PELLIAN REPRESENTATIONS

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1. INTRODUCTION

We define the Pellian numbers by means of

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+1} = 2P_n + P_{n-1} \qquad (n \geq 1).$$

By a Pellian representation of the positive integer N we mean a representation of the form

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \cdots , \tag{1.1}$$

where the ϵ_i are non-negative integers. If the ϵ_i are restricted to the values 0, 1, not all integers N are representable. Indeed we have the sequence of "missing" numbers:

$$4, 9, 10, 11, 16, 21, 22, 23, 24, 25, 26, 27, 28, \cdots .$$

On the other hand we prove that every positive integer N is uniquely representable in the form (1.1) where the ϵ_i satisfy the following conditions:

$$\begin{aligned} &\epsilon_1 = 0 \text{ or } 1; \quad \epsilon_i = 0, 1 \text{ or } 2; \\ &\text{if } \epsilon_i = 2 \text{ then } \epsilon_{i-1} = 0 . \end{aligned} \tag{1.2}$$

It follows that the sequence of "missing" numbers is infinite.

When (1.2) is satisfied we call (1.1) the canonical representation of N. Let A_k denote the set of integers N such that

$$\epsilon_1 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k \neq 0 .$$

and let B_k denote the set of integers N such that

$$\epsilon_1 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k = 2.$$

As in the previous papers of this series [1, 2, 3, 4], we shall characterize the sets A_k, B_k) in terms of certain arithmetic functions. As we shall see below, the discussion is considerably

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more elaborate than that in the case of Fibonacci representations. The number of functions necessary to describe the sets A_k, B_k is greater than that needed for the corresponding Fibonacci results; moreover some of the relations are more intricate.

To begin with, if N has the canonical representation (1.1) we define

(1.3) $$e(N) = \epsilon_2 P_1 + \epsilon_3 P_2 + \epsilon_4 P_3 + \cdots$$

and

(1.4) $$p(N) = \epsilon_1 P_2 + \epsilon_2 P_3 + \epsilon_3 P_4 + \cdots .$$

Then

$$e\bigl(p(n)\bigr) = n \qquad (n = 1, 2, 3, \cdots)$$

however, for some n,

$$p\bigl(e(n)\bigr) \neq n .$$

Note that the right member of (1.3) need not be canonical.

Next we define the following six functions:

$$a(n) = [\sqrt{2}\, n], \qquad b(n) = [(2 + \sqrt{2})n]$$
$$d(n) = [(1 + \sqrt{2})n], \qquad d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n],$$
$$\delta(n) = b(n) + d(n), \quad \epsilon(n) = \text{complement of } \delta(n) .$$

Two (strictly monotone) functions f_1, f_2 from $\underset{\sim}{N}$ to $\underset{\sim}{N}$ are <u>complementary</u> if the sets

$$f_1(\underset{\sim}{N}), \qquad f_2(\underset{\sim}{N})$$

constitute a disjoint partition of $\underset{\sim}{N}$, the set of positive integers. In particular a, b; d, d'; δ, ϵ are complementary pairs of functions.

Of the numerous relations satisfied by these functions we mention in particular the following:

$$b(n) = a(n) + 2n, \qquad d(n) = a(n) + n ,$$
$$ab(n) = a(n) + b(n), \qquad d'(2n) = b(n) ,$$
$$d(n) = a\bigl(b(n) - d'(n)\bigr), \quad a^2b(n) = 2b(n) = 1,$$
$$\epsilon(2n) = \epsilon(2n - 1) + 1 = d(n), \quad d'\bigl(b(n)\bigr) = \delta(n),$$
$$a(n + 1) = e(n) + n + 1, \quad b(n + 1) = p(n) + n + 3,$$
$$e\bigl(d(n)\bigr) = n, \quad e\bigl(b(n)\bigr) = a(n), \quad e\bigl(\delta(n)\bigr) = d(n),$$
$$p\bigl(d(n)\bigr) = \delta(n), \qquad p\bigl(\delta(n)\bigr) = d\bigl(\delta(n)\bigr).$$

The sets A_k, B_k are described by the following formulas:

$$A_1 = d(\underset{\sim}{N}) - 1 ,$$

$$A_{2k} = d\delta^{k-1}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) ,$$

$$A_{2k+1} = \delta^{k}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) ,$$

$$B_{2k} = d\delta^{k-1}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) ,$$

$$B_{2k+1} = \delta^{k}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) .$$

This summarizes the first half of the paper. In the remaining sections of the paper we discuss various other functional relations. For the most part these relations are motivated by the introduction of certain supplementary functions $f, f'; g, g'$ now to be defined. To begin with, we note that the function

$$s(n) = ab(n) - ba(n)$$

takes on only the values 1, 2; similarly the function

$$t(n) = ad'(n) - d'a(n)$$

takes on only the values 0, 1. We define f, f' by means of

$$s\bigl(f(n)\bigr) = 1, \qquad s\bigl(f'(n)\bigr) = 2 ;$$

similarly we define g, g' by means of

$$t\bigl(g(n)\bigr) = 0, \qquad t\bigl(g'(n)\bigr) = 1 .$$

Thus $f, f'; g, g'$ are complementary pairs.

Alternatively we may define these functions by means of

$$a^2\bigl(f(n)\bigr) \equiv 1, \qquad a^2\bigl(f'(n)\bigr) \equiv 0 \pmod 2$$

and

$$a\bigl(g(n)\bigr) \equiv 1, \qquad a\bigl(g'(n)\bigr) \equiv 0 \pmod 2 .$$

In addition, the complementary pair c, c' should also be mentioned:

$$c(n) = b(n) - d'(n) ;$$

as noted above,

$$d(n) = a\bigl(c(n)\bigr) .$$

Of the relations satisfied by these functions we note the following:

$$g(n) = a(f(n)), \qquad f'(n) = d(f(n))$$

$$b(f(n)) - a(f'(n)) = 1$$

$$c'(n) = \begin{cases} d(n) & (n = f'(k)) \\ d(n) - 1 & (n = f(k)) \end{cases}$$

$$c(n) = \begin{cases} d'(n) + 1 & (n = g(k)) \\ d'(n) & (n = g'(k)) \end{cases}$$

