then we have

(1.14) $$ea(n) = n, \quad eb(n) = a(n) .$$

Comparison of (1.9) and (1.10) with (1.3) suggests that it would be of interest to introduce the function

(1.15) $$c(n) = a(n) + 2n = b(n) + n .$$

It is not difficult to show that $bc(n) - cb(n) = 0$ or 1. We accordingly define two strictly monotonic functions $r(n)$, $s(n)$ by means of

(1.16) $$bcr(n) = cbr(n), \quad bcs(n) = cbs(n) + 1 .$$

The functions $r(n)$ and $s(n)$ are <u>complementary</u>, that is, the sets $\{r(n)\}$ and $\{s(n)\}$ constitute a (disjoint) partition of the positive integers.

The present paper is concerned with the properties of $r(n)$ and $s(n)$ and various related functions. In particular we define

$$u'(n) = bs(n) + 1 \tag{1.17}$$

and

$$t'(n) = as(n) + n . \tag{1.18}$$

It then follows that

$$(s) = (ab) \cup (a^2u') ; \tag{1.19}$$

more precisely

$$st = ab, \qquad st' = a^2u' , \tag{1.20}$$

where t and t' are complementary functions. Also

$$\begin{cases} c(n) \in (a) \rightleftarrows n \in (a^2u) \cup (bs) \\ c(n) \in (b) \rightleftarrows n \in (br) \cup (s) \end{cases} ;$$

this is equivalent to

$$\begin{cases} ca^2(n) \in (a) & (n \in (r)) \\ cb(n) \in (b) & (n \in (r)) \end{cases} . \tag{1.21}$$

It should be noted that the unions above are disjoint unions.

In these formulas we have used the symbol (f) to denote the range of the function f. If f and g are two strictly monotonic functions such that $(f) \subset (g)$, it is clear that there exists a strictly monotonic function h such that f = gh. In particular since $(b) \subset (a)$, there exists a function v such that b = av. Also since $(cs) \subset (b)$, there exists a function z such that cs = bz. Similarly we define functions p, x, y, w by means of

$$es(n) = rp(n) = ux(n) = uwy(n) , \tag{1.22}$$

so that x = wy. Among various relations among these functions we cite in particular the following.

$$z(n) = c's(n) \tag{1.23}$$

$$zt(n) = ca(n) + 1, \qquad zt'(n) = b^2a(n) \tag{1.24}$$

$$tb^2(n) = t(n) + b^2(n) \tag{1.25}$$

$$t't(n) = tb^2(n) - 1 \tag{1.26}$$

$$yt(n) = 2n \tag{1.27}$$

$$v(n) = w(2n) . \tag{1.28}$$

The formula

$$eca(n) = c(n) - 1 \tag{1.29}$$

proved in Section 3 can be thought of as one of the basic results of the paper. It was originally proved in an entirely different way.

We may also note the formula

$$(1.30) \qquad (s) = \bigcup_{k=0}^{\infty} (a(a^2b)^k b) ,$$

which is a consequence of (1.19). There are similar formulas for (r), (u), (u').

For the convenience of the reader a summary of formulas is included at the end of the paper as well as several brief numerical tables.

It should be remarked that almost all the theorems in this paper were suggested by numerical data. Thus it seems plausible that further numerical data may suggest additional theorems. The authors have prepared rather extensive tables which will be available from the Fibonacci Bibliographical and Research Center.

2. NOTATION AND PRELIMINARIES

If f is a function on the set $\mathbf{N}$ of positive integers, we let (f) denote the range of f, that is

$$(f) = \{f(n) \mid n \in \mathbf{N}\} .$$

If $n, m \in \mathbf{N}$, then

$$\eta(n < f < m)$$

is the number of integers j such that $n < f(j) < m$.

If f has the property that $f(n+1) - f(n) > 1$ for all $n \in \mathbf{N}$, then we say that f is <u>separated.</u>

If f is a function such that $\mathbf{N}/(f)$ is infinite, we may define a strictly monotonic function f' by:

$$(f') = \mathbf{N}/(f) .$$

This function f' is called the <u>complement</u> of f.

2.1. <u>Theorem</u>. If f is a strictly monotonic function from $\mathbf{N}$ to $\mathbf{N}$ such that $\mathbf{N}/(f)$ is infinite, then

$$f(n) = n + \eta(f' < f(n))$$

$$f'(n) = n + \eta(f < f'(n)) .$$

<u>Proof</u>. Suppose that $f(n) = k$. Since f is strictly monotonic we have $k \geq n$. Clearly $\eta(f < k) = n - 1$ and by definition of f', some $\eta(f' < k) = k - n$. Thus

$$f(n) = k = n + \eta(f' < k) = n + \eta(f' < f(n)) .$$

A similar argument shows that

$$f'(n) = n + \eta(f < f'(n)) .$$

2.2 Theorem. If f is a strictly monotonic function and, for some n and j we have $f'(j) = f(n) - 1$, then $j = f(n) - n$.

Proof. We have

$$f'(j) = f(n) - 1 = n + \eta(f' < f(n)) - 1$$
$$= n + \eta(f' < (j) + 1) - 1$$
$$= n + j - 1 .$$

Then $f(n) - 1 = n + j - 1$ and $j = f(n) - n$.

2.3. Corollary. [3, Th. 3.1]. If f is a separated function then for all $n > 1$,

$$f'(f(n) - n) = f(n) - 1$$

and

$$f'(f(n) - n + 1) = f(n) + 1 .$$

Proof. This is a direct consequence of the fact that if f is separated, then $f(n) - 1 \in (f')$ and $f(n) + 1 \in (f')$.

2.4. Theorem. If f is separated, then for all $n > 1$,

$$\eta\big(f'(n) < f < f(f'(n)) + 1\big) = n .$$

Proof.

$$\eta\big(f < f(f'(n)) + 1\big) = f'(n) = n + \eta(f < f'(n))$$
$$= \eta(f < f'(n)) + \eta\big(f'(n) < f < f(f'(n)) + 1\big)$$

and the theorem follows.

2.5. Definition. If f and g are functions, we use juxtaposition to mean composition of functions, that is,

$$fg(n) \equiv f(g(n)) .$$

2.6. Theorem. If f, g, h and k are all strictly monotonic, and if $f = g'h$ and $g = f'k$, then

$$f'k' = g'h' .$$

Proof. We have

$$(f') = (g) \cup (g'h') = (f'k) \cup (g'h')$$

and

$$(f') = (f'k) \cup (f'k') .$$

Since all functions are strictly monotonic, these are disjoint unions, and hence

$$(f'k') = (g'h') .$$

Again using strict monotonicity, we must have

$$f'k' = g'h' .$$

2.7. Definition. If

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists, we set

$$c_f = \lim_{n\to\infty} \frac{f(n)}{n} .$$

2.8. Theorem. If f and f' are complementary strictly monotonic functions and

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists and $\neq 0$, then

$$\lim_{n\to\infty} \frac{f'(n)}{n}$$

exists, and we have

(i) $$\frac{1}{c_f} + \frac{1}{c_{f'}} = 1$$

(ii) $$c_f + c_{f'} = c_f \cdot c_{f'} .$$

2.9. Definition. If ρ is any real number, then $[\rho]$ is defined to be the greatest integer less than or equal to ρ, and $\{\rho\}$ denotes $\rho - [\rho]$.

2.10. We shall make extensive use of the functions a, b, c, e defined in [1]. For convenience, we recall the definitions, and state some properties, of these functions.

(2.11) $$a(n) = [\alpha n] \quad \text{where} \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5})$$

(2.12) $$b(n) = [\alpha^2 n] = a(n) + n$$

(2.13) $$c(n) = [(\alpha + 2)n] = a(n) + 2n .$$

(2.14) $$b(n) = a^2(n) + 1$$

(2.15) $$ab(n) = a(n) + b(n) = ba(n) + 1$$

(2.16) $$ab(n) + 1 = a(b(n) + 1)$$

(2.17) $$a^2(n) + 1 = a(a(n) + 1)$$

(2.18) $$n \in (a) \rightleftarrows \begin{array}{l} a(n + 1) = a(n) + 2 \\ b(n + 1) = b(n) + 3 \\ c(n + 1) = c(n) + 4 \end{array}$$

(2.19) $$n \in (b) \rightleftarrows \begin{array}{l} a(n + 1) = a(n) + 1 \\ b(n + 1) = b(n) + 2 \\ c(n + 1) = c(n) + 3 \end{array} .$$

2.20. Theorem. Suppose that

$$\alpha n = m + \epsilon_1 \qquad (0 < \epsilon_1 < 1)$$

$$\alpha m = k + \epsilon_2 \qquad (0 < \epsilon_2 < 1)$$

then

$$\epsilon_2 + \alpha\epsilon_1 = 1 + \epsilon_1 .$$

Proof. We have

$$\begin{aligned}\alpha^2 n &= \alpha m + \alpha\epsilon_1 = (\alpha + 1)n = \alpha n + n \\ &= m + \epsilon_1 + n \\ &= k + \epsilon_2 + \alpha\epsilon_1 .\end{aligned}$$

From the definition, $m = a(n)$, $k = a^2(n)$ and $k + 1 = b(n) = m + n = [\alpha^2 n]$. Thus $k + 1 = m + n$ and so

$$k + 1 + \epsilon_1 = k + \epsilon_2 + \alpha\epsilon_1$$

and the result follows.

2.21. Theorem [4]. For all n,

(i) $$n \in (a) \rightleftarrows \{\alpha n\} > \frac{1}{\alpha^2}$$

(ii) $$n \in (b) \rightleftarrows \{\alpha n\} < \frac{1}{\alpha^2} .$$

2.22. Theorem. We have

$$\eta(a < n) = a(n) - n .$$

Proof. This follows from the fact that $a(n + 1) - a(n)$ is 1 if $n \in (b)$ and 2 if $n \in (a)$. Since $a(1) = 1$, we have $a(n) = n + \eta(a < n)$.

The following formulas follow from 2.22.

(2.23) $$\begin{aligned}\eta(b < n) &= n - 1 - \eta(a < n) \\ &= n - 1 - (a(n) - n) \\ &= 2n - 1 - a(n) .\end{aligned}$$

(2.24) $$\eta(b < a(n)) = a(n) - n = \eta(a < n) .$$

Recall that the function e was originally defined in terms of the Zeckendorf representation of n. In [1, Th. 6] it is shown that

$$\text{(2.25)} \qquad \begin{aligned} eb(n) &= a(n) \\ ea(n) &= n \ . \end{aligned}$$

We list some properties of e.

$$\text{(2.26)} \qquad e(n) = \eta(a \le n)$$

$$\text{(2.27)} \qquad e(n) = \eta(a < n + 1) = a(n + 1) - (n + 1)$$

$$\text{(2.28)} \qquad e(n) = \left[\frac{n + 1}{\alpha}\right] \ .$$

3. BASIC RESULTS

3.1. Theorem. For all n, $0 \le bc(n) - cb(n) \le 1$.

Proof. Recall that $b(n) = [\alpha^2 n]$ by (2.9). Thus

$$\begin{aligned} bc(n) &= [\alpha^2 c(n)] = [\alpha^2(b(n) + n)] \\ &= [\alpha^2 b(n) + \alpha^2 n] \\ &= [\alpha^2[\alpha^2 n] + \alpha^2 n] \end{aligned}$$

and

$$\begin{aligned} cb(n) &= b(b(n)) + b(n) \\ &= [\alpha^2[\alpha^2 n]] + [\alpha^2 n] \ . \end{aligned}$$

It is evident that $bc(n) \ge cb(n)$, and $0 \le bc(n) - cb(n) \le 1$.

3.2. Corollary. If $cb(n) \in (b)$, then $cb(n) = bc(n)$. If $cb(n) \in (a)$, then $cb(n) = bc(n) - 1 = a^2c(n)$.

Proof. Since for all r, $b(r + 1) - b(r) \ge 2$, then if $cb(n) \in (b)$, it follows that $cb(n) = bc(n)$. If $cb(n) \in (a)$, then $cb(n) = bc(n) - 1 = a^2c(n)$.

3.3. Definition. We define two strictly monotonic complementary functions r and s by means of

$$\text{(3.4)} \qquad \begin{cases} (r) = \{n \mid cb(n) = bc(n)\} \\ (s) = \{n \mid cb(n) = bc(n) - 1\} \end{cases} .$$

3.5. Theorem. For all n, $car(n) = acr(n) - 1$ and $cas(n) = acs(n) - 2$.

Proof. By definition, $cbr(n) = bcr(n)$, that is,

$$abr(n) + 2br(n) = acr(n) + cr(n) \ .$$

Then

$$\begin{aligned} acr(n) &= abr(n) + 2br(n) - cr(n) = abr(n) + br(n) - r(n) \\ &= abr(n) + ar(n) \end{aligned}$$

and

$$\begin{aligned} car(n) &= bar(n) + ar(n) \\ &= abr(n) - 1 + ar(n) \\ &= acr(n) - 1 \quad . \end{aligned}$$

Similarly, $cas(n) = acs(n) - 2$.

3.6. Theorem. If $ca(n) \in (a)$, then $ca(n) = a(c(n) - 1)$.

Proof. Case 1. if $n \in (r)$, then $ca(n) = ac(n) - 1$, and evidently if $ca(n) \in (a)$, we must have $ca(n) = a(c(n) - 1)$.

Case 2. If $n \in (s)$, then $ca(n) = ac(n) - 2$. Thus if $ca(n) \in (a)$, it must be that $ca(n) + 1 \in (b)$ (by (2.15)), and hence $ca(n) = a(c(n) - 1)$.

3.7. Theorem. For all n, $cs(n) \in (b)$.

For the proof of this theorem, we require some preliminary lemmas.