$$a(c'(n)) = c'(n) + n - 1 = d'(2n - 1)$$

$$c(n) = \epsilon(a(n)) + 1$$

$$e(c'(n) + 1) = n .$$

The last section of the paper contains some theorems involving the functions of σ, τ defined as follows by means of (1.1):

$$\sigma(N) \equiv \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots \pmod 2$$

$$\tau(N) \equiv k \pmod 2 \qquad (N \in A_k) .$$

In particular we show that

$$b(\underset{\sim}{N}) = \{n \mid \sigma(n) = 0, \quad \tau(n) = 1\}$$

$$g(\underset{\sim}{N}) = \{n \mid \sigma(n) = \tau(n)\}$$

$$= \{n \mid \sigma(n - 1) = 0\} ,$$

$$dg(\underset{\sim}{N}) = \{n | n \in (d), \quad \sigma(n) = 1\}$$

$$dg'(\underset{\sim}{N}) = \{n | n \in (d), \quad \sigma(n) = 0\} .$$

For the convenience of the reader a summary of formulas appears at the end of the paper, as well as several numerical tables.

It should be remarked that most of the theorems in this paper were suggested by numerical data. Thus further numerical data may well suggest additional theorems, particularly in the case of some of the functions defined in the latter part of the paper and not explicitly mentioned in this Introduction.

2. THE CANONICAL REPRESENTATIONS

As above, the Pellian numbers P_n are defined by

(2.1) $$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} ,$$

so that

$$P_2 = 2, \quad P_3 = 5, \quad P_4 = 12, \quad P_5 = 29, \quad P_6 = 70, \quad \cdots .$$

We consider sequences

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \tag{2.2}$$

of length n, where the ϵ_i satisfy the conditions

$$\begin{cases} \epsilon_1 = 0 \text{ or } 1; \quad \epsilon_i = 0, 1, 2 \quad (i > 1) \\ \text{if } \epsilon_i = 2 \text{ then } \epsilon_{i-1} = 0 . \end{cases} \tag{2.3}$$

It is easily seen by induction on n that the number of sequences (2.2) is precisely P_{n+1}. We prove next that if N is given by

$$N = \epsilon_1 P_1 + \cdots + \epsilon_n P_n ,$$

where the ϵ_i satisfy the conditions (2.3), then $N < P_{n+1}$. For otherwise we would have

$$\begin{aligned} N - \epsilon_n P_n - \epsilon_{n-1} P_{n-1} &\geq P_{n+1} - \epsilon_n P_n - \epsilon_{n-1} P_{n-1} \\ &= (2 - \epsilon_n) P_n + (1 - \epsilon_{n-1}) P_{n-1} \geq P_{n-1} \quad , \end{aligned}$$

which eventually leads to a contradiction. See Keller [7] for similar results.

Theorem 2.1. Every positive integer N can be written uniquely in the form

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \cdots , \tag{2.4}$$

where

$$\begin{cases} \epsilon_1 = 0 \text{ or } 1; \quad \epsilon_i = 0, 1 \text{ or } 2; \\ \text{if } \epsilon_i = 2 \text{ then } \epsilon_{i-1} = 0 \end{cases} \tag{2.5}$$

Proof. In view of the preceding remarks, it is enough to prove that no integer N can have more than one representation (2.4), because if this can be established, the P_{n+1} numbers corresponding to the sequences (2.2) of length n will be precisely

$$0, 1, 2, \cdots, P_{n+1} - 1 .$$

Now suppose N is given by

$$N = \epsilon_1 P_1 + \cdots + \epsilon_n P_n, \qquad \epsilon_n \neq 0 ,$$

where the ϵ_i satisfy (2.5). Then $P_n \leq N < P_{n+1}$, so that n is uniquely determined by N. Now by considering $N - \epsilon_n P_n$ we see that ϵ_n itself is determined uniquely by N. Hence, by induction, the theorem is proved.

In a similar manner we can prove the following theorem.

Theorem 2.2. Every positive integer N can be written uniquely in the form

(2.6)
$$N = \delta_1 P_1 + \delta_2 P_2 + \cdots ,$$

where

(2.7)
$$\begin{cases} \delta_i = 0, 1 \text{ or } 2 & (i = 1, 2, 3, \cdots) \\ \text{if } \delta_1 = \cdots = \delta_{i-1} \neq 0, \quad \delta_i \neq 0, & \text{then } i \text{ is odd.} \end{cases}$$

The form (2.4) will be called the first canonical representation for N (or simply the canonical representation); the form (2.6) will be called the second canonical representation.

It will be convenient to abbreviate the formula

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \cdots$$

as follows:

$$N = .\epsilon_1 \epsilon_2 \epsilon_3 \cdots .$$

We shall say that N is a missing number if $\epsilon_i = 2$ for some i. Hence the missing numbers are those which are not the sum of distinct Pell numbers.

Theorem 2.3. The number of missing numbers less than P_{n+1} is equal to $P_{n+1} - 2^n$. Moreover if

$$N = \epsilon_0 + 2\epsilon_1 + \cdots + 2^k \epsilon_k \qquad (\epsilon_i = 0, 1)$$

is the binary representation of N, then

$$R_N = \epsilon_0 P_1 + \epsilon_1 P_2 + \cdots + \epsilon_k P_{k+1}$$

is the N^{th} number that can be represented as a sum of distinct Pell numbers.

Proof. The number of sequences

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$$

in which each $\epsilon_i = 0$ or 1 is clearly 2^n. Since the total number of sequences is P_{n+1}, it follows that the number of sequences containing at least one 2 is $P_{n+1} - 2^n$.

For the second half of the theorem it suffices to observe that the proof of Theorem 2.1 shows that R_N is a strictly monotone function of N.

The first few missing numbers are

(2.8)
$$4, 9, 10, 11, 16, 21, 22, 23, 24, 25, 26, 27, 28, \cdots .$$

Let N have the first canonical representation

$$N = \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \cdots .$$

We define the functions $e(N)$, $p(N)$ by means of

(2.9) $$e(N) = \epsilon_2 P_1 + \epsilon_3 P_2 + \epsilon_4 P_3 + \cdots$$

and

(2.10) $$p(N) = \epsilon_1 P_2 + \epsilon_2 P_3 + \epsilon_3 P_4 + \cdots .$$

Theorem 2.4. The functions e and p satisfy the following identities:

(2.11) $$p(n) = e(n) + 2n$$

(2.12) $$e\bigl(p(n)\bigr) = n$$

(2.13) $$e\bigl(p(n) + 1\bigr) = n$$

(2.14) $$e\ p(n) + 2 = n + 1 .$$

Moreover e and p are monotone.

Proof. Let n be given canonically by

$$n = \cdot\,\epsilon_1\epsilon_2\epsilon_3 \cdots .$$

Then by definition

$$p(n) = \cdot\,0\epsilon_1\epsilon_2\epsilon_3 \cdots$$

and

$$e(n) = \cdot\,\epsilon_2\epsilon_3\epsilon_4 \cdots .$$

Hence (2.11), (2.12), (2.13) follow at once. If $\epsilon_2 < 2$, $p(n) + 2$ is given canonically by

$$p(n) + 2 = \cdot\,0(\epsilon_1 + 1)\epsilon_2\epsilon_3 \cdots$$

and (2.14) follows. Now suppose $\epsilon_2 = 2$. Then $\epsilon_1 = 0$ and

$$p(n) + 2 = (\epsilon_3 + 1)P_4 + \epsilon_4 P_5 + \cdots .$$

As before this is canonical if $\epsilon_4 < 2$ and (2.14) follows. Otherwise we continue until, for some k, $\epsilon_{2k} < 2$, and again (2.14) follows.

To prove the monotonicity of e and p, we again take the canonical representation

$$n = \cdot\,\epsilon_1\epsilon_2\epsilon_3 \cdots .$$

if $\epsilon_1 = 1$, then

$$n - 1 = \cdot\,0\epsilon_2\epsilon_3 \cdots ,$$

so that $e(n - 1) = e(n)$. If $\epsilon_1 = 0$ and $\epsilon_2 \neq 0$, then

$$n - 1 = \cdot 1(\epsilon_2 - 1)\epsilon_3 \cdots$$

and $e(n-1) = e(n) - 1$. If

$$\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k \neq 0, \tag{2.15}$$

then, for k odd,

$$n - 1 = eP_2 + 2P_4 + \cdots + 2P_{k-1} + (\epsilon_k - 1)P_k + \epsilon_{k+1}P_{k+1} + \cdots \tag{2.16}$$

and

$$e(n-1) = 2P_1 + 2P_3 + \cdots + 2P_{k-2} + (\epsilon_n - 1)P_{k-1} + \epsilon_{k+1}P_k + \cdots .$$

This gives $e(n-1) = e(n)$. If in (2.15) k is even, we have

$$n - 1 = P_1 + 2P_3 + \cdots + 2P_{n-1} + (\epsilon_k - 1)P_k + \epsilon_{k+1}P_{k+1} + \cdots \tag{2.17}$$

and

$$e(n-1) = 2P_2 + \cdots + 2P_{k-2} + (\epsilon_k - 1)P_{k-1} + \epsilon_{k+1}P_k + \cdots ,$$

which gives $e(n-1) = e(n) - 1$.

This proves that e is monotone and therefore, by (2.12), p is also monotone.

As a corollary we have the following theorem.

Theorem 2.5. For any n, the equation $e(x) = n$ has at most three solutions.

Proof. Assume

$$e(x_1) = e(x_2) = e(x_3) = e(x_4)$$

with

$$x_1 < x_2 < x_3 < x_4 .$$

It follows from the definition of p that any n must be of at least one of the three forms $p(j)$, $p(j) + 1$ or $p(j) + 2$. Take $n = x_2$. Then by Theorem 2.4 we have

$$e(x_1) \neq e(x_4) .$$

3. NEWMAN-SKOLEM PAIRS

By a Newman-Skolem pair we shall mean a pair of functions (a, b) defined on the positive integers N and satisfying the conditions

$$a(\underset{\sim}{N}) \cup b(\underset{\sim}{N}) = \underset{\sim}{N}, \tag{3.1}$$

$$a(\underset{\sim}{N}) \cap b(\underset{\sim}{N}) \text{ vacuous}, \tag{3.2}$$

$$a, b \text{ strictly monotone.} \tag{3.3}$$

Hence a and b are complementary functions. The Newman-Skolem pair (a, b) defined uniquely by the condition

$$b(n) = a(n) + n$$

was introduced in [5].

We shall say that (a, b) is ordered if

$$a(n) < b(n) \qquad (n = 1, 2, 3, \cdots) \tag{3.4}$$

and that (a, b) is separated if (a, b) is ordered and

$$b(n + 1) > b(n) + 1 \qquad (n = 1, 2, 3, \cdots) . \tag{3.5}$$

Define

$$d(n) = b(n) - n . \tag{3.6}$$

Theorem 3.1. If (a, b) is separated then

$$ad(n) = b(n) - 1$$

and

$$a(d(n) + 1) = b(n) + 1 . \tag{3.8}$$

Proof. By (3.5) we must have, for some k,

$$b(n) - 1 = a(k), \qquad b(n) + 1 = a(k + 1) .$$

Hence the $k + n$ numbers

$$a(1), a(2), \cdots, a(k); b(1), \cdots, b(n)$$

comprise all the numbers less than or equal to b(n), so that

$$k + n = b(n), \qquad k = b(n) - n = d(n) .$$

This evidently completes the proof of the theorem.

Theorem 3.2. If (a, b) is separated then

$$a(n + 1) = a(n) + 2 \rightleftarrows n \in (d) , \tag{3.9}$$

where (d) denotes the range of the function d.

Proof. Since (a, b) is separated it is clear that, for any n, either $a(n + 1) = a(n) + 1$ or $a(n + 1) = a(n) + 2$. Also we have

$$d(n + 1) = d(n) = b(n + 1) - b(n) - 1 \geq 1 ,$$

so that d is strictly monotone.

Now assume

$$n \neq d(k) \qquad (k = 1, 2, 3, \cdots).$$

Then, for some k,

$$d(k) + 1 \le n < d(k+1).$$

If $a(n+1) = a(n) + 2$ then $a(n) + 1 = b(j)$ for some (j). But

$$b(k) + 2 = a\bigl(d(k) + 1\bigr) + 1 \le a(n) + 1 < ad(k+1) + 1 = b(k+1),$$

so that $a(n) + 1 = b(j)$ is impossible.