3.8. Lemma. If $n \in (s)$, then $a(n) \notin (s)$.

Proof. Let $n \in (s)$. Then $cb(n) = a^2c(n)$ and $ac(n) = ca(n) + 2$. Thus

$$a^2c(n) = a(ac(n)) = a(ca(n) + 2) \quad .$$

But also

$$cb(n) = ca^2(n) + 4 = c(ca(n)) + 4 \quad ,$$

by (2.15).

Now suppose that $ca^2(n) \in (a)$. Then by Theorem 3.6, $ca^2(n) = a(ca(n) - 1)$. Since three consecutive integers cannot all be in (a), it must be that $ca^2(n) + 2 \in (b)$, $ca^2(n) + 1 \in (a)$, and $ca^2(n) + 3 \in (a)$. Then

$$\begin{aligned} ca^2(n) + 3 &= a(ca(n) + 1) \\ ca^2(n) + 1 &= a(ca(n)) \\ ca^2(n) &= aca(n) - 1 = ca(a(n)) \end{aligned}$$

and by Theorem 3.5, $a(n) \in (r)$.

On the other hand, if $ca^2(n) \in (b)$, we must have $ca^2(n) + 1 \in (a)$, and precisely one of $ca^2(n) + 2$, $ca^2(n) + 3$ must be in (b). Thus $ca^2(n) + 1 = aca(n)$ and so $aca(n) - 1 = ca(a(n))$ and by Theorem 3.5, $a(n) \in (r)$.

3.9. Lemma. For all n, $cabr(n) = bacr(n)$ and $cabs(n) = bacs(n) - 2$.

Proof. The proof is manipulative. We show first for all n, $cab(n) = 7a(n) + 4n - 1$, as follows:

$$\begin{aligned} cab(n) &= bab(n) + ab(n) && \text{by (2.10)} \\ &= ab^2(n) - 1 + ab(n) && \text{by (2.12)} \\ &= ab(n) + b^2(n) - 1 + ab(n) && \text{by (2.12)} \\ &= 2ab(n) - 1 + ab(n) + b(n) && \text{by (2.9)} \end{aligned}$$

$$
\begin{aligned}
cab(n) &= 3ab(n) - 1 + a(n) + n && \text{by (2.9)} \\
&= 3(a(n) + b(n)) - 1 + a(n) + n && \text{by (2.12)} \\
&= 4a(n) + 3(a(n) + n) - 1 + n && \text{by (2.9)} \\
&= 7a(n) + 4n - 1 .
\end{aligned}
$$

Similarly, using the fact that $acr(n) = car(n) + 1$, we get $cabr(n) = 7ar(n) + 4r(n) - 1$, and from $acs(n) = cas(n) + 2$, we get $cabs(n) = 7as(n) + 4s(n) + 1$, and the result follows.

3.10. Lemma. For all n, $bcb(n) = cb^2(n)$, that is, $(b) \subset (r)$.

Proof. We first have, for all n,

$$
\begin{aligned}
cb^2(n) &= ab^2(n) + 2b^2(n) \\
&= ab(n) + 3b^2(n) \\
&= ab(n) + 3(ab(n) + b(n)) \\
&= 4ab(n) + 3b(n) .
\end{aligned}
$$

Case 1. $n \in (r)$. Then $bcb(n) = b^2c(n)$ and

$$
\begin{aligned}
b^2c(n) &= abc(n) + bc(n) \\
&= ac(n) + 2bc(n) \\
&= ca(n) + 1 + 2cb(n) && \text{(by Corollary 3.2 and Theorem 3.5)} \\
&= ab(n) + a(n) + 2(ab(n) + 2b(n)) \\
&= 3ab(n) + a(n) + 4b(n) \\
&= 4ab(n) + 3b(n) .
\end{aligned}
$$

Case 2. $n \in (s)$. Then $cb(n) = a^2c(n)$ and we have

$$
\begin{aligned}
bcb(n) &= ba^2c(n) = ba(ac(n)) \\
&= ab(ac(n)) - 1 \\
&= a^2c(n) + bac(n) - 1 \\
&= cb(n) + abc(n) - 2 \\
&= cb(n) + ac(n) + bc(n) - 2 \\
&= cb(n) + ca(n) + cb(n) + 1 && \text{(by Corollary 3.2 and Theorem 3.5)} \\
&= 2(ab(n) + 2b(n)) + (ba(n) + a(n)) + 1 \\
&= 3ab(n) + 4b(n) + a(n) \\
&= 4ab(n) + 3b(n) .
\end{aligned}
$$

3.11. Lemma. We have $(s) \subset (a)$.

The proof follows immediately from Lemma 3.10.

Proof of Theorem 3.7. If $n \in (s)$, then $n = a(j)$, for some integer j, where $j \notin (s)$ by Lemma 3.8. Since $n \in (s)$, we have

$$\begin{aligned} bc(n) - 1 &= cb(n) \\ bc(n) + 2 &= cb(n) + 3 \\ &= c(b(n) + 1) && \text{by (2.16)} \\ &= c(ba(j) + 1) = cab(j) = bac(j) && \text{by (2.12)} . \end{aligned}$$

Thus $bc(n) + 2 \in (b)$, which implies that $c(n) \in (b)$ by (2.19). This completes the proof of Theorem 3.7.

3.12. Corollary. For all n, $cas(n) \in (a)$.

Proof. By Theorem 3.8, $cs(n) = b(j)$ for some integer j. Then

$$\begin{aligned} cas(n) &= acs(n) - 2 && \text{by Theorem 3.5} \\ &= ab(j) - 2 \\ &= a^3(j) . \end{aligned}$$

3.13. Theorem. If $ca(n) \in (b)$, then $ca(n) = a(c(n) - 1) + 1$.

Proof. We need only consider $n \in (r)$, since if $n \in (s)$, $ca(n) \in (a)$. Thus, if $n \in (r)$, $ca(n) = ac(n) - 1$. If $ca(n) \in (b)$, then $ca(n) - 1 = a(c(n) - 1)$ and the result follows.

Recall that the function e satisfies

$$\begin{aligned} e(a(n)) &= n \\ e(b(n)) &= a(n) . \end{aligned}$$

Note that $e(n) = \eta(a \leq n)$.

3.14. Theorem. For all n, $eca(n) = c(n) - 1$.

Proof. We have shown that if $ca(n) \in (a)$, then $ca(n) = a(c(n) - 1)$, and if $ca(n) \in (b)$, $ca(n) = a(c(n) - 1) + 1$. In either case, $eca(n) = c(n) - 1$.

3.15. Theorem. For all n, $ecb(n) = ac(n)$.

Proof. Case 1. If $n \in (r)$, $cb(n) = bc(n)$ and $ecb(n) = ebc(n) = ac(n)$.

Case 2. If $n \in (s)$, $cb(n) = a^2c(n)$ and $ecb(n) = e(a^2c(n)) = ac(n)$.

4. THE FUNCTIONS c', ϕ, ϕ', ψ, ψ'

In this section, we consider some functions which arise in a natural way from the results of Section 3, and give some of their properties. Recall that c' denotes the complementary function to c. We shall require some properties of c', given in the following.

4.1. Theorem. We have

(i) $$c'b(n) = c(n) - 1$$
$$c'(b(n) - 1) = c(n) - 2$$
$$c'(b(n) + 1) = c(n) + 1$$
$$c'(b(n) + 2) = c(n) + 2$$

(ii) $$c(n) + c'(n) = 5n - 1$$

(iii) $$c'(n) = n + \eta(b < n)$$

(iv) $$c'a(n) = c'(a(n) + 1) - 1$$

(v) $$c'ab(n) = ca(n) + 1\ .$$

Proof. Since $c'(c(n) - n) = c(n) - 1$ (by Theorem 2.2) and $c(n) - n = b(n)$, we have $c'b(n) = c(n) - 1$. The rest of (i) follows from the fact that $c(n + 1) - c(n) \geq 3$ for all n (see (2.15) and (2.16)).

The proof of (ii) is straightforward. For example,

$$\begin{aligned} cb(n) + c'b(n) &= ab(n) + 2b(n) + c(n) - 1 \\ &= a(n) + 4b(n) + n - 1 \\ &= 5b(n) - 1\ . \end{aligned}$$

To see (iii), use (i) and the fact that $\eta(b < b(n)) = n - 1$ and $\eta(b < a(n)) = a(n) - n$ (see (2.20)). Both (iv) and (v) follow from (i).

4.2. Theorem. For all n, $ec(n) \in (c')$.

Proof. Case 1. If $n \in (a)$, say $n = a(j)$, then $ec(n) = c(j) - 1 = c'b(j)$.

Case 2. If $n \in (b)$, say $n = b(j)$, then $ec(n) = ac(j)$. We have seen that $ac(j)$ is either $ca(j) + 1$ or $ca(j) + 2$; in either case, $ac(j) \in (c')$.

We may now define a strictly monotonic function ϕ by the equality

$$ec(n) = c'\phi(n)\ .$$

The complementary function ϕ' is also of interest.

4.3. Theorem. For all n,

(i) $$\phi a(n) = b(n)$$

(ii) $$\phi br(n) = abr(n)$$

(iii) $$\phi bs(n) = abs(n) + 1\ .$$

Proof.

(i) $$eca(n) = c(n) - 1 = c'b(n) = c'\phi a(n)$$

(ii) $$\begin{aligned} ecbr(n) &= acr(n) = car(n) + 1 = c'(bar(n) + 1) \\ &= c'(abr(n)) = c'\phi br(n) \end{aligned}$$

(iii)
$$ecbs(n) = acs(n) = cas(n) + 2 = c'(bas(n) + 2)$$
$$= c'(abs(n) + 1) = c'\phi bs(n) .$$

4.4. Theorem. The function ϕ' is separated.

Proof. We show that for all n, $\phi(n+1) - \phi(n) \leq 2$. It then follows that for all n, $\phi'(n+1) - \phi'(n) \geq 2$. Since $(b) \subset (\phi)$, then for all n, $\phi(n+1) - \phi(n) \leq 3$. If $\phi(n) \in (b^2)$, then $\phi(n+1) - \phi(n) = 2$. If $\phi(n) \in (ba)$, then $\phi(n) + 1 \in (ab)$ and $\phi(n) + 2 \in (ab+1)$, and it follows that either $\phi(n) + 1$ or $\phi(n) + 2$ is in (ϕ). If $\phi(n) \in (a^2)$, then $\phi(n) + 1 = \phi(n+1)$ and if $\phi(n) \in (ab)$, then $\phi(n) + 2 = \phi(n+1)$. This completes the proof.

4.5. Theorem. If n is any integer not of the form $n = as(j) + 1$, then $\phi'(n) = a^2(n)$. If $n = as(j) + 1$ for some j, then $\phi'(n) = a(a(n) - 1)$.

Proof. Since by Theorem 4.3 we have

$$(\phi) = (b) \cup (abr) \cup (abs + 1)$$

it follows that

$$(\phi') = (a^2) \cup (abs)/(abs + 1) .$$

We next show that $\phi'(as(j) + 1) = abs(j)$ for all j. We have $\phi bs(j) = abs(j) + 1$. Since $abs(j) \in (\phi')$, then by Theorem 2.2 we have

$$\begin{aligned} \phi bs(j) - 1 &= \phi'(\phi bs(j) - bs(j)) \\ &= \phi'(abs(j) + 1 - bs(j)) \\ &= \phi'(as(j) + 1) \\ &= abs(j) . \end{aligned}$$

Next, if $n = as(j) + 1$, then

$$a(a(n) - 1) = a(a(as(j) + 1) - 1) = a(a^2s(j) + 2 - 1) = a(a^2s(j) + 1) = a(bs(j)) .$$

Thus if $n = as(j) + 1$, $\phi'(n) = a(a(n) - 1)$. Now (ϕ') differs from (a^2) only in that

$$\phi'(as(n) + 1) = a(a(n) - 1) = abs(n)$$

while $a^2(as(n) + 1) = abs(n) + 1$. Thus since ϕ' and a^2 are strictly monotonic, we must have $\phi'(n) = a^2(n)$ for all n not of the form $as(j) + 1$, and $\phi'(n) = a(a(n) - 1)$ for $n = as(j) + 1$ for some j.

It is now possible to define a new strictly monotonic function ψ by $\psi(n) = e\phi'(n)$.

4.6. Theorem. $\psi(n) = a(n)$ for all n not of the form $as(j) + 1$, and $\psi(n) = a(n) - 1$ for $n = as(j) + 1$. Thus $\psi(as(j) + 1) = a(as(j) + 1) - 1 = bs(j)$.

Proof. This follows from Theorem 4.5, and the definition of the function e.

Now ψ', the complementary function to ψ, is also strictly monotonic, and we have

$$(\psi') = (b) \cup (bs + 1)/(bs) .$$

4.7. Theorem. $\psi' r(n) = br(n)$ and $\psi' s(n) = bs(n) + 1$.

Proof. We first show that $\psi' s(n) = bs(n) + 1$. By Corollary 2.3, we have, since $bs(n) + 1 \in (\psi')$, and $bs(n) \in (\psi)$,

$$\begin{aligned} bs(n) + 1 &= \psi'(\psi(as(n) + 1) - (as(n) + 1) + 1) \\ &= \psi'(bs(n) - as(n)) \\ &= \psi'(s(n)) . \end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 4.5.

Using the results of this section, we can easily derive various formulas.

4.8. Theorem.

(i) $$\phi s(n) = bes(n)$$

(ii) $$\phi a(n) + \phi' a(n) = \phi\phi' a(n) - 1$$

(iii) $$\phi'\phi s(n) = abs(n)$$

(iv) $$\phi\phi' s(n) = abs(n) - 1$$

(v) $$\phi\phi' r(n) = bar(n)$$

(vi) $$\phi s(n) + c' s(n) = 3s(n) .$$

5. THE FUNCTIONS r AND s AND SOME RELATED FUNCTIONS

In this section, we consider the functions r and s in detail, and introduce some important auxiliary functions.