Theorem 3.3. If (a,b) is separated and

$$d(n+1) > d(n) + 1 \qquad (n = 1, 2, 3, \cdots)$$

then

$$a\bigl(d(k) - 1\bigr) = b(k) - 2 \qquad \bigl(d(k) \ge 2\bigr).$$

Proof. Since

$$d(k) - 1 \neq d(j) \qquad (j = 1, 2, 3, \cdots),$$

by Theorem 3.1,

$$b(k) - 1 = ad(k) = a\bigl(d(k) - 1\bigr) + 1.$$

Theorem 3.4. If (a,b) is a Newman-Skolem pair and if, for all n, we have

$$ba(n) < ab(n) < b\bigl(a(n) + 1\bigr),$$

then

$$ab(n) = a(n) + b(n). \tag{3.10}$$

Proof. Using the hypothesis we see that the $a(n) + b(n)$ numbers

$$b(1),\ b(2),\ \cdots,\ ba(n);\quad a(1),\ a(2),\ \cdots,\ ab(n)$$

coincide with the numbers less than or equal to $ab(n)$. Hence (3.10) follows at once.

It is well known that if α, β are positive irrational numbers satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \qquad \alpha < \beta, \tag{3.11}$$

the pair (a,b) defined by

$$a(n) = [\alpha n], \qquad b(n) = [\beta n] \tag{3.12}$$

is a separated Newman-Skolem pair. For the remainder of this paper we define

$$a(n) = [\sqrt{2}\,n]$$

$$b(n) = a(n) + 2n = [(2 + \sqrt{2})n]$$

$$d(n) = b(n) - n = [(1 + \sqrt{2})n]$$

$$d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n] .$$

Thus (a,b) and (d',d) are separated Newman-Skolem pairs. Making use of the preceding theorems we get

Theorem 3.5. The functions a, b, d, d' as defined above, satisfy the following relations:

$$ad(n) = b(n) - 1$$

$$a\big(d(n) + 1\big) = b(n) + 1$$

$$a\big(d(n) - 1\big) = b(n) - 2$$

$$d'\big(a(n)\big) = d(n) - 1$$

$$d'\big(a(n) + 1\big) = d(n) + 1$$

$$a(n + 1) = a(n) + 2 \rightleftarrows n \in (d)$$

$$d'(n + 1) = d'(n) + 2 \rightleftarrows n \in (a) .$$

Here we have let (f) denote the range of the function f.

Theorem 3.6. For all positive integers n, we have

$$ab(n) = a(n) + b(n) . \tag{3.13}$$

Proof. Since

$$a(n) < \sqrt{2}\,n < a(n) + 1 ,$$

we see that

$$2a(n) + \sqrt{2}\,a(n) < \sqrt{2}\big(2n + a(n)\big) \le 2\big(a(n) + 1\big) + \sqrt{2}\big(a(n) + 1\big).$$

Hence, taking greatest integers,

$$b\big(a(n)\big) \le ab(n) \le b\big(a(n) + 1\big) .$$

Equality is obviously impossible. Hence, by Theorem 3.4, we get (3.13).

Suppose (d',d) is any separated Newman-Skolem pair and suppose f is any increasing function. Let $d'f = b$ and let a be such that (a,b) is a Newman-Skolem pair. Then since $d'(\underset{\sim}{N})$ $b(\underset{\sim}{N})$, it follows that $d(\underset{\sim}{N})$ $a(\underset{\sim}{N})$. Hence there exists an increasing function c such that

$$d(n) = ac(n) . \tag{3.14}$$

Now, since (d',d) is separated, we have

$$d'(d(n) - n) = d(n) - 1 .$$

Hence, among the numbers

$$1, 2, 3, \cdots, d(n) ,$$

there are exactly j members of $b(\underset{\sim}{N})$, namely

$$d'f(1), \quad d'f(2), \quad \cdots, \quad d'f(j) ,$$

where j is the largest integer such that

$$f(j) \leq d(n) - n .$$

We may write (symbolically)

$$(3.15) \qquad j = \left[\frac{d(n) - n}{f}\right] .$$

The remaining $d(n) - j$ members in

$$\{1, 2, 3, \cdots, d(n)\}$$

are members of $a(N)$, so that

$$d(n) = a(d(n) - j),$$

that is

$$(3.16) \qquad c(n) = d(n) - \left[\frac{d(n) - n}{f}\right] .$$

Theorem 3.7. For the functions a, b, c, d' previously defined, we have

$$(3.17) \qquad d(n) = a(b(n) - d'(n)) .$$

Proof. Since $d'(2n) = b(n)$, the above remarks apply with $f(n) = 2n$. Hence

$$c(n) = d(n) - \left[\frac{d(n) - n}{2}\right] = b(n) - n - \left[\frac{a(n)}{2}\right]$$

But

$$n + \left[\frac{a(n)}{2}\right] = n + \left[\tfrac{1}{2}[2n]\right] = n + \left[\frac{\sqrt{2}\, n}{2}\right]$$

$$= [\tfrac{1}{2}(2 + \sqrt{2})n] = d'(n) ,$$

so that

$$c(n) = b(n) - d'(n) .$$

This evidently completes the proof of the theorem.

4. RELATIONS BETWEEN a, b, d, d' AND e AND p

Theorem 4.1. The functions a, b, c and p are related by the following formulas:

$$a(n + 1) = e(n) + n + 1 \tag{4.1}$$

$$b(n + 1) = p(n) + n + 3 . \tag{4.2}$$

These formulas imply

$$e(n) = [(\sqrt{2} - 1)(n + 1)], \qquad e(0) = 0 \tag{4.3}$$

$$p(n) = [\sqrt{2}(n + 1)] + n - 1, \qquad p(0) = 0 . \tag{4.4}$$

Proof. It is clear by induction that (a, b) is the unique Newman-Skolem pair satisfying

$$b(n) = a(n) + 2n \qquad (n = 1, 2, 3, \cdots) . \tag{4.5}$$

Now let

$$a'(n + 1) = e(n) + n + 1$$

and

$$b'(n + 1) = p(n) + n + 3 .$$

We shall show that (a', b') is a Newman-Skolem pair satisfying

$$b'(n) = a'(n) + 2n . \tag{4.6}$$

This will evidently prove the theorem.

By (2.11) we have

$$p(n) = e(n) + 2n.$$

Hence

$$b'(n + 1) - a'(n + 1) = p(n) - e(n) + 2 = 2n + 2 ,$$

so that (4.6) is satisfied.

Since, by Theorem (2.4),

$$e\bigl(p(n)\bigr) = e\bigl(p(n) + 1\bigr) = n, \qquad e\bigl(p(n) + 2\bigr) = n + 1 ,$$

we get

$$a'\bigl(p(n) + 2\bigr) = p(n) + n + 2 = b'(n + 1) - 1$$

and

$$a'\bigl(p(n) + 3\bigr) = p(n) + n + 4 = b'(n + 1) + 1.$$

Hence the ranges of a' and b' are disjoint. Furthermore we see that

$$a'(1), \quad a'(2), \quad \cdots, \quad a'(p(n)) + 2 \; ; \quad b'(1), \quad b'(2), \quad \cdots, \quad b'(n+1)$$

are $p(n) + n + 3$ distinct numbers less than or equal to

$$b'(n+1) = p(n) + n + 3 .$$

Hence all numbers in this range must be included and the theorem is proved.

Theorem 4.2. We have, for all n,

$$e(b(n)) = a(n) , \tag{4.7}$$

$$e(d(n)) = n . \tag{4.8}$$

Proof. By Theorems 3.5 and 4.1 we have

$$b(n) + 1 = a(d(n) + 1) = d(n) + 1 + e(d(n)) .$$

Hence, since $b(n) - d(n) = n$, we get (4.8).

Since $d(n) = [(1 + \sqrt{2})n]$, it follows that

$$d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n] .$$

Hence

$$d'(2n) = b(n) .$$

In particular

$$b(n) \notin d(\underset{\sim}{N}) ,$$

so that, by Theorem 3.2,

$$a(b(n) + 1) = ab(n) + 1 .$$

Then

$$b(n) + 1 + e(b(n)) = a(n) + b(n) + 1$$

and therefore

$$e(b(n)) = a(n) .$$

This completes the proof of the theorem.

Further relations between a, b, d, d', e and p will be established in the next section.

5. THE SETS A_k AND B_k

We define the sets A_k and B_k as follows:

$$A_k = \{N \mid \epsilon_1 = \cdots = \epsilon_{k-1} = 0, \quad \epsilon_k \neq 0\} , \tag{5.1}$$

(5.2) $$B_k = \{N \mid \epsilon_1 = \cdots = \epsilon_{k-1} = 0, \quad \epsilon_k = 2\},$$

where

(5.3) $$N = \cdot\, \epsilon_1 \epsilon_2 \epsilon_3 \cdots$$

is the canonical representation of N.

We also define

(5.4) $$\delta(n) = b(n) + d(n) = 2a(n) + 3n$$

and define $\epsilon(n)$ by the requirement

(5.5) $$(\epsilon, \delta) \text{ is a Newman-Skolem pair.}$$

Theorem 5.1. Let the non-negative integer n have the canonical representation

(5.6) $$n = \cdot\, \epsilon_1 \epsilon_2 \epsilon_3 \cdots .$$

Then

(5.7) $$d(n + 1) - 1 = p(n) + 1 = \cdot 1 \epsilon_1 \epsilon_2 \epsilon_3 \cdots .$$

Hence

(5.8) $$A_1 = d(\underset{\sim}{N}) - 1 .$$

Proof. The theorem follows from the relations

$$b(n + 1) = d(n + 1) + n + 1 = p(n) + n + 3 .$$

Since it is clear that (ϵ, δ) is a separated Newman-Skolem pair, it follows from Theorem 3.1 that

(5.9) $$\epsilon(2d(n)) = \delta(n) - 1$$

(5.10) $$\epsilon(2d(n) + 1) = \delta(n) + 1 .$$

Since $\delta(n) - n = 2d(n)$, it follows from Theorem 3.3 that

(5.11) $$\epsilon(2d(n) - 1) = \delta(n) - 2 .$$

Moreover we have

$$d^2(n) = d(n) + ad(n) = d(n) \neq b(n) - 1 = \delta(n) - 1 ,$$

so that

(5.12) $$e(\delta(n) - 1) = d(n) .$$

Also we have

$$3 + d + pd = b(d + 1) = 2(d + 1) + a(d + 1) = 2d + 2 + b + 1 ,$$

so that

(5.13) $$pd(n) = d(n) + b(n) = \delta(n) .$$

Applying e, we get

(5.14) $$e\delta(n) = d(n) .$$

Now using (4.1) and (4.2) we get

$$p\delta = d(\delta + 1) - 2 = (\delta + 1) + a(\delta + 1) - 2$$
$$= (\delta + 1) + (\delta + 1 + e) - 2 = d + 2\delta$$
$$= \delta + \delta + e(\delta - 1) = \delta + a\delta = d\delta \quad ,$$

so that

(5.15) $$p\delta = d\delta .$$

Theorem 5.2. We have

(5.16) $$B_2 = d^2(\underset{\sim}{N})$$

(5.17) $$B_{2k+1} = \delta^k d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$$

(5.18) $$B_{2k} = d\delta^{k-1} d(\underset{\sim}{N}) \qquad (k = 2, 3, 4, \cdots) .$$

Proof. It is only necessary to prove (5.16) since (5.17) will then follow by (5.13) and (5.15).

Applying Theorem 5.1 to $d(n + 1) - 1$ we obtain

$$d^2(n + 1) - 1 = \cdot 11\epsilon_1\epsilon_2\epsilon_3 \cdots ,$$

so that

$$d^2(n + 1) = \cdot 02\, \epsilon_1\epsilon_2\epsilon_3 \cdots .$$

This evidently proves (5.16) and therefore the proof of Theorem 5.2 is complete.

Note that if n has the canonical representation

$$n = \cdot \epsilon_1\epsilon_2\epsilon_3 \cdots ,$$

then

(5.19) $$d(n + 1) - 1 = \cdot 1\epsilon_1\epsilon_2\epsilon_3 \cdots$$

is also canonical. Since $\delta(n) = 2d(n) + n$, it follows that

(5.20) $$\delta(n + 1) - 1 = \cdot 02\, \epsilon_1\epsilon_2\epsilon_3 \cdots$$

and

(5.21) $$d\big(\delta(n + 1)\big) - 1 = \cdot 102\, \epsilon_1\epsilon_2 \cdots$$

are both canonical.