5.1. Theorem. The function s is separated.

Proof. Suppose, on the contrary, that consecutive integers $n, n + 1$ are in (s). Then both $n, n + 1$ must be in (a) by Lemma 3.10, and both $c(n)$ and $c(n + 1)$ must be in (b) by Theorem 3.6. Since $n, n + 1$ are both in (a), we must have $n = ab(j)$ for some j. Then

$$c(n + 1) = c(ab(j) + 1) = cab(j) + 4 .$$

If $j \in (r)$, then $cab(j) + 4 = bac(j) + 4$, and if $j \in (s)$, then $cab(j) + 4 = bac(j) + 2$. In either case, since for any integer k, $ba(k) + 2$ and $ba(k) + 4$ are both in (a), we must have $c(n + 1) \in (a)$, which is a contradiction. Thus s is a separated function.

5.2. Lemma. $(ab) \subset (s)$.

Proof. The proof is manipulative like the proof of Lemma 3.10. Using the definition $c(n) = a(n) + 2n$, one first shows that, for all n,

$$cbab(n) + 1 = 7ab(n) + 4b(n) - 3 .$$

Case 1. $n \in (r)$. Then we have

$$cb(n) = bc(n) \qquad \text{(Corollary 3.2)}$$
$$ac(n) = ca(n) + 1 \qquad \text{(Theorem 3.5)}$$
$$cab(n) = bac(n) \qquad \text{(Lemma 3.9)} .$$

Then

$$\begin{aligned} bcab(n) &= b^2ac(n) \\ &= a^2c(n) + 2bac(n) \\ &= bc(n) - 1 + 2(abc(n) - 1) \\ &= cb(n) - 1 + 2(ac(n) + bc(n) - 1) \\ &= cb(n) - 1 + 2(ca(n) + 1) + 2cb(n) - 2 \\ &= 3(ab(n) + 2b(n)) + 2(ba(n) + a(n) + 1) - 3 \\ &= 5ab(n) + 6b(n) + 2a(n) - 3 \\ &= 7ab(n) + 4b(n) - 3 \\ &= cbab(n) + 1 . \end{aligned}$$

Case 2. $n \in (s)$. Then

$$cb(n) = bc(n) - 1 \qquad \text{(Corollary 3.2)}$$
$$ac(n) = ca(n) + 2 \qquad \text{(Theorem 3.5)}$$
$$cab(n) = bac(n) - 2 \qquad \text{(Lemma 3.9)} .$$

By Corollary 3.12 we have $ca(n) \in (a)$, and since $ca(n) + 2 = ac(n)$, we have $ac(n) - 1 \in (b)$. By Theorem 3.7, we have $c(n) \in (b)$, say $c(n) = b(j)$. Then

$$\begin{aligned} cab(n) &= bab(j) - 2 \\ &= b(ba(j) + 1) - 2 \\ &= b^2a(j) + 2 - 2 \\ &= b^2a(j) . \end{aligned}$$

Thus $bac(n) - 2 \in (b)$, and so $bac(n) - 2 = b(ac(n) - 1)$. Now

$$\begin{aligned} bcab(n) &= b^2(ac(n) - 1) \\ &= ab(ac(n) - 1) + b(ac(n) - 1) \\ &= a(ac(n) - 1) + 2b(ac(n) - 1) \\ &= a(ca(n) + 1) + 2b(cs(n) + 1) . \end{aligned}$$

Since $ca(n) \in (a)$, we have $ca(n) = a(c(n) - 1)$, and so

$$\begin{aligned} bcab(n) &= a(a(c(n) - 1) + 1) + 2b(a(c(n) - 1) + 1) \\ &= a^2(c(n) - 1) + 2 + 2[a(a(c(n) - 1) + 1) + a(c(n) - 1) + 1] \\ &= 3(a^2(c(n) - 1) + 2) + 2a(c(n) - 1) + 2 \\ &= 3(b(c(n) - 1) + 1) + 2a(c(n) - 1) + 2 \\ &= 3b(c(n) - 1) + 2a(c(n) - 1) + 5 \\ &= 3b(c(n) - 1) + 2ca(n) + 5 \quad . \end{aligned}$$

Since $c(n) \in (b)$, then $c(n) - 1 \in (a)$ and we have $bc(n) = b(c(n) - 1) + 3$. Then

$$\begin{aligned} bcab(n) &= 3(bc(n) - 3) + 2ca(n) + 5 \\ &= 3(cb(n) - 2) + 2(ba(n) + a(n)) + 5 \\ &= 3(ab(n) + 2b(n) - 2) + 2a(n) + 2ab(n) - 2 + 5 \\ &= 5ab(n) + 6b(n) + 2a(n) - 3 \\ &= 7ab(n) + 4b(n) - 3 \\ &= cbab(n) + 1 \ . \end{aligned}$$

This completes the proof.

5.3. Lemma. For all n, $a^2(bs(n) + 1) \in (s)$.

Proof.

$$\begin{aligned} cb(a^2(bs(n) + 1)) &= cb(b^2s(n) + 1) \\ &= c(b^3s(n) + 2) \\ &= cb^3s(n) + 7 \\ &= b^2cbs(n) + 7 \\ &= [b^2a(acs(n)) + 3] + 4 \\ &= ab^2(acs(n)) + 4 \quad . \end{aligned}$$

But $ab^2(n) + 4 = a(bab(n) + 1)$ for all n, so $cb(a^2(bs(n) + 1)) \in (a)$, and this completes the proof.

5.4. Lemma. If $a^2(n) \in (s)$, then $n = bs(k) + 1$ for some integer k.

Proof. First note that $ab(b(n) + 1) = ab^2(n) + 3$, and $ab(a(n) + 1) = aba(n) + 5$. Since s is separated, if $a^2(n) \in (s)$, there must be some integer $a(j)$ so that $aba(j) + 1 < a^2(n) < ab(a(j) + 1) - 1$. Thus

$$a^2(n) = aba(j) + 3$$

and

$$a^2(n) + 2 = a(a(n) + 1) = ab(a(j) + 1) .$$

Then

$$a(n) + 1 = b(a(j) + 1) = ba(j) + 3 = ab(j) + 2$$

and

$$a(n) = a(b(j) + 1)$$

so that

$$n = b(j) + 1 .$$

We now show that $j = s(k)$ for some integer k. First $ca^2(n) \in (b)$ gives:

$$ca^2(n) = ca^2(b(j) + 1) = a(ca(b(j) + 1) - 1) + 1 \in (b)$$

so that $ca(b(j) + 1) - 1 \in (a)$. We also have:

$$ca(b(j) + 1) = c(ab(j) + 1) = cab(j) + 4$$

and we have seen before that $cab(j) + 4$ is always in (a). Thus

$$ca(b(j) + 1) = a(c(b(j) + 1) - 1) .$$

Now it must be that

$$ca(b(j) + 1) - 1 = a(c(b(j) + 1) - 2)$$

and hence

$$c(b(j) + 1) - 2 \in (b)$$

by (2.7, (iii).). But

$$c(b(j) + 1) - 2 = cb(j) + 3 - 2 = cb(j) + 1 \in (b) .$$

Thus $bc(j) = cb(j) + 1$ and $j \in (s)$. We now have the following:

5.5. Theorem. $(s) = (ab) \cup (a^2(bs + 1))$.

We can now prove

5.6. Theorem. $c(n) \in (a)$ if and only if $n \in [(a) \cup (bs)]/(s)$.

Proof. Clearly if $n \in (bs)$, $c(n) \in (a)$, and if $n \in (s)$, then $c(n) \in (b)$. Also, if $n \in (br)$, then $c(n) \in (b)$. Thus, suppose $n \in (a)/(s)$. Then suppose $ca^2(j) \in (b)$. Then $a(j) \notin (s)$, since $cas(n) \in (a)$ for all n. Thus $cba(j) = bca(j)$. Now if $ca^2(j) \in (b)$, then $ca^2(j) = a(ca(j) - 1) + 1$ so that $ca(j) - 1 \in (a)$. Then $b(ca(j) - 1) + 3 = bca(j)$ and

$$cba(j) = c(a^3(j) + 1) = ca^3(j) + 4 = b(ca(j) - 1) + 3 .$$

Then

$$ca^3(j) = b(ca(j) - 1) - 1 \in (a) .$$

Now, since $a^2(j) \notin (a)$, we have

$$ca(a^2(j)) + 1 = aca^2(j) ,$$

so that if $ca^2(j) \in (b)$, then $ca^3(j) \notin (b)$. This is a contradiction, and the proof is complete.

5.7. Corollary. $c(n) \in (b) \rightleftarrows n \in (br) \cup (s)$.

We now introduce some additional functions, defined as follows:

$$\text{(5.8)} \quad \begin{aligned} &\text{(i)} & u'(n) &= bs(n) + 1 \\ &\text{(ii)} & t'(n) &= as(n) + n \\ &\text{(iii)} & bz(n) &= cs(n) . \end{aligned}$$

We also have the corresponding complementary functions u, t and z'.

The reasons for considering these functions are made evident in the following theorem.

5.9. Theorem. We have

$$\begin{aligned} &\text{(i)} & (u') &= \{n \mid a^2(n) \in (s)\} \\ &\text{(ii)} & (t') &= \{n \mid s(n) \in (a^2)\} \\ && (t) &= \{n \mid s(n) \in (ab)\} \\ &\text{(iii)} & st(n) &= ab(n) \\ &\text{(iv)} & st'(n) &= a^2(bs(n) + 1) = a^2u'(n) \\ &\text{(v)} & zt(n) &= cs(n) + 1 \\ &\text{(vi)} & zt'(n) &= cbs(n) + 1 \\ &\text{(vii)} & z(n) &= c's(n) . \end{aligned}$$

Proof. (i) is clear from Theorem 5.5. To see (ii) we take (ii) as the definition of t and t' and show that we then have $t'(n) = as(n) + n$. In the proof of Lemma 5.4, it was shown that

$$abas(n) < a^2(bs(n) + 1) < ab(as(n) + 1)$$

for all integers n. From (ii), we have $a^2(bs(n) + 1) = st'(n)$, and $stas(n) = abas(n)$ (that is, $a^2(bs(n) + 1)$ is the n^{th} value of s of the form $a^2(bs(j) + 1)$, and $abas(n)$ is the $as(n)^{th}$ value of s of the form $ab(j)$). Now $stas(n) = s(t'(n) - 1)$, so that $t'(n) = tas(n) + 1$. From Theorem 2.1, we have

$$\begin{aligned} t'(n) &= n + \eta(t < t'(n)) \\ &= n + \eta(t < tas(n) + 1) \\ &= n + as(n) . \end{aligned}$$

Parts (iii) and (iv) follow from (ii).

To see (v): Case 1. $n \in (r)$. Then from the definition, $bzt(n) = cst(n) = cab(n) = bac(n)$. Then $zt(n) = ac(n) = ca(n) + 1$ since $n \in (r)$.

Case 2. $n \in (s)$. Then $bzt(n) = cst(n) = cab(n) \in (b)$ and since $n \in (s)$, $cab(n) = bac(n) - 2 = b(ac(n) - 1)$. Then $zt(n) = ac(n) - 1 = ca(n) + 1$ since $n \in (s)$.

To see (vi):

$$\begin{aligned} cst'(n) &= ca^2(bs(n) + 1) \\ &= c(b(bs(n) + 1) - 1) \\ &= c(b^2s(n) + 1) \\ &= cb^2s(n) + 3 \\ &= bcbs(n) + 3 \\ &= ba^2cs(n) + 3 \\ &= b(a^2cs(n) + 1) \quad . \end{aligned}$$

So $zt'(n) = a^2cs(n) + 1 = cbs(n) + 1$.

For (vii), we have first

$$c'st(n) = c'ab(n) = ca(n) + 1 = zt(n) \quad .$$

On the other hand,

$$\begin{aligned} cbs(n) + 1 = c'(b^2s(n) + 1) &= c'(b(bs(n) + 1) - 1) \\ &= c'(a^2(bs(n) + 1)) \\ &= c'st'(n) = zt'(n) \; . \end{aligned}$$

5.10. Theorem. $s(n) = c(n)$ if and only if $t'(n) = b^2(n)$.

Proof. We use the fact that $t'(n) = as(n) + n$, and consider the cases $n \in (r)$ and $n \in (s)$. If $n \in (r)$ and $s(n) = c(n)$, then

$$\begin{aligned} t'(n) = ac(n) + n &= ca(n) + 1 + n \\ &= ba(n) + 1 + a(n) + n \\ &= ab(n) + b(n) \\ &= b^2(n) \quad . \end{aligned}$$

If $n \in (s)$, then $c(n) \in (b)$ and $c(n) = s(n)$ is not possible.

Now suppose $t'(n) = b^2(n)$. Then

$$\begin{aligned} as(n) + n &= ab(n) + b(n) \\ &= ab(n) + a(n) + n \end{aligned}$$

and

$$\begin{aligned} as(n) &= ab(n) + a(n) \\ &= ca(n) + 1 \quad . \end{aligned}$$

Then $ca(n) + 1 \in (a)$. If $ca(n) + 1 = ac(n)$, then we have $c(n) = s(n)$ as required. If $ca(n) + 2 = ac(n)$, then we have $n \in (s)$, and $ca(n) \in (a)$, $ca(n) + 1 \in (b)$ (Theorem 3.5). Thus if $as(n) = ca(n) + 1$, then $n \in (r)$ and $c(n) = s(n)$.