Theorem 5.3. We have

(5.22) $$A_1 = d(\underset{\sim}{N}) - 1$$

(5.23) $$A_{2k} = d\delta^{k-1}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$$

(5.24) $$A_{2k+1} = \delta^k\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots) .$$

Proof. We have already proved (5.22). It will therefore suffice to establish

(5.25) $$A_2 = d\epsilon(\underset{\sim}{N}) .$$

Now A_2 consists of all N in the canonical form

$$N = \cdot 0\ \epsilon_2\epsilon_3\epsilon_4 \cdots \qquad (\epsilon_2 \neq 0) .$$

Hence $A_2 - 1$ consists of all N in the canonical form

$$N = \cdot 1\ (\epsilon_2 - 1)\epsilon_3\epsilon_4 \cdots \qquad (\epsilon_3 \neq 2) .$$

Furthermore $d(\underset{\sim}{N}) - 1$ consists of all N in the canonical form

$$N = \cdot 1\ f_2f_3f_4 \cdots$$

and by (5.21), $d\delta(N) - 1$ consists of all N in the canonical form

$$N = \cdot 102\ g_4g_5 \cdots .$$

Therefore since $d(\underset{\sim}{N}) - 1$ is the disjoint union of $d\delta(N) - 1$ and $d\epsilon(N) - 1$, we see that

$$d\epsilon(\underset{\sim}{N}) - 1 = A_2 - 1 ,$$

that is,

$$A_2 = d\epsilon(\underset{\sim}{N}) .$$

This completes the proof of the Theorem.

Theorem 5.4. We have

(5.26) $$d(\underset{\sim}{N}) = \bigcup_1^{\infty} A_{2k}$$

(5.27) $$(\underset{\sim}{N}) = \bigcup_1^{\infty} A_{2k+1}$$

(5.28) $$(\underset{\sim}{N}) = d(\underset{\sim}{N}) \cup \left(d(\underset{\sim}{N}) - 1\right) .$$

Proof. Since every integer is of the form $\delta^k \epsilon(n)$ for some $k \geq 0$, (5,26) and (5.27) follow from the previous theorem. Since $\epsilon(\underset{\sim}{N})$ is the complement of $\delta(\underset{\sim}{N})$, (5.28) follows from (5.22) and (5.26).

We have seen above that

(5.29) $$\epsilon(\underset{\sim}{N}) = d(\underset{\sim}{N}) \cup \left(d(\underset{\sim}{N}) - 1\right) .$$

Hence the numbers in $\epsilon(\underset{\sim}{N})$ are, in order,

$$d(1) - 1, \quad d(1), \quad d(2) - 1, \quad d(2), \quad d(3) - 1, \quad d(3), \quad \cdots .$$

It follows that

(5.30) $$\epsilon(2n) = d(n), \qquad \epsilon(2n - 1) = d(n) - 1 .$$

Applying e, we have

(5.31) $$e(\epsilon(n)) = [n/2] .$$

The following remark concerning the second canonical form is useful. If

$$n = \cdot f_1 f_2 f_3 \cdots \qquad \text{(second canonical)}$$

then

$$d(n) = \cdot 0\, f_1 f_2 f_3 \cdots \qquad \text{(first canonical)}$$

and

$$\delta(n) = \cdot 00\, f_1 f_2 f_3 \cdots \qquad \text{(first and second canonical)} .$$

6. ADDITIONAL RELATIONS INVOLVING a AND b

Theorem 6.1. We have

(6.1) $$a^2 b = 2b - 1 .$$

For the proof we require

Theorem 6.2. The integer n is in (d) if and only if

(6.2) $$\left\{ \frac{n}{1 + \sqrt{2}} \right\} > 2 - \sqrt{2} ,$$

where $\{\alpha\}$ denotes the fractional part of the real number α.

Proof. Let

$$n = d(k) = [(1 + \sqrt{2})k] ,$$

so that

$$(1 + \sqrt{2})k - 1 < n < (1 + \sqrt{2})k, \qquad k - \frac{1}{1 + \sqrt{2}} < \frac{n}{1 + \sqrt{2}} < k .$$

This is equivalent to

$$\left\{ \frac{n}{1 + \sqrt{2}} \right\} > 1 - \frac{1}{1 + \sqrt{2}} = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2} .$$

Proof of Theorem 6.1. It follows from

$$a(n) = [\sqrt{2}\, n]$$

that

(6.3) $$n - 2 \le a^2(n) \le n - 1 .$$

It therefore suffices to show that

$$(6.4)\qquad a^2b(n) \equiv 1 \pmod 2 \qquad (n = 1, 2, 3, \cdots).$$

Assume that there exists an integer k such that

$$a^2b(k) \equiv 0 \pmod 2,$$

that is

$$a(2d(k)) \equiv 0 \pmod 2.$$

Then

$$[2\sqrt{2}\, d(k)] = 2j$$

for some integer j. Hence

$$2j < 2\sqrt{2}\, d(k) < 2j + 1,$$

$$j < \sqrt{2}\, d(k) < j + \tfrac{1}{2},$$

so that

$$(6.5)\qquad \{\sqrt{2}\, d(k)\} < \tfrac{1}{2}.$$

By Theorem 6.2,

$$\left\{\frac{d(k)}{1 + \sqrt{2}}\right\} > 2 - \sqrt{2},$$

that is

$$\{(\sqrt{2} - 1)d(k)\} > 2 - \sqrt{2}.$$

Hence

$$\{\sqrt{2}\, d(k)\} > 2 - \sqrt{2}.$$

This contradicts (6.5) and so completes the proof of the theorem.