5.11. Corollary. If $s(n) = c(n)$, then $n \in (r)$.

5.12. Theorem. If $s(n) = c(n)$, for some n, then

$$s(t'(n) + 1) = c(t'(n) + 1 \, .$$

Proof. If $s(n) = c(n)$, then we have $n \in (r)$ and $t'(n) = b^2(n)$. Then

$$\begin{aligned} ca^2b(n) &= a(cab(n) - 1) \\ &= a(bac(n) - 1) = a(a^3c(n)) \\ &= a^2(a^2c(n)) = ba^2c(n) - 1 \\ &= abac(n) - 2 \\ &= stac(n) - 2 \\ &= stas(n) - 2 \\ &= st'(n) - 5 \quad . \end{aligned}$$

On the other hand,

$$c(a^2b(n)) = c(b^2(n) - 1) = cb^2(n) - 4 \, .$$

Thus we have

$$\begin{aligned} cb^2(n) &= ab^2(n) - 1 \\ cb^2(n) + 3 &= sb^2(n) + 2 \\ &= st'(n) + 2 \\ &= s(t'(n) + 1) \quad \text{(by Theorem 6.5 (iv))} \, . \end{aligned}$$

Since $cb^2(n) + 3 = c(b^2(n) + 1) = c(t'(n) + 1)$, the proof is complete.

5.13. Theorem. If $s(n) = c(n)$, then $z(n) = 5n - 1$.

Proof. By Theorem 6.1 (iv) we have

$$\begin{aligned} z(n) &= 2s(n) - es(n) = 3s(n) - (s(n) + es(n)) = 3s(n) - (as(n) + 1) \\ &= 3c(n) - ac(n) - 1 = 3a(n) + 6n - (ca(n) + 1) - 1 \quad \text{since } n \in (r) \\ &= 3a(n) + 6n - 1 - b(n) - 2a(n) \\ &= 5n - 1 \quad . \end{aligned}$$

Other results of this nature are easily obtained; for example:

5.14. Corollary. If $s(n) = c(n)$, then

(i) $$rb(n) = c'b(n) = c(n) - 1$$

(ii) $$z'(4n - 1) = 5n - 2 .$$

It should be noted that since $s(1) = c(1)$, for example, it follows that there are infinitely many values of n for which $s(n) = c(n)$. We list the values of $n \leq 101$ for which $s(n) = c(n)$:

Table 1

n	1	6	9	22	40	43	48	56	61	64
s(n) = c(n)	3	21	32	79	144	155	173	202	220	231

We note that $t'(1) + 1 = 6$ and $t'(6) + 1 = 40$, and $t'(9) + 1 = 61$, while $t'(1) + 4 = 9$, $t'(6) + 4 = 43$, $t'(9) + 4 = 64$. One might conjecture that if $s(n) = c(n)$, we have not only $s(t'(n) + 1) = c(t'(n) + 1)$, but also $s(t'(n) + 4) = c(t'(n) + 4)$.

Using the fact that $(s) = (ab) \cup (a^2u')$ where $(u') = (bs + 1)$, we may express (s) as an infinite union as follows:

5.15. Theorem. We have

$$(s) = \bigcup_{k=0}^{\infty} (a(a^2b)^k b) .$$

Proof. The proof is by induction. We first show that every $x \in (s)$ satisfies

(5.16) $$x = a(a^2b)^k b(j)$$

for some integers k, j.

For $n = 1$, we have $s(1) = ab(1)$. Suppose n given, and for all $k < n$ we have

(5.17) $$s(k) = a(a^2b)^j b(m)$$

for some integers j and m (depending on k). Now $s(n)$ might be of the form $ab(N)$, for some N, in which case $s(n)$ satisfies (5.16), or else $s(n) = a^2u'(N)$ for some N. In the latter case, we have

$$u'(N) = bs(N) + 1$$

and since $s(n) = a^2u'(N)$, it must be that $N < n$. By the induction assumption,

$$s(N) = a(a^2b)^j b(m)$$

for some integers j and m, and so

$$\begin{aligned} s(n) &= a^2u'(N) \\ &= a^2(bs(N) + 1) \\ &= a^2(ba(a^2b)^j b(m) + 1) \\ &= a^2(ab(a^2b)^j b(m)) \\ &= a(a^2b)^{j+1}b(m) \quad . \end{aligned}$$

This completes the induction, and we have

$$(5.18) \qquad (s) \subseteq \bigcup_{k=0}^{\infty} (a(a^2b)^k b) \quad .$$

To show inclusion in the other direction, let m be a fixed integer. We show by induction that every integer of the form

$$(5.19) \qquad K = a(a^2b)^k b(m) \qquad (k = 0, 1, 2, \cdots)$$

satisfies $K \in (s)$.

When $k = 1$, we have $ab(m) \in (s)$. Suppose for some integer $k > 1$ we have $a(a^2b)^k b(m) \in (s)$, say

$$s(N) = a(a^2b)^k b(m) \; .$$

Then $a^2u'(N) \in (s)$, and since

$$\begin{aligned} a^2(bs(N) + 1) &= a^2(ba(a^2b)^k b(m) + 1) \\ &= a^2(ab(a^2b)^k (b(m)) \\ &= a(a^2b)^{k+1}b(m) \end{aligned}$$

we have $a(a^2b)^{k+1}b(m) \in (s)$. Thus for all m, we have

$$\left\{ a(a^2b)^k b(m) \mid k = 0, 1, 2, \cdots \right\} \subset (s).$$

This completes the proof.

Using Theorem 5.15 and the fact that $(r) \cup (s) = \mathbf{N}$, it is easy to prove

5.20. Corollary.

$$(r) = (b) \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k ab) \right] \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k a^3) \right] \; .$$

Since $u'(n) = bs(n) + 1 \quad (n = 1, 2, \cdots)$, we have

5.21. Corollary.

$$(u') = \bigcup_{k=0}^{\infty} (ab(a^2b)^k b) .$$

In a similar fashion, one can find "infinite union" formulas for many of the other functions mentioned in this paper. However, we have not been able to give any such formula for t'.

Theorem 5.15 suggests the definition of a set of functions $\{f_k\}$ as follows:

$$s(f_k(n)) = a(a^2b)^k b(n) . \tag{5.22}$$

It is evident, for example, that $f_0(n) = t(n)$. The functions f_k are completely described in the next theorem.

5.23. Theorem. For all n, $f_k(n) = (t')^k t(n)$.

Proof. The proof is by induction on k. It is clear that $f_0(n) = t(n)$. Suppose for some $k > 0$, we have

$$f_{k-1}(n) = (t')^{k-1} t(n) \qquad (n = 1, 2, 3, \cdots) . \tag{5.24}$$

Then

$$\begin{aligned} st'(f_{k-1}(n)) &= a^2(bs)f_{k-1}(n)) + 1) \\ &= a^2(ba(a^2b)^{k-1}b(n) + 1) \end{aligned}$$

by the induction assumption. This gives

$$\begin{aligned} st'(f_{k-1}(n)) &= a^2(ab(a^2b)^{k-1}b(n)) \\ &= a(a^2b)^k b(n) \end{aligned}$$

and it follows that

$$f_k(n) = t'f_{k-1}(n) .$$

Then for all k, we have $f_k(n) = (t')^k t(n)$. This completes the proof.

In Section 6, we shall show that $t't(n) = tb^2(n) - 1$ and also $t't(n) = t(b^2(n) - 1) + 1$ (Theorem 6.3). In view of this, we have the following inequalities for the functions f_k.

5.25. Theorem. For all integers j and k, $k > 0$,

$$f_{k-1}(b^2(j) - 1) < f_k(j) < f_{k-1}(b^2(j)) .$$

In addition, if

$$f_{k-1}(b^2(j) - 1) < f_k(m) < f_{k-1}(b^2(j))$$

then $m = j$.

Proof. Since $t't(n) = t(b^2(n) - 1) + 1$, we have

$$f_{k-1}(b^2(j) - 1) = (t')^{k-1}t(b^2(j) - 1) = (t')^{k-1}(t't(j) - 1) < (t')^k t(j) ,$$

and since $t't(n) = tb^2(n) - 1$, we have

$$f_{k-1}(b^2(j)) = (t')^{k-1}t(b^2(j))$$
$$= (t')^{k-1}(t't(j) + 1) > (t')^k t(j) .$$

To see that $f_k(j)$ is the only value of f_k between $f_{k-1}(b^2(j) - 1)$ and $f_{k-1}(b^2(j))$, consider for example $f_k(j - 1)$. By the preceding argument,

$$f_k(j - 1) < f_{k-1}(b^2(j - 1)) < f_{k-1}(b^2(j) - 1)$$

since f_{k-1} is strictly monotonic and $b^2(j - 1) < b^2(j) - 1$. On the other hand, we have

$$f_k(j + 1) > f_{k-1}(b^2(j + 1) - 1) > f_{k-1}(b^2(j))$$

since f_{k-1} is strictly monotonic, and $b^2(j + 1) > b^2(j) + 1$. This completes the proof.

6. CONTINUATION

In this section, we give various formulas involving the functions introduced in Section 5.

6.1. Theorem.

(i) $$z(n) = c(z(n) - s(n)) + 1$$

(ii) $$s(n) = b(z(n) - s(n)) + 1$$

(iii) $$az(n) - as(n) = es(n)$$

(iv) $$z(n) + es(n) = 2s(n)$$

(v) $$az(n) - as(n) = a(z(n) - s(n)) + 1 .$$

Proof. (i) Case 1. $zt(n) = ca(n) + 1$ and $st(n) = ab(n)$. Then

$$zt(n) - st(n) = ca(n) + 1 - ab(n)$$
$$= ba(n) + a(n) + 1 - ab(n)$$
$$= a(n)$$

and so $zt(n) = c(zt(n) - st(n)) + 1$.

Case 2. $zt'(n) = cbs(n) + 1$ and $st'(n) = a^2(bs(n) + 1)$. As above, we show that $zt'(n) - st'(n) = bs(n)$, and (i) follows.

To see (ii), we have

$$z(n) = c(z(n) - s(n)) + 1$$
$$= b(z(n) - s(n)) + z(n) - s(n) + 1$$

and this proves (ii).

For (iii), we have (since $as(n) = s(n) + es(n) - 1$)

$$\begin{aligned} az(n) &= ebz(n) = ecs(n) = ecaes(n) \\ &= ces(n) - 1 = bes(n) + es(n) - 1 \\ &= aes(n) + es(n) + es(n) - 1 \\ &= (s(n) + es(n) - 1) + es(n) \\ &= as(n) + es(n) . \end{aligned}$$

To see (iv), we use $z(n) = c's(n) = s(n) + \eta(b < s(n))$. Then

$$\begin{aligned} z(n) - s(n) &= (s(n) - 1) - \eta(a < s(n)) \\ &= s(n) - [\eta(a < s(n)) + 1] \\ &= s(n) - es(n) . \end{aligned}$$

Finally, (v) follows from

$$z(n) = a(z(n) - s(n)) + 2(z(n) - s(n)) + 1$$

$$\begin{aligned} a(z(n) - s(n)) &= 2s(n) - z(n) - 1 \\ &= es(n) - 1 \\ &= az(n) - as(n) - 1 . \end{aligned}$$

6.2. Remark. Theorem 6.1 could also have been proved by noting that $z(n) - s(n)$ is a monotonic function satisfying

$$(z - s) = (a) \cup (bs) .$$

6.3. Theorem.

(i) $$t't(n) = a^2b(n) + t(n)$$

(ii) $$tb^2(n) = t(n) + b^2(n)$$

(iii) $$t't(n) = tb^2(n) - 1$$

(iv) $$\eta(t(n) < t' < t(n) + b^2(n)) = n .$$

Proof.

(i): by definition,

$$t't(n) = ast(n) + t(n) = a^2b(n) + t(n) .$$

(ii): We know $t(as(n)) = t(t'(n) - n) = t'(n) - 1$ by Theorem 2.2. Then

$$\begin{aligned} t(ast(n)) &= t't(n) - 1 \\ t(a^2b(n)) &= t't(n) - 1 \\ t(a^2b(n) + 1) &= t't(n) + 1 = a^2b(n) + t(n) + 1 \\ tb^2(n) &= b^2(n) + t(n) . \end{aligned}$$

Statement (iii) follows directly from (i) and (ii), and statement (iv) follows from (iii) and Theorem 2.4.

6.4. Theorem.

(i) $$\mathrm{tsta}(n) = t(n) - 1 + \mathrm{sta}(n)$$

(ii) $$\mathrm{tstb}(n) = \mathrm{ta}(n) + \mathrm{stb}(n)$$

(iii) $$\mathrm{tab}(n) = \mathrm{te}(n) + \mathrm{ab}(n) - \delta$$

where $\delta = 0$ if $n \in (b)$ and $\delta = 1$ if $n \in (a)$.

(iv) $$\mathrm{taba}(n) = t(n) - 1 + \mathrm{aba}(n)$$

(v) $$\mathrm{tab}^2(n) = \mathrm{ta}(n) + \mathrm{ab}^2(n).$$

Proof. For the proof, we require the following identities (See Section 2):

$$b^2(n) = \mathrm{aba}(n) + 2$$

$$\mathrm{ab}^2(n) = b^2a(n) + 3 .$$

Since $t'(n + 1) - t'(n) \geq 4$ for all n, we have

$$t(b^2 - k) = tb^2 - k - 1 \quad \text{for} \quad k = 1, 2, 3$$

and

$$t(b^2 + k) = tb^2 + k \quad \text{for} \quad k = 1, 2, 3 .$$

To see (i), we have

$$\begin{aligned} \mathrm{tsta}(n) &= \mathrm{tab}(n) = t(b^2(n) - 2) \\ &= tb^2(n) - 3 \\ &= t(n) + b^2(n) - 3 \\ &= t(n) + (b^2(n) - 2) - 1 \\ &= t(n) + \mathrm{aba}(n) - 1 \\ &= t(n) + \mathrm{sta}(n) - 1 . \end{aligned}$$

Statement (ii) follows similarly. Statements (iv) and (v) are simply restatements of (i) and (ii). For (iii), note that $\mathrm{ea}(n) = n$ and $\mathrm{eb}(n) = a(n)$ and apply (i) and (ii).

It is of some interest to determine for what values of n the difference $s(n + 1) - s(n)$ takes on the value 2 (or 3, or 5), and similarly for $t'(n + 1) - t'(n)$. The next theorem gives a complete description of this.