It follows from $ab = a + b$ that

$$b^2 = ab + 2b = a + 3b,$$

$$\begin{aligned} b^3 &= ab + 3b^2 \\ &= a + b + 3(a + 3b) \\ &= 4a + 10b, \end{aligned}$$

$$\begin{aligned} b^4 &= 4(a + b) + 10(a + 3b) \\ &= 14a + 34b. \end{aligned}$$

Put

$$(6.6)\qquad b^k = u_k a + v_k b, \quad u_1 = 0, \quad v_1 = 1, \quad u_2 = 1, \quad v_2 = 3.$$

Then

$$\begin{aligned} b^{k+1} &= u_k(a + b) + v_k(a + 3b) \\ &= (u_k + v_k)a + (u_k + 3v_k)b, \end{aligned}$$

so that

(6.7)
$$\begin{cases} u_{k+1} = u_k + v_k \\ v_{k+1} = u_k + 3v_k \end{cases} .$$

It follows that

$$\begin{cases} u_{k+2} - 4u_{k+1} + 2u_k = 0 \\ v_{k+2} - 4v_{k+1} + 2v_k = 0 \end{cases} .$$

Then

$$U(x) = \sum_1^\infty u_k x^k = x^2 + \sum_3^\infty (4u_{k-1} - 2u_{k-2}) x^k$$

$$= x^2 + (4x - 2x^2)U(x) ,$$

so that

$$U(x) = \frac{x^2}{1 - 4x + x^2} .$$

We find that

(6.8)
$$u_k = \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta}, \qquad v_k = u_{k+1} - u_k ,$$

where

$$\alpha = 2 + \sqrt{2}, \qquad \beta = 2 - \sqrt{2} .$$

Theorem 6.3. The function b^k is evaluated by means of (6.6) and (6.8). In the next place,

$$ab = a + b ,$$

$$(ab)^2 = a^2b + bab$$
$$= 2a^2b + 2ab$$
$$= 2(2b - 1) + 2(a + b)$$
$$= 2a + 6b - 2 ,$$

$$(ab)^3 = 2a^2b + 6bab - 2$$
$$= 8a^2b + 12ab - 2$$
$$= 8(2b - 1) + 12(a + b) - 2$$
$$= 12a + 28b - 10 ,$$

$$(ab)^4 = 56a + 136b - 50 .$$

Put

(6.9)
$$(ab)^k = u_k a + v_k b - t_k ,$$

$$u_1 = v_1 = 1, \quad t_1 = 0, \quad u_2 = 2, \quad v_2 = 6, \quad t_2 = 2 .$$

Then

$$(ab)^{k+1} = u_k a^2b + v_k bab - t_k$$
$$= (u_k + v_k) a^2b + 2v_k ab - t_k$$

$$= (u_k + v_k)(2b - 1) + 2v_k(a + b) - t_k$$
$$= 2v_k a + (2u_k + 4v_k)b - (u_k + v_k + t_k),$$

so that

$$u_{k+1} = 2v_k$$
$$v_{k+1} = 2u_k + 4v_k = 4v_k + 4v_{k-1}$$
$$t_{k+1} = u_k + v_k + t_k \quad .$$

Let

$$Q_0 = Q_1 = 1, \quad Q_2 = 3, \quad Q_3 = 7, \quad Q_{k+1} = 2Q_k + Q_{k-1}.$$

It is easily verified that

(6.10) $$Q_k = P_{k-1} + P_k$$

k	0	1	2	3	4	5	6
P_k	0	1	2	5	12	29	70
Q_k	1	1	3	7	17	41	99

We find that

(6.11) $$u_k = 2^{k-1}Q_{k-1}, \qquad v_k = 2^{k-1}Q_k$$

(6.12) $$t_k = \frac{1}{7}(2^{k+1}P_{k+1} - 3\cdot 2^k P_k - 2).$$

Theorem 6.4. The function $(ab)^k$ is evaluated by means of (6.9), (6.10), (6.11) and (6.12).

7. THE FUNCTIONS f, f', g, g', c, c'

It follows from

$$a(n) = [\sqrt{2}\, n], \qquad b(n) = [(2 + \sqrt{2})n]$$

that

(7.1) $$ab(n) - ba(n) = 1 \text{ or } 2 \qquad (n = 1, 2, 3, \cdots).$$

We may accordingly define the pair of complementary functions f, f' by means of

(7.2) $$ab(n) - ba(n) = \begin{cases} 1 & (n \in (f)) \\ 2 & (n \in (f')) \end{cases}.$$

An equivalent definition is

(7.3) $$\begin{cases} a^2f(n) \equiv 1 \pmod 2 \\ a^2f'(n) \equiv 0 \pmod 2 \end{cases}.$$

It is also easily verified that

(7.4) $$ad'(n) - d'a(n) = 0 \text{ or } 1 \qquad (n = 1, 2, 3, \cdots).$$

Hence we may define the pair g, g' by means of

(7.5) $$ad'(n) - d'a(n) = \begin{cases} 0 & (n \in (g)) \\ 1 & (n \in (g')) \end{cases}.$$

It is somewhat more convenient to take as definition

(7.6) $$\begin{cases} ag(n) \equiv 1 \pmod 2 \\ ag'(n) \equiv 0 \pmod 2 \end{cases}.$$

We shall show that (7.5) and (7.6) are equivalent.

For brevity put

(7.7) $$s = ab - ba, \qquad t = ad' - d'a.$$

It is easily verified that

(7.8) $$s(n) = 2n - a^2(n)$$

from which the equivalence of (7.2) and (7.3) is immediate.

It is also immediate from (7.3) and (7.6) that

(7.9) $$g = af.$$

In the next place

$$t = ad' - d'a = ad' - a - n + 1,$$

$$\begin{aligned} ta &= ad'a - a^2 - a + 1 \\ &= a(d - 1) - a^2 - a + 1 \\ &= b - a^2 - a - 1 \quad , \end{aligned}$$

$$ta(n) = 2n - a^2 - 1,$$

(7.10) $$\begin{cases} taf \equiv a^2f + 1 \equiv 0 \pmod 2, \\ taf' \equiv a^2f' + 1 \equiv 1 \pmod 2. \end{cases}$$

Also

$$\begin{aligned} tb &= ad'b - db + 1 \\ &= a\delta - ab - b + 1 \\ &= d + \delta - a - 2b + 1 \\ &= b + 2d - a - 2b + 1 \end{aligned}$$

(7.11) $$\equiv 1 \pmod 2 \quad .$$

It follows from (7.10) and (7.11) that

(7.12) $$t(n) \equiv 0 \pmod 2 \rightleftarrows n \in (g).$$

This evidently establishes the equivalence of (7.5) and (7.6).

Note that the pair g, g' is not separated.

Theorem 7.1. We have

(7.13) $$df = f' .$$

The proof of this theorem requires a number of preliminary results.