6.5. Theorem.

(i) $$s(\mathrm{tb}(n) + 1) = \mathrm{stb}(n) + 3$$

(ii) $$s(\mathrm{tas}(n) + 1) = \mathrm{stas}(n) + 3$$

(iii) $$s(tar(n) + 1) = star(n) + 5$$

(iv) $$s(t'(n) + 1) = st'(n) + 2$$

(v) $$t'(t'(n) + 1) = t't'(n) + 4$$

(vi) $$t'(tar(n) + 1) = t'tar(n) + 9$$

(vii) $$t'(tb(n) + 1) = t'tb(n) + 6$$

(viii) $$t'(tas(n) + 1) = t'tas(n) + 6 .$$

Proof. (i). $stb(n) = ab^2(n)$. Since $ab^2(n) + 3 = ab(b(n) + 1)$, s is a separated function, and $(ab) \subset (s)$, and we must have $ab(b(n) + 1) = s(tb(n) + 1)$.

(ii). $stas(n) + 3 = abas(n) + 3 = a^2(bs(n) + 1)$, and so $stas(n) + 3 = st'(n)$. This proves (ii).

(iii). $star(n) + 5 = abar(n) + 5 = ab(ar(n) + 1)$. We have seen that if $a^2(j) \in (s)$ and $ab(n) < a^2(j) < ab(n + 1)$, then we must have $n \in (as)$. This proves (iii).

(iv). $s(t'(n) + 1) = s(tas(n) + 2)$, and we have

$$stas(n) = abas(n)$$

$$stas(n) + 3 = a^2(bs(n) + 1) = st'(n)$$

$$stas(n) + 5 = ab(as(n) + 1) = s(t'(n) + 1) .$$

This proves (iv).

(v). $t't'(n) = ast'(n) + t'(n)$ and

$$t'(t'(n) + 1) = as(t'(n) + 1) + (t'(n) + 1)$$
$$= a(st'(n) + 2) + t'(n) + 1 .$$

Now $st'(n) \in (a^2)$, so $st'(n) + 1 \in (b)$ and we have

$$a(st'(n) + 2) = a(st'(n) + 1) + 1$$
$$= (ast'(n) + 2) + 1 = ast'(n) + 3 .$$

Then

$$t'(t'(n) + 1) = ast'(n) + t'(n) + 4 = t't'(n) + 4 ,$$

and (v) is proved.

(vi). $t'(tar(n) + 1) = as(tar(n) + 1) + tar(n) + 1$

$$= a(star(n) + 5) + tar(n) + 1$$

$$= a(abar(n) + 5) + tar(n) + 1 .$$

Now $abar(n) + 2 \in (b^2)$, so $abar(n) + 4 \in (b)$ and $abar(n)$, $abar(n) + 1$, $abar(n) + 3$ are all in (a), while $abar(n) + 2$ and $abar(n) + 4$ are in (b). Then by 2.18 and 2.19,

$$a(abar(n) + 5) = a(abar(n)) + 8$$

and we have

$$\begin{aligned} t'(tar(n) + 1) &= a(star(n)) + tar(n) + 9 \\ &= as(tar(n)) + tar(n) + 9 \\ &= t'tar(n) + 9 . \end{aligned}$$

In a similar manner one proves (vii) and (viii).

6.6. Corollary.

(i) $$\begin{aligned} s(n) &= 3 + 3\eta(tb < n) + 3\eta(tas < n) \\ &\quad + 5\eta(tar < n) + 2\eta(t' < n) \end{aligned}$$

(ii) $$\begin{aligned} t'(n) &= 5 + 4\eta(t' < n) + 6\eta(tb < n) \\ &\quad + 6\eta(tas < n) + 9\eta(tar < n) \end{aligned}$$

(iii) $$2s(n) - t'(n) = 1 + \eta(tar < n) .$$

6.7. Theorem. $$b(n) = r(2n - \eta(u' \leq n)) .$$

Proof.

$$\begin{aligned} (r) &= (b) \cup (a^2)/(a^2u') \\ &= (b) \cup (b - 1)/(bu' - 1) . \end{aligned}$$

Thus

$$\begin{aligned} \eta(r < b(n)) &= 2n - 1 - \eta(bu' - 1 < b(n)) \\ &= 2n - 1 - \eta(u' \leq b(n)) . \end{aligned}$$

The result follows, since $b(n) \in (r)$.

6.8. Corollary. $bu'(n) = r(2u'(n) - n)$, and

$$b(u'(n) - 1) = r(2u'(n) - n - 1) = b^2s(n) .$$

Proof. The first statement is clear from Theorem 6.5. Since $bu'(n) - 1 = a^2u'(n) \in (s)$, it follows from the definition of r that $r(2u'(n) - n - 1) = b(u'(n) - 1)$. Since $u'(n) - 1 = bs(n)$, we have $r(2u'(n) - n - 1) = b^2s(n)$.

7. PROPERTIES OF OTHER RELATED FUNCTIONS

There are many additional functions which come about naturally from the consideration of relations between the functions r, s, t, t', u, u', and z, and the functions a, b, c, e. In this section we define the most interesting of these functions and list some of their properties.

7.1. Definitions.

(i). Since $(es) \subset (r)$, we define a strictly monotonic function p by: $es(n) = rp(n)$.

(ii). Since $(b) \subset (u)$, we define a strictly monotonic function v by

$$b(n) = uv(n).$$

(iii). Since $(es) = (b) \cup (abs + 1) \subset (u)$, we define a strictly monotonic function x by

$$es(n) = ux(n) .$$

(iv). Since $(ab)' = (a^2) \cup (b) \subset (u)$, we define a strictly monotonic function w by

$$(ab)' = (uw) .$$

(v). Since $(es) \subset (ab)' = (uw)$, we define a strictly monotonic function y by

$$es(n) = uwy(n) .$$

(vi). Since $(z) \subset (c')$ by Theorem 5.9 (vii), we define a strictly monotonic function λ by

$$c(n) = z'\lambda(n) .$$

(vii). Put $\sigma(n) = pt(n)$. Define a monotonic function τ by:

$$\tau(u(n)) = \sigma(u(n)) - 1$$
$$\tau(u'(n)) = \sigma(u'(n)) \quad .$$

(viii). Define a function K by:

$$K(n) = \eta(b < c(n)) .$$

7.2. <u>Theorem.</u> $pt(n) = 2n - \eta(u' \leq n)$

$$pt'(n) = 2as(n) + 1 - \eta(u' \leq as(n) + 1) .$$

<u>Proof</u>. Since $est(n) = b(n) = rpt(n)$, it follows from Theorem 6.8 that $pt(n) = 2n - \eta(u' \leq n)$. Recall that for all n, $tas(n) = t'(n) - 1$. Thus $t(as(n) + 1) = t'(n) + 1$. We know $rp(tas(n)) = b(as(n))$ and $rpt(as(n) + 1) = b(as(n) + 1)$. Since $as(n) + 1 \notin (u')$, it follows that $b(as(n) + 1) - 1 \in (r)$, and so

$$rp(t(as(n) + 1) - 1) = b(as(n) + 1) - 1$$

that is,

$$rpt'(n) = b(as(n) + 1) - 1.$$

By Theorem 6.8, we have

$$b(as(n) + 1) = r(2(as(n) + 1) - \eta(u' \leq as(n) + 1))$$

and so

$$b(as(n) + 1) - 1 = r(2(as(n) + 1) - \eta(u' \leq as(n) + 1) - 1)$$

which gives

$$pt'(n) = 2as(n) + 1 - \eta(u' \leq as(n) + 1)$$

and this completes the proof.

7.3. <u>Theorem</u>. $v(n) = b(n) - \eta(s < n)$.

<u>Proof</u>. We will show that, for all n,

(7.4) $$v(b(n) + 1) = vb(n) + 2$$

(7.5) $$v(s(n) + 1) = vs(n) + 2$$

(7.6) $$v(a(n) + 1) = va(n) + 3 \quad \text{if} \quad a(n) \notin (s).$$

Then since $v(1) = 2$, we have

$$\begin{aligned} v(n) &= 2 + 2\eta(b < n) + 2\eta(s < n) \\ &\quad + 3\eta(a < n) - 3\eta(s < n) \\ &= 2 + 2(n - 1) + \eta(a < n) - \eta(s < n) \qquad (7.7) \\ &= 2n + (a(n) - n) - \eta(s < n) \\ &= b(n) - \eta(s < n). \end{aligned}$$

We first prove (7.4). Since $b(n) = uv(n)$, we have $b^2(n) = uvb(n)$. Now $b^2(n) + 1 \in (u)$, since $(u') = (bs + 1)$ and $b^2(n) + 1 \neq bs(j) + 1$ for any n, j because $(s) \subset (a)$. Thus

$$u(vb(n) + 1) = uvb(n) + 1$$

and

$$u(vb(n) + 2) = uvb(n) + 2 .$$

Since $uvb(n) = b^2(n)$ and $uvb(n) + 2 = b^2(n) + 2 = b(b(n) + 1) = uv(b(n) + 1)$, it must be that $v(b(n) + 1) = vb(n) + 2$, as required. To see (7.5), note that $uvs(n) = bs(n)$, and $uvs(n) + 1 = u'(n)$. Then $u(vs(n) + 1) = uvs(n) + 2$ and $u(vs(n) + 2) = bs(n) + 3$. But $bs(n) + 3 = b(s(n) + 1)$ by (2.15), that is,

$$u(vs(n) + 2) = uv(s(n) + 1)$$

and so

$$vs(n) + 2 = v(s(n) + 1) .$$

As for (7.6), suppose $a(n) \notin (s)$. Then we show that none of $uva(n) + 1$, $uva(n) + 2$, $uva(n) + 3$ are in (u'). Since $a(n) \notin (s)$, $uva(n) + 1 = ba(n) + 1 \notin (u')$, since $(u') = (bs + 1)$. Also, $uva(n) + 2 = ba(n) + 2 = ab(n) + 1$ by (2.13) and clearly $ab(n) + 1 \neq bs(j) + 1$ for any n, j. Finally $uva(n) + 3 = ba(n) + 3 = b(a(n) + 1) \notin (u')$ and we have $uva(n) + 3 = uv(a(n) + 1)$. Thus $v(a(n) + 1) = va(n) + 3$ for $a(n) \notin (s)$. This completes the proof.

7.8. Theorem.

$$\begin{cases} xt(n) = v(n) \\ xt'(n) = au'(n) - \eta(s < as(n)) . \end{cases}$$

Proof. From the definition, $est(n) = uxt(n)$. Since $st(n) = ab(n)$, we have $b(n) = uxt(n)$. On the other hand, $uv(n) = b(n)$, and so $xt(n) = v(n)$.

For the second statement, we require the fact that $tas(n) + 1 = t'(n)$ (this follows from Theorem 2.2 and the fact that $t'(n) - n = as(n)$). Then

$$xtas(n) = vas(n) = bas(n) - \eta(s < as(n))$$

and also

$$uxtas(n) = uvas(n) = bas(n)$$

$$uxtas(n) + 1 = bas(n) + 1 = abs(n)$$

$$uxtas(n) + 2 = abs(n) + 1 = au'(n) .$$

Since $est'(n) = uxt'(n)$ and $est'(n) = au'(n)$, we must have $uxtas(n) + 2 = uxt'(n)$. Since $uxtas(n) + 1 = abs(n) \in (u)$, we have

$$uxt'(n) = u(xtas(n) + 2) ,$$

and so

$$\begin{aligned} xt'(n) &= xtas(n) + 2 \\ &= bas(n) + 2 - \eta(s < as(n)) \\ &= au'(n) - \eta(s < as(n)) . \end{aligned}$$

This completes the proof.

Table 2

n	1	2	3	4	5	6	7	30	50
t'(n)	5	14	20	29	35	39	45	207	342
xt'(n)	11	30	42	60	72	79	91	416	686

7.9. Theorem. $w(2n) = v(n)$ and $w(2n - 1) = v(n) - 1$.

Proof. Since $(ab)' = (b) \cup (b - 1)$ and $(ab)' = (uw)$ it is clear that $uw(2n) = b(n)$

and $uw(2n - 1) = b(n) - 1$. On the other hand, $b(n) = uv(n)$, and so $w(2n) = v(n)$. Since $b(n) - 1 \in (n)$ for all n, we have $b(n) - 1 = uv(n) - 1 = u(v(n) - 1) = uw(2n - 1)$, and $w(2n - 1) = v(n) - 1$.

7.10. Theorem. $w(n) = n + \eta(2a^2u < n)$.

Proof. We require the fact that

$$(u') = (abes) = (ab^2) \cup (abau') .$$

It follows that $(u) = (ab)' \cup (abau)$.

Note that $abau(j) = ba^2u(j) + 1$. Then

$$ba^2u(j) = uv(a^2u(j)) = uw(2a^2u(j)) .$$

Since $abau(j) \notin (uw)$ and

$$abau(j) + 1 = b^2u(j) - 1 \in (uw) ,$$

we have

$$w(2a^2u(j) + 1) = w(2a^2u(j)) + 2 . \tag{7.11}$$

On the other hand, since $(u) = (ab)' \cup (abau)$, if n is not of the form $2a^2u(j)$ for some j, then $w(n + 1) = w(n) + 1$. The theorem follows.

7.12. Corollary. $w'(n) = 2a^2u(n) + n$.

Proof. By Theorem 7.10, $w(2a^2u(n)) = 2a^2u(n) + n - 1$ and

$$w(2a^2u(n) + 1) = 2a^2u(n) + n + 1.$$

For all n not of the form $n = 2a^2u(j)$, we have $w(n + 1) = w(n) + 1$ and thus

$$(w') = \{2a^2u(n) + n \mid n = 1, 2, 3, \cdots\} .$$

As usual, w' is taken to be monotonic, and the theorem follows.

7.13. Corollary.

(i) $$b(n) = uv(n)$$

(ii) $$b(n) - 1 = uv'(v(n) - n)$$

(iii) $$b(n) - 1 = u(v(n) - 1)$$

(iv) $$abau(n) = uv'(a^2u(n) + n) .$$

Proof. (i) is evident from the definition of v. Statements (ii) and (iii) follow from the fact that $b(n) - 1 = uv(n) - 1 = u(v(n) - 1)$, and then by Theorem 2.2 $v(n) - 1 = v'(v(n) - n)$ since v is a separated function. To see (iv), note that $abau(n) = ba^2u(n) + 1$. Then

$$\begin{aligned} abau(n) &= uv(a^2u(n)) + 1 \\ &= u(v(a^2u(n)) + 1) . \end{aligned}$$

By Corollary 2.3 we have

$$v'(v(a^2u(n)) - a^2u(n) + 1) = v(a^2u(n)) + 1 .$$

By Theorem 7.9 we have

$$\begin{aligned} v(a^2u(n)) + 1 &= v'(w(2a^2u(n)) - a^2u(n) + 1) \\ &= v'(2a^2u(n) + n - 1 - a^2u(n) + 1) \\ &= v'(a^2u(n) + n) . \end{aligned}$$

This completes the proof.

7.14. Theorem. $yt(n) = 2n$ and $yt'(n) = 2eu'(n) - 1$.

Proof. By the definition of y, we have

$$est(n) = b(n) = uwyt(n) = uv(n) .$$

We know by Theorem 7.9 that $v(n) = w(2n)$, and so $wyt(n) = w(2n)$, that is, $yt(n) = 2n$. Secondly, $est'(n) = au'(n) = a(bs(n) + 1) = b(as(n) + 1) - 1 = uwyt'(n)$. Now

$$b(as(n) + 1) = uv(as(n) + 1) = uw(2as(n) + 2)$$

and since

$$b(as(n) + 1) - 1 \in (uw) ,$$

we have

$$w(2as(n) + 1) = wyt'(n) .$$

Now $eu'(n) = as(n) + 1$, and so

$$2as(n) + 1 = 2eu'(n) - 1 = yt'(n) .$$

This completes the proof.

7.15. Theorem. $\lambda(n) = 3n - \eta(u' \le n)$.

Proof. We show that if $n = bs(j)$ for some j, then $\lambda(n + 1) = \lambda(n) + 2$, and for all other n we have $\lambda(n + 1) = \lambda(n) + 3$. Then clearly

$$\begin{aligned} \lambda(n) &= 3n - \eta(bs < n) \\ &= 3n - \eta(bs + 1 \le n) \qquad (7.16) \\ &= 3n - \eta(u' \le n) . \end{aligned}$$

In the proof, we shall require the fact that $zt(n) = ca(n) + 1$ and $zt'(n) = cbs(n) + 1$.

Case 1. $n = a(j)$. Then $c(n) = ca(j) = z'\lambda a(j)$, and $ca(j) + 1 = zt(j)$. By 2.18,

$$c(a(j) + 1) = ca(j) + 4 = z'\lambda(a(j) + 1) .$$

Since $(z) \subset (c + 1)$, we have $ca(j) + 2$ and $ca(j) + 3 \in (z')$, and so $\lambda(a(j) + 1) = \lambda a(j) + 3$.

Case 2. $n = b(j)$ where $j \notin (s)$. Then $b(j) = a^2(j) + 1$, and $zta(j) = ca^2(j) + 1$. Also $z'\lambda a^2(j) = ca^2(j)$ and $z'\lambda b(j) = cb(j) = ca^2(j) + 4$. Again $ca^2(j) + 2$ and $ca^2(j) + 3 \in (z')$, and we have $\lambda b(j) = \lambda(b(j) - 1) + 3$. Since $j \notin (s)$, $cb(n) + 1 \in (z')$, and so are $cb(n) + 2$ and $cb(n) + 3$. By 2.19, $cb(n) + 3 = c(b(n) + 1) = z'\lambda(b(n) + 1)$. Thus $\lambda(b(n) + 1) = \lambda b(j) + 3$.

Case 3. $n = bs(j)$. Then $cbs(j) + 1 = zt'(j)$, $cbs(j) = z'\lambda bs(j)$, and $cbs(j) + 3 = c(bs(j) + 1) = z'\lambda(bs(j) + 1)$. Since $cbs(j) + 2 \in (z')$, we have $\lambda(bs(j) + 1) = \lambda bs(j) + 2$. This completes the proof.

The function τ is of interest for the following reasons. Recall that the function r satisfies $(r) = (b) \cup (a^2u)$. Thus we get $(er) = (a) \cup (au)$, and er is not strictly monotonic. In particular, if $r(k) = b(n) - 1$ and $r(k + 1) = b(n)$, then $er(k) = er(k + 1) = a(n)$; $\sigma(n) = k + 1$ and $\tau(n) = k$. On the other hand, if $r(k) = bu'(n)$, then $er(k - 1) \neq er(k)$ and $er(k + 1) \neq er(k)$; that is, the value $er(k) = au'(n)$ is not repeated, and we have

$$ptu'(n) = \sigma u'(n) = \tau u'(n) .$$

7.17. <u>Theorem</u>. We have, for all n,

$$\begin{aligned} r\sigma(n) - \sigma(n) &= b(n) - \sigma(n) \\ &= r\tau(n) - \tau(n) . \end{aligned}$$

<u>Proof</u>. We need only note that if $\tau(n) = \sigma(n) - 1$, then $r\tau(n) = r\sigma(n) - 1$.

7.18. <u>Theorem</u>. The function K defined by $K(n) = \eta(b < c(n))$ is strictly monotonic. Furthermore, we have

(7.19)

(i) $$Kb(n) = c(n) - 1$$

(ii) $$Ks(n) = z(n) - 1$$

(iii) $$Ka^2u(n) = cu(n) - 2$$

(iv) $$(K') = (z) \cup (cbr) .$$

<u>Proof</u>. Since $c(n + 1) - c(n) \geq 3$, and of the three consecutive integers $c(n)$, $c(n) + 1$, $c(n) + 2$ at least one must be in (b), it is evident that $K(n + 1) \geq K(n) + 1$, so that K is strictly monotonic. To see (i), we have

$$\eta(b < cbr(n)) = \eta(b < bcr(n)) = cr(n) - 1$$

$$\eta(b < cbs(n)) = \eta(b < bcs(n) - 1) = cs(n) - 1 .$$

For (ii), $\eta(b < cs(n)) = \eta(b < bz(n)) = z(n) - 1$. For (iii), we have

$$\eta(b < ca^2u(n)) = ca^2u(n) - 1 - \eta(a < ca^2u(n)).$$

By Theorem 3.6, we know $ca^2u(n) = a(cau(n) - 1)$, and so

$$\begin{aligned} Ka^2u(n) &= ca^2u(n) - 1 - \eta(a < a(cau(n) - 1)) \\ &= ca^2u(n) - 1 - [cau(n) - 2] \\ &= ca^2u(n) - cau(n) + 1 \\ &= a^3u(n) + 2a^2u(n) - bau(n) - au(n) + 1 \\ &= [bau(n) - 1] + 2a^2u(n) - au(n) - [bau(n) - 1] \\ &= a^2u(n) + [bu(n) - 1] - au(n) \\ &= a^2u(n) + u(n) - 1 \\ &= bu(n) - 1 + u(n) - 1 \\ &= cu(n) - 2 . \end{aligned}$$

Finally to see (iv), first we note that

$$Kb(n) = c(n) - 1 \notin (z) \quad \text{and} \quad Ks(n) = z(n) - 1 \notin (z).$$

We show that $cu(n) - 2 \notin (z)$. If $u(n) - 1 \in (a)$, then $c(u(n) - 1) + 4 = cu(n)$. Since $cu(n) - 2 \neq c(j) + 1$ for any j, then in this case $cu(n) - 2 \notin (z)$.

Suppose $u(n) - 1 = b(j)$ for some j. Then (since $(u') = (bs + 1)$) we must have $j \in (r)$, say $j = r(k)$ for some k. Then $c(u(n) - 1) + 3 = cu(n)$, and $cu(n) - 2 = cbr(k) + 1$ and this is not a value of z.

Now from (i), (ii) and (iii) we have, for all n,

$$\begin{gathered} cb(n) + 2 \in (c - 1) \subset (K) \\ ca(n) + 3 \in (c - 1) \subset (K) \\ cs(n) + 2 = c(s(n) + 1) - 2 \in (cu - 2) \subset (K) \\ ca^2u(n) + 2 = c(bu(n) - 2) \in (cu - 2) \subset (K) \\ ca(n) = zt(n) - 1 \in (K) \\ cbs(n) = zt'(n) - 1 \in (K) \\ cbr(n) + 1 \in (cu - 2) \in (K) \end{gathered}$$

while $cbr(n) \in (K')$, $ca(n) + 1 \in (K')$, and $cbs(n) + 1 \in (K')$. Thus (iv) holds.

7.20. Theorem.

(i) $$K'br(n) = cbr(n)$$

(ii) $$K'a(n) = ca(n) + 1$$

(iii) $$K'bs(n) = cbs(n) + 1 .$$

Proof. This is evident from the fact that

$$(K') = (cbr) \cup (ca + 1) \cup (cbs + 1) .$$

7.21. Theorem. The following are equivalent.

(a) $$K'(j) = z(n)$$

(b) $$n = j - \eta(br < j)$$

(c) $$n = c(j) + 1 - \lambda(j) .$$

Proof. Since $(K') = (z) \cup (cbr)$, it is evident that (a) and (b) are equivalent. To see that (b) and (c) are equivalent, we show that, for all j,

(7.22) $$j - \eta(br < j) = c(j) + 1 - \lambda(j) .$$

Recall that $\lambda(j) = 3j - \eta(u' \leq j)$ (Theorem 7.15). That is,

$$\lambda(j) = 3j - \eta(bs + 1 \leq j)$$
$$= 3j - \eta(bs < j) .$$

Now

$$\eta(bs < j) = \eta(b < j) - \eta(br < j)$$
$$= j - 1 - \eta(a < j) - \eta(br < j)$$
$$= j - 1 - (a(j) - j) - \eta(br < j)$$
$$= 2j - 1 - a(j) - \eta(br < j) .$$

Thus

$$\lambda(j) = 3j - [2j - 1 - a(j) - \eta(br < j)]$$
$$= j + 1 + a(j) + \eta(br < j)$$
$$= b(j) + 1 + \eta(br < j) .$$

Now $c(j) + 1 - \lambda(j) = j - \eta(br < j)$, and this completes the proof.

7.23 Theorem.

(i) $$tar(n) = br(n) - n$$

(ii) $$K'(br(n) - 1) = ztar(n)$$

(iii) $$cbr(n) = z'(b^2r(n) + n) .$$

Proof. To see (i) we use Theorems 7.20 and 7.21. We have $K'br(n) = cbr(n)$ and

$$\begin{aligned} K'(br(n) + 1) &= c(br(n) + 1) + 1 \\ &= c(a^2r(n) + 2) + 1 \\ &= ca(ar(n) + 1) + 1 \\ &= zt(ar(n) + 1) . \end{aligned}$$

Then by Theorem 7.20(b) we have

$$\begin{aligned} t(ar(n) + 1) &= (br(n) + 1) - \eta(br < br(n) + 1) \\ &= br(n) + 1 - n. \end{aligned}$$

Since $t(n) - t(n - 1) = 1$ unless $n - 1 \in (as)$, we have $t(ar(n)) = br(n) - n$. This proves (i).

To see (ii), we have $K'(br(n) + 1) = zt(ar(n) + 1)$ and $K'(br(n)) = cbr(n)$. Thus $K'(br(n) - 1) \in (z)$ and by (i), $K'(br(n) - 1) = ztar(n)$.

Finally, (iii) follows from the fact that

$$\begin{aligned} ztar(n) - 1 &= z'(ztar(n) - tar(n)) \\ &= z'(ca^2r(n) + 1 - br(n) + n) \\ &= z'(cbr(n) - 4 + 1 - br(n) + n) \\ &= z'(b^2r(n) - 3 + n) . \end{aligned}$$

Now $ztar(n) + k \in (z')$ for $k = 1, 2, 3$ and so

$$z'(b^2r(n) + n) = ztar(n) + 3 = ca^2r(n) + 4 = cbr(n) .$$

This completes the proof.

7.24. Corollary. $\lambda br(n) = b^2r(n) + n$.

Proof. By definition, $cbr(n) = z'\lambda br(n)$, and the result follows from Theorem 7.23 (iii).

7.25. Theorem. p' is separated.

Proof. We show that $p(n + 1) - p(n) = 1$ or 2.

Case 1. $n = t(j)$ and $n + 1 = t(j + 1)$. Then $p(n) = pt(j) = 2j - \eta(u' \leq j)$ by Theorem 7.2, and $p(n + 1) = pt(j + 1) = 2j + 2 - \eta(u' \leq j + 1)$. Then $p(n + 1) - p(n) = 2 - \delta$ where $\delta = 1$ if $j + 1 \in (u')$ and $\delta = 0$ if $j + 1 \notin (u')$.

Case 2. $n = t(j)$, and $n + 1 = t'(k)$. Then $j = as(k)$ and we have

$$p(n) = pt(as(k)) = 2as(k) - \eta(u' \leq as(k))$$
$$p(n+1) = pt'(k) = 2as(k) + 1 - \eta(u' \leq as(k) + 1) .$$

Since $as(k) + 1 \in (b)$, it follows that $as(k) + 1 \notin (u')$ and so $p(n+1) - p(n) = 1$ in this case.

Case 3. $n = t'(j)$ and $n + 1 = t(k)$. Then $k = as(j) + 1$ and we have

$$p(n) = 2as(j) + 1 - \eta(u' \leq as(j) + 1)$$
$$p(n+1) = 2as(j) + 2 - \eta(a' \leq as(j) + 1)$$

and

$$p(n+1) - p(n) = 1.$$

This completes the proof.