Theorem 7.2

(7.14) $$bf - 1 = dg .$$

Proof.

$$bf - dg - 1 = af + 2f - ag - g - 1$$
$$= 2f - a^2f - 1 = 0 .$$

Theorem 7.3

(7.15) $$n \in (f) \rightleftarrows \{\sqrt{2}\,n\} < \frac{1}{\sqrt{2}} .$$

Proof. By (7.2) or (7.3)

$$n \in (f) \rightleftarrows a^2n = 2n - 1 .$$

Consider

$$\left[\sqrt{2}\,[\sqrt{2}\,n]\right] = 2n - 1, \qquad 2n - 1 < \sqrt{2}\,[\sqrt{2}\,n] < 2n .$$

Put $k = [\sqrt{2}\,n]$, so that

$$\sqrt{2}\,n - 1 < \sqrt{2}\,k < 2n$$

$$\sqrt{2}\,n - \frac{1}{\sqrt{2}} < k < \sqrt{2}\,n$$

$$0 < \sqrt{2}\,n - k < \frac{1}{\sqrt{2}} ,$$

that is

(7.16) $$\{\sqrt{2}\,n\} < \frac{1}{\sqrt{2}} .$$

Hence if $n \in (f)$, Eq. (7.6) is satisfied.

Next let $n \in (f')$, so that $a^2(n) = 2n - 2$. Consider

$$\left[\sqrt{2}\,[\sqrt{2}\,n]\right] = 2n - 2$$

$$2n - 2 < \sqrt{2}\,[\sqrt{2}\,n] < 2n - 1$$

$$2n - 2 < \sqrt{2}\,k < 2n - 1 \qquad (k = [\sqrt{2}\,n])$$

$$\sqrt{2}\,n - \sqrt{2} < k < \sqrt{2}\,n - \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} < \sqrt{2}\,n - k < \sqrt{2} ,$$

that is

$$\{\sqrt{2}\,n\} > \frac{1}{\sqrt{2}} \quad . \tag{7.17}$$

Hence if $n \in (f')$, Eq. (7.17) is satisfied.

Combining (7.16) and (7.17), we get (7.15).

<u>Proof of Theorem 7.1.</u> By Theorem 6.2, $n \in (d)$ if and only if

$$\left\{\frac{n}{1+\sqrt{2}}\right\} > 2 - \sqrt{2} \quad . \tag{7.18}$$

Put

$$(1 + \sqrt{2})f = df + \epsilon;$$

by Theorem 7.3, we have $\epsilon < 1/\sqrt{2}$. Moreover

$$\begin{aligned} f &= \frac{df}{1+\sqrt{2}} + \frac{\epsilon}{1+\sqrt{2}} \\ &= J + \left\{\frac{df}{1+\sqrt{2}}\right\} + \frac{\epsilon}{1+\sqrt{2}} \quad , \end{aligned}$$

where

$$J = \left[\frac{df}{1+\sqrt{2}}\right] .$$

Then

$$\left\{\frac{df}{1+\sqrt{2}}\right\} + \frac{\epsilon}{1+\sqrt{2}} = 1 \quad ,$$

$$\{\sqrt{2}\,df\} + \epsilon(\sqrt{2} - 1) = 1 \quad ,$$

$$\{\sqrt{2}\,df\} > 1 - \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \quad ,$$

so that

$$(df) \subset (f') \quad . \tag{7.19}$$

We shall now show that

$$(f') \subset (df) \quad . \tag{7.20}$$

Let n satisfy $\{\sqrt{2}\,n\} > 1/\sqrt{2}$, so that $n \in (f')$. Then, by (7.18), $n \in (d)$, that is

$$n = d(k) = [(1 + \sqrt{2})k] \quad ,$$

for some integer k. Thus

$$(1 + \sqrt{2})k = n + \{\sqrt{2}\,k\}$$

$$(1 + \sqrt{2})k + (\sqrt{2} - 1)n = \sqrt{2}\,n + \{\sqrt{2}\,k\}$$

$$(1 + \sqrt{2})k + (\sqrt{2} - 1)d(k) = ad(k) + \{\sqrt{2}\,n\} + \{\sqrt{2}\,k\} > b(k) - 1 + \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$\sqrt{2}\,k - (2 - \sqrt{2})d(k) + 1 > \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$\sqrt{2}\,k - (2 - \sqrt{2})\big((1 + \sqrt{2})k - \{\sqrt{2}\,k\}\big) + 1 > \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$(2 - \sqrt{2})\{\sqrt{2}\,k\} + 1 > \frac{1}{\sqrt{2}} + \{\sqrt{2}\,k\}$$

$$\frac{\sqrt{2} - 1}{\sqrt{2}} > (\sqrt{2} - 1)\{\sqrt{2}\,k\}$$

$$\frac{1}{\sqrt{2}} > \{\sqrt{2}\,k\} .$$

Therefore $k \in (f)$, $n \in (df)$.

This proves (7.20) and so completes the proof of the theorem.

Theorem 7.4. We have

(7.21) $$bf - af' = 1 .$$

Proof. By (7.14), Eq. (7.21) may be replaced by

(7.22) $$af' = dg = daf ,$$

which by Theorem 7.1 is the same as

(7.23) $$adf = daf .$$

Now

$$\begin{aligned} ad - da &= b - 1 - a^2 - a \\ &= 2n - 1 - a^2 , \\ adf - daf &= 2f - 1 - a^2 f = 0 . \end{aligned}$$

This proves (7.23) and therefore proves (7.21).

Theorem 7.5. The pair (f, f') is separated.

Proof. By (7.13)

$$f'(n) = df(n) > f(n) ,$$

so that the pair (f, f') is ordered. Since the pair (d', d) is separated, it follows that

$$f'(n + 1) - f'(n) = df(n + 1) - df(n) > 1.$$

Define

(7.24) $$c(n) = b(n) - d'(n) ,$$

so that by (3.17)

(7.25) $$d = ac.$$

Theorem 7.6. We have

(7.26) $$f' = acf = caf.$$

Proof. It suffices to show that

(7.27) $$acf - caf = 0.$$

Now

$$\begin{aligned} ac - ca &= d - ba + d'a \\ &= d - a^2 - 2a + d - 1, \\ acf - caf &= 2df - 2af - a^2f - 1 \\ &= 2f - a^2f - 1 = 0. \end{aligned}$$

Theorem 7.7

(7.28) $$\begin{cases} n \in (g) \rightleftarrows \left\{\dfrac{n}{\sqrt{2}}\right\} < \dfrac