7.26. Theorem. σ' is separated.

Proof. We show that $\sigma(n+1) - \sigma(n) \leq 2$. By Theorem 7.2, we have

$$\sigma(n) = pt(n) = 2n - \eta(u' \leq n).$$

Then

$$\sigma(n+1) - \sigma(n) = 2 - \delta$$

where $\delta = 1$ if $n + 1 \in (u)'$ and $\delta = 0$ if $n + 1 \notin (u')$. This completes the proof.

7.27. Theorem. $$\sigma(n) = \lambda(n) - n .$$

Proof. This is evident from the fact that $\sigma(n) = 2n - \eta(u' \leq n)$, while (by Theorem 7.15) $\lambda(n) = 3n - \eta(u' \leq n)$.

7.28. Theorem. $$(\tau) = (\sigma') \cup (\sigma u') .$$

Proof. If $n > 1$, we have $\sigma u(n) - \sigma(u(n) - 1) = 2$, as above, and thus for all n, $\sigma u(n) - 1 \in (\sigma')$. On the other hand, $\sigma u'(n) - 1 = \sigma(u'(n) - 1)$. Since for all n, $\sigma(n+1) - \sigma(n) \leq 2$, we know $(\sigma') \subset (\sigma - 1)$. It follows that $\tau u(n) = \sigma u(n) - 1 \in (\sigma')$ and further $(\tau u) = (\sigma')$. Since $\tau u'(n) = \sigma u'(n)$, the proof is complete.

7.29. Theorem. (a) $$\sigma u(n) = u(n) + n$$

(b) $$\tau u(n) = u(n) + n - 1$$

(c) $$\sigma u'(n) = 2u'(n) - n = \tau u'(n) .$$

Proof. (a) We have $\sigma(n) = 2n - \eta(u' \leq n)$ for all n. Then $\sigma u(n) = 2u(n) - \eta(u' \leq u(n))$. By Theorem 2.1, we know $u(n) = n + \eta(u' \leq u(n))$. Thus

$$\sigma(u(n)) = 2u(n) - (u(n) - n) = u(n) + n .$$

Statement (b) follows from (a) and the fact that $\tau u(n) = \sigma u(n) - 1$. To see (c), we have

$$\sigma u'(n) = 2u'(n) - \eta(u' \leq u'(n)) .$$

Then $\sigma u'(n) = 2u'(n) - n$.

7.30. Theorem. (a) $\tau'(n) = \sigma u(n)$

(b) $\sigma'(n) = \tau' u(n)$.

Proof. First, $(\tau) = (\sigma u - 1) \cup (\sigma u')$, in particular $\tau u(n) = \sigma u(n) - 1$ and $\tau u'(n) = \sigma u'(n)$. Since $\sigma u'(n) - 1 = \sigma(u'(n) - 1)$, we have $(\tau') = (\sigma u)$, and (a) follows. Statement (b) is proved similarly.

7.31. Theorem. For each integer $n > 0$, put $J_n = \eta(eu'(j) - j < n)$. Then

(a) y' is a separated function

(b) $$y'(eu'(n) - n) = 2(eu'(n) - 1) - 1$$

(c) $$y'(eu'(n) - n + 1) = y'(eu'(n) - n) + 4$$

(d) If j is not of the form $eu'(n) - n$, then $y'(j + 1) = y'(j) + 2$.

(e) $$y'(n) = 2(n + J_n) - 1 .$$

(f) $$y'(n - \eta(eu' < n)) = 2n - 1.$$

Proof. By Theorem 7.14, we know $yt(n) = 2n$ and $yt'(n) = 2eu'(n) - 1$. It follows that

$$(y') = (2n - 1)/(2eu' - 1)$$

and it is evident that y' is separated. Clearly if $y'(j) = 2eu'(n) - 3$ for some n, then $y'(j + 1) = y'(j) + 4$, and otherwise $y'(j) = y'(j) + 2$. We prove statement (f). Clearly if $y'(j) = 2n - 1$, then since

$$\{y'(k) : y'(k) \leq 2n - 1\} = \{2j - 1 : j = 1, 2, \cdots, n \text{ and } j \neq eu'\}$$

we must have $j = n - \eta(2eu' - 1 < 2n - 1) = n - \eta(eu' < n)$. To see (b), it follows from (f) that

$$y'((eu'(n) - 1) - (n - 1)) = 2(eu'(n) - 1) - 1 = 2eu'(n) - 3 .$$

Statements (c) and (d) are evident from (f) and the fact that numbers of the form $2eu'(n) - 1$ are the only odd numbers in (y). To see (e), suppose that

$$eu'(j) - j < n \leq eu'(j + 1) - (j + 1) ,$$

say $n = eu'(j) - j + k$. Then

$$\begin{aligned} y'(n) &= y'(eu'(j) - j) + 2 + 2k \\ &= 2eu'(j) - 3 + 2 + 2k \\ &= 2(eu'(j) + k) - 1 \\ &= 2([eu'(j) - j] + [k + j]) - 1 \\ &= 2n + 2j - 1 \\ &= 2n + 2\eta(eu'(m) - m < n) - 1 \\ &= 2(n + J_n) - 1 . \end{aligned}$$

This completes the proof.

8. ASYMPTOTIC PROPERTIES

In this section, we show that the function s is asymptotic to the function c. In particular, $s(n) \sim (\alpha + 2)n$. Similar asymptotic results follow at once for all the auxiliary functions introduced so far.

8.1. Theorem. $n \in (s)$ if and only if

$$\frac{\alpha}{\sqrt{5}} \le \{\alpha n\} < 1 .$$

Proof. Recall that $n \in (s)$ if and only if $ac(n) = ca(n) + 2$. By definition, we have

(8.2) $$ca(n) = [\alpha[\alpha n]] + 2[\alpha n]$$

(8.3) $$ac(n) = [\alpha([\alpha n] + 2n)] .$$

Put

$$\alpha n = m + \epsilon_1 \qquad (0 < \epsilon_1 < 1)$$
$$\alpha m = k + \epsilon_2 \qquad (0 < \epsilon_2 < 1) .$$

By 2.20, we have $\epsilon_2 = 1 + (1 - \alpha)\epsilon_1$.

Thus we have

(8.4) $$ca(n) = [\alpha m] + 2m = k + 2m$$

and

(8.5) $$\begin{aligned} ac(n) &= [\alpha(m + 2n)] = [\alpha m + 2\alpha n] \\ &= [k + \epsilon_2 + 2m + 2\epsilon_1] \\ &= k + 2m + [\epsilon_2 + 2\epsilon_1] . \end{aligned}$$

Then $n \in (s)$ if and only if $[\epsilon_2 + 2\epsilon_1] = 2$. Now

$$\begin{aligned}\epsilon_2 + 2\epsilon_1 &= 1 + (1 - \alpha)\epsilon_1 + 2\epsilon_1 \\ &= 1 + (3 - \alpha)\epsilon_1\end{aligned}$$

and so $n \in (s)$ if and only if

(8.6) $$1 \le (3 - \alpha)\epsilon_1 < 2 ,$$

that is, if and only if

$$\frac{1}{3 - \alpha} \le \epsilon_1 < \frac{2}{3 - \alpha} .$$

Note that

$$\frac{2}{3 - \alpha} > 1$$

and $\epsilon_1 < 1$, so this reduces to:

(8.7) $$n \in (s) \leftrightarrows \frac{1}{3 - \alpha} \le \{\alpha n\} < 1 .$$

Since $\alpha = \frac{1}{2}(1 + \sqrt{5})$, it is easy to show that

$$\frac{1}{3 - \alpha} = \frac{\alpha}{\sqrt{5}} ,$$

and this completes the proof.

8.8. Theorem. We have $s(n) \sim c(n)$.

Proof. We require the fact that the values of $\{\alpha n\}$ are uniformly distributed in $(0,1)$ (see [5, Th. 6.3]). It follows from the previous theorem that

(8.9) $$\eta(r < n) \sim \frac{\alpha}{\sqrt{5}} n$$
$$\eta(s < n) \sim \left(1 - \frac{\alpha}{\sqrt{5}}\right) n .$$

Since $s(n) = n + \eta(r < s(n))$, we have

(8.10) $$s(n) \sim n + \frac{\alpha}{\sqrt{5}} s(n) , \qquad s(n) \sim \frac{n}{1 - \dfrac{\alpha}{\sqrt{5}}} ,$$
$$\frac{s(n)}{n} \to \alpha + 2 ,$$

On the other hand, $c(n) = [(\alpha + 2)n]$, and it follows that $s(n) \sim c(n)$. This completes the proof.

8.11. Corollary. (i) $t'(n) \sim b^2(n)$

(ii) $u'(n) \sim bc(n) + 1$.

Proof. This is evident from the fact that $t'(n) = as(u) + n$ and $u'(n) = bs(n) + 1$. The result follows from Theorem 8.8.

Clearly, similar results could be stated for most of the functions considered previously, since these were defined in terms of a, b, and s.

Recall (Definition 2.7) if

$$\lim_{n \to \infty} \frac{f(n)}{n}$$

exists and $\neq 0$ we set

$$c_f = \lim_{n \to \infty} \frac{f(n)}{n} .$$

In view of Theorem 8.8, we have

$$\lim_{n \to \infty} \frac{s(n)}{n}$$

exists, and is not 0. Then all of the functions introduced so far also satisfy

$$\lim_{n \to \infty} \frac{f(n)}{n}$$

exists, since they are defined in terms of s, a, b, c, and e. Then we have the following:

8.12. Theorem. We have

(i) $c_a = \alpha, \quad c_b = \alpha + 1$

(ii) $c_c = \alpha + 2, \quad c_{c'} = 3 - \alpha$

(iii) $c_s = \alpha + 2, \quad c_r = 3 - \alpha$

(iv) $c_w = \dfrac{\alpha + 1}{\sqrt{5}} = \dfrac{\alpha^2}{\sqrt{5}}, \quad c_{w'} = 3\alpha + 1$

(v) $c_u = \dfrac{\sqrt{5}}{2}, \quad c_{u'} = 4\alpha + 3$

(vi) $c_v = \dfrac{3 + \sqrt{5}}{\sqrt{5}}, \quad c_{v'} = \dfrac{2\alpha^2}{3}$

(vii) $c_t = \dfrac{2\alpha + 1}{\alpha + 2}, \quad c_{t'} = \alpha^4$

(viii) $c_z = 5, \quad c_{z'} = 5/4$

(ix) $$c_\sigma = \frac{2+\sqrt{5}}{\sqrt{5}}, \qquad c_{\sigma'} = \frac{1}{2} + \alpha$$

(x) $$c_\tau = \frac{2+\sqrt{5}}{\sqrt{5}}, \qquad c_{\tau'} = \frac{1}{2} + \alpha$$

(xi) $$c_\lambda = \frac{4}{5}(\alpha + 2) = \frac{4\alpha}{\sqrt{5}}, \qquad c_{\lambda'} = \frac{4(\alpha+2)}{4\alpha+3}$$

9. CONJECTURES

Many of the results in the preceding sections were first arrived at empirically, using extensive numerical data. We list here some conjectures, also arrived at "by inspection," which remain unproved.

(9.1) $$tt'(n) = t(n) + t'(n) \quad \text{except for} \quad n \in (bs)$$

(9.2) $$ts(n) = et'(n) \quad \text{except for} \quad n \in (bs)$$

(9.3) $$t'b(n) \in (a^2) \quad \text{except for} \quad n \in (s)$$

(9.4) $$|xt'(n) - 2t'(n)| \le 2 .$$

It has been shown that the functions s and c are asymptotic (Theorem 8.8) and also that there are infinitely many values of n for which $s(n) = c(n)$ (Theorem 5.12). It remains an open question whether the difference $|s - c|$ is bounded. More generally, what is the smallest Ⓗ ≥ 0 such that

$$|s(n) - c(n)| = 0(n^{Ⓗ+\epsilon}) \qquad (\epsilon > 0) .$$

Numerically, we have for $n \le 101$, $|s(n) - c(n)| \le 5$.

Of course any such result for s and c implies corresponding results for other pairs (e.g., for t' and b^2, or for u' and bc).

We could define another function, say g(n), where g(n) is the n^{th} integer k such that $s(k) = c(k)$. It is evident from Theorem 5.12 that if $n \in (g)$, then $t'(n) + 1 \in (g)$. The numerical data indicate a possibility that if $n \in (g)$, then also $t'(n) + 4 \in (g)$, but this remains unproved.

Finally, it would be very interesting to have an "infinite union" formula for the function t', similar to that for s given by Theorem 5.15.

10. SUMMARY

1. $$cbs(n) = bcs(n) - 1 = a^2cs(n)$$

2. $$cbr(n) = bcr(n)$$

3. $$cs(n) = bz(n)$$

4. $$\mathrm{cas}(n) = a(\mathrm{cs}(n) - 1) = \mathrm{acs}(n) - 2$$

5. $$\mathrm{car}(n) = \mathrm{acr}(n) - 1$$

6. $$\mathrm{ca}(n) = \begin{cases} a(c(n) - 1) & \text{if } \mathrm{ca}(n) \in (a) \\ a(c(n) - 1) + 1 & \text{if } \mathrm{ca}(n) \in (b) \end{cases}$$

7. $$\mathrm{cabr}(n) = \mathrm{bacr}(n)$$

8. $$\mathrm{cabs}(n) = \mathrm{bacs}(n) - 2$$

9. $$\mathrm{cab}(n) = b(\mathrm{ca}(n) + 1)$$

10. $$\mathrm{cb}^2(n) = \mathrm{bcb}(n)$$

11. $$\mathrm{ecb}(n) = \mathrm{ac}(n)$$

12. $$\mathrm{eca}(n) = c(n) - 1$$

13. $$\begin{cases} c'b(n) = c(n) - 1 \\ c'(b(n) - 1) = c(n) - 2 \\ c'(b(n) + 1) = c(n) + 1 \\ c'(b(n) + 2) = c(n) + 2 \end{cases}$$

14. $$c(n) + c'(n) = 5n - 1$$

15. $$c'(n) = n + \eta(b < n)$$

16. $$c'a(n) = c'(a(n) + 1) - 1$$

17. $$c'ab(n) = \mathrm{ca}(n) + 1$$

18. $$\mathrm{ec}(n) = c'\phi(n)$$

19. $$\phi a(n) = b(n)$$

20. $$\phi\, \mathrm{br}(n) = \mathrm{abr}(n)$$

21. $$\phi \mathrm{bs}(n) = \mathrm{abs}(n) + 1$$

22. $$\phi s(n) = \mathrm{bes}(n) = \mathrm{as}(n) + 1$$

23. $$\phi'(n) = \begin{cases} a^2(n), & n \in (\mathrm{as} + 1) \\ a(a(n) - 1), & n \in (\mathrm{as} + 1) \end{cases}$$

24. $$\phi a(n) + \phi' a(n) = \phi\phi' a(n) - 1$$

25. $$\psi(n) = e\phi'(n)$$

26. $$\psi' r(n) = \mathrm{br}(n)$$

27. $$\psi' s(n) = \mathrm{bs}(n) + 1 = u'(n)$$

28. $$\phi \mathrm{br}(n) - \phi' \mathrm{ar}(n) = 2$$

29. $$\phi'\phi s(n) = \mathrm{abs}(n)$$

30. $$\phi\phi' s(n) = \mathrm{abs}(n) - 1$$

31. $$\phi\phi'r(n) = bar(n)$$

32. $$\phi s(n) + c's(n) = 3s(n)$$

33. $$ab(n) = st(n)$$

34. $$st'(n) = a^2u'(n) = a^2(bs(n) + 1)$$

35. $$z(n) = c's(n)$$

36. $$es(n) = rp(n)$$

37. $$aes(n) = s(n) = arp(n)$$

38. $$b(n) = rpt(n)$$

39. $$pt(n) = \sigma(n)$$

40. $$b(n) = r\sigma(n)$$

41. $$b(n) = uv(n)$$

42. $$u'(n) = bs(n) + 1$$

43. $$t'(n) = as(n) + n$$

44. $$tas(n) = t'(n) - 1$$

45. $$z(n) = c(z(n) - s(n)) + 1$$

46. $$s(n) = b(z(n) - s(n)) + 1$$

47. $$az(n) - as(n) = a(z(n) - s(n)) + 1$$

48. $$zt(n) = ca(n) + 1$$

49. $$zt'(n) = cbs(n) + 1 = b^2z(n)$$

50. $$s(n) = c(n) \Longleftrightarrow t'(n) = b^2(n)$$

51. $$zt(n) - st(n) = a(n)$$

52. $$zt'(n) - st'(n) = bs(n)$$

53. $$bas(n) = rp(t'(n) - 1)$$

54. $$az(n) - as(n) = es(n)$$

55. $$z(n) + es(n) = 2s(n)$$

56. $$u'(n) + z(n) = 4s(n)$$

57. $$tb^2(n) = t(n) + b^2(n)$$

58. $$t't(n) = t(n) + b^2(n) - 1 = tb^2(n) - 1$$

59. $$t't(n) = a^2b(n) + t(n)$$

60. $$tab(n) = te(n) + ab(n) - \delta$$

where $\delta = 0$ if $n \in (b)$ and $\delta = 1$ if $n \in (a)$

61. $$tst'(n) = ts(n) + bu'(n) - 1$$

62. $$tst'(n) - st'(n) = ts(n)$$

63. $$r(2u'(n) - n) = bu'(n) = st'(n) + 1$$

64. $$b(n) = r(2n - \eta(u' \le n))$$

65. $$c^2r(n) = 5br(n)$$

66. $$c^2s(n) = \mathbf{5bs(n)} + 1$$

67. $$es(n) = ux(n)$$

68. $$\tau u(n) = \sigma u(n) - 1$$

69. $$\tau u'(n) = \sigma u'(n)$$

70. $$K(n) = \eta(b < c(n))$$

71. $$(ab)'(n) = uw(n)$$

72. $$es(n) = uwy(n)$$

73. $$c(n) = z'\lambda(n)$$

74. $$pt(n) = 2n - \eta(u' \le n) = \sigma(n) = \lambda(n) - n$$

75. $$pt'(n) = 2as(n) + 1 - \eta(u' \le as(n) + 1)$$

76. $$v(n) = b(n) - \eta(s < n)$$

77. $$xt(n) = v(n)$$

78. $$xt'(n) = au'(n) - \eta(s < as(n))$$

79. $$v(n) = w(2n)$$

80. $$v(n) - 1 = w(2n - 1)$$

81. $$w(n) = n + \eta(2a^2u < n)$$

82. $$w'(n) = 2a^2u(n) + n$$

83. $$b(n) - 1 = uv'(v(n) - n) = u(v(n) - 1)$$

84. $$abau(n) = uv'(a^2u(n) + n)$$

85. $$yt(n) = 2n$$

86. $$yt'(n) = 2eu'(n) - 1$$

87. $$\lambda(n) = 3n - \eta(u' \le n)$$

88. $$r\sigma(n) - \sigma(n) = b(n) - \sigma(n) = r\tau(n) - \tau(n)$$

89. $$Kb(n) = c(n) - 1$$
$$Ks(n) = z(n) - 1$$
$$Ka^2u(n) = cu(n) - 2$$

90. $$K'br(n) = cbr(n)$$

$$K'a(n) = ca(n) + 1$$

$$K'bs(n) = cbs(n) + 1$$

91. $$K'(j) = z(n) \Leftrightarrow n = j - \eta(br < j) = c(j) + 1 - \lambda(j)$$

92. $$tar(n) = br(n) - n$$

93. $$K'(br(n) - 1) = ztar(n)$$

94. $$cbr(n) = z'(b^2r(n) + n)$$

95. $$\lambda br(n) = b^2r(n) + n$$

96. $$s(n) \sim c(n)$$

$$t'(n) \sim b^2(n)$$

$$u'(n) \sim bc(n)$$

97. $$\lambda u(n) = 2u(n) + n$$

98. $$\lambda u'(n) = 3u'(n) - n$$

99. $$vs(n) = u'(n) - n$$

100. $$vr(n) = ar(n) + n$$

101. $$(s) = (ab) \cup (a^2u')$$

102. $$(r) = (b) \cup (bu - 1) = (b) \cup (a^2u)$$

103. $$(es) = (au)' = (au') \cup (b)$$

104. $$(u') = (ab^2) \cup (abau')$$

105. $$(u') \cup (abau) = (ab)$$

106. $$(u) = (a^2) \cup (b) \cup (abau) = (b) \cup (b - 1) \cup (abau)$$

107. $$(u) = (uv) \cup (uv') = (b) \cup (uv')$$

108. $$(es) = (b) \cup (au') = (ux)$$

109. $$(ux) = (uxt) \cup (au') = (uxt) \cup (uxt')$$

110. $$(\phi) = (b) \cup (abr) \cup (abs + 1) = (b) \cup (abr) \cup (au')$$

111. $$(\phi') = (abs) \cup (a^2(bes)') = (abs) \cup (a^2(eu')')$$

112. $$(\psi) = (e\phi') = (bs) \cup (a(eu')')$$

113. $$(eu') = (b^2) \cup (bau')$$

114. $$(eu') \cup (bau) = (b^2) \cup (ba) = (b)$$

115. $$(eu')' = (a) \cup (bau)$$

116. $$(\psi) = (a^2) \cup (abau) \cup (bs)$$

117. $(\psi') = (u') \cup (br)$

118. $(e\psi') = (eu') \cup (ar)$

119. $(u') = ab(az - as)$

120. $(a) = \{ n \mid aen = n \}$

$(b) = \{ n \mid aen = n - 1 \}$

121. $c(n) \in (a) \Longleftrightarrow n \in [(a)/(s)] \cup (bs) = (a^2u) \cup (bs)$

122. $a(n) = n + \eta(a < n)$

123. $e(n) = \eta(a \leq n)$

124. $a(n) = t(n) + \eta(ar < n)$

125. $t(n) = n + \eta(as < n)$

126. $t(n) = \eta(as < b^2(n))$

127. $(\eta(a < s(n)))) = (a^2) \cup (abs) = (a(z - s))$

128. $z(n) = as(n) - n + 1 + \eta(t' < n)$

129. $z(n) + 2n = t'(n) + 1 + \eta(t' < n)$

130. $\eta(t(n) < t' < t(n) + b^2(n)) = n$

131. $\lambda(n) = 3n - \eta(bs < n)$

132. $\lambda(n) = n + 1 - \eta(r < b(n))$

133. $\eta(r < b(n)) = 2n - 1 - \eta(u' \leq n)$

134. $2s(n) = t'(n) + 1 + \eta(tar < n)$

135. $2ab(n) = t(n) + b^2(n) + \eta(ar < n)$

136. $\eta(t' < t(n)) = \eta(as < n)$

137. $$(s) = \bigcup_{k=0}^{\infty} (a(a^2b)^k b)$$

138. $$(r) = (b) \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k ab) \right] \cup \left[\bigcup_{k=0}^{\infty} (a(a^2b)^k a^3) \right]$$

139. $$(u') = \bigcup_{k=0}^{\infty} (ab(a^2b)^k b)$$

140. $$(u) = (b) \cup (b - 1) \cup \left[\bigcup_{k=0}^{\infty} (ab(a^2b)^k ab) \right] \cup \left[\bigcup_{k=0}^{\infty} (ab(a^2b)^k a^3) \right]$$

141. $s(f_k(n)) = a(a^2b)^k b(n)$

142. $$s(t')^k t(n) = a(a^2b)^k b(n)$$

143. $$f_k(n) = (t')^k t(n) \qquad (k = 0, 1, 2, \cdots)$$

144. $$u'(f_k(n)) = ab(a^2b)^k b(n)$$

145. $$f_{k-1}(b^2(n) - 1) < f_k(n) < f_{k-1}(b^2(n)) \qquad (k = 1, 2, 3, \cdots) .$$

The following functions are separated:

$$b,\ c,\ s,\ \phi',\ \psi',\ t',\ z,\ u',$$
$$K',\ v,\ y',\ w',\ \lambda,\ \rho',\ \sigma',\ \tau' .$$

Table 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27
b	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44
c	3	7	10	14	18	21	25	28	32	36	39	43	47	50	54	57	61
s	3	8	11	16	19	21	24	29	32	37	42	45	50	53	55	58	63
r	1	2	4	5	6	7	9	10	12	13	14	15	17	18	20	22	23
z	4	11	15	22	26	29	33	40	44	51	58	62	69	73	76	80	87
z'	1	2	3	5	6	7	8	9	10	12	13	14	16	17	18	19	20
t'	5	14	20	29	35	39	45	54	60	69	78	84	93	99	103	109	118
t	1	2	3	4	6	7	8	9	10	11	12	13	15	16	17	18	19
u'	8	21	29	42	50	55	63	76	84	97	110	118	131	139	144	152	164
u	1	2	3	4	5	6	7	9	10	11	12	13	14	15	16	17	18

Table 2

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
p	2	4	6	8	9	10	12	14	15	17	19	21	23	24	25	27	29
p'	1	3	5	7	11	13	16	18	20	22	26	28	30	32	36	38	41
v	2	5	7	9	12	14	17	19	21	24	26	28	31	33	36	38	40
v'	1	3	4	6	8	10	11	13	15	16	18	20	22	23	25	27	29
w	1	2	4	5	6	7	8	9	11	12	13	14	16	17	18	19	20
w'	3	10	15	22	29	34	41	52	59	64	71	78	83	90	95	102	109

(continued)

Table 2 (continued)

x	2	5	7	9	11	12	14	17	19	21	24	26	28	30	31	33	36
x'	1	3	4	6	8	10	13	15	16	18	20	22	23	25	27	29	32
y	2	4	6	8	9	10	12	14	16	18	20	22	24	25	26	28	30
y'	1	3	5	7	11	13	15	17	19	21	23	27	29	31	33	37	39
λ	3	6	9	12	15	18	21	23	26	29	32	35	38	41	44	47	50
λ'	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	24	25

Table 3

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
e	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9	10	11
ϕ	2	3	5	7	8	10	12	13	15	16	18	20	21	23	24	26	28
ϕ'	1	4	6	9	11	14	17	19	22	25	27	30	32	35	38	40	43
ψ	1	3	4	6	7	9	11	12	14	16	17	19	20	22	24	25	27
ψ'	2	5	8	10	13	15	18	21	23	26	29	31	34	36	39	42	44
σ	2	4	6	8	10	12	14	15	17	19	21	23	25	27	29	31	33
σ'	1	3	5	7	9	11	13	16	18	20	22	24	26	28	30	32	34
τ	1	3	5	7	9	11	13	15	16	18	20	22	24	26	28	30	32
τ'	2	4	6	8	10	12	14	17	19	21	23	25	27	29	31	33	35
K	1	2	3	5	6	8	9	10	12	13	14	16	17	19	20	21	23
K'	4	7	11	15	18	22	26	29	33	36	40	44	47	51	54	58	62

Fifty pages of extended data tables are available (for $2.50) from Brother Alfred Brousseau, St. Mary's College, Maraga, California 94575.

REFERENCES

1. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Fibonacci Representations," Fibonacci Quarterly, Vol. 10 (1972), pp. 1-28.
2. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Lucas Representations," Fibonacci Quarterly, Vol. 10 (1972), pp. 29-42.
3. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Pellian Representations," Fibonacci Quarterly, Vol. 10 (1972), pp. 449-488.
4. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Addendum to the Paper, 'Fibonacci Representations'," Fibonacci Quarterly, Vol. 10 (1972), pp. 527-530.
5. Ivan Niven, Irrational Numbers, Carus Mathematical Monographs, No. 11, 1963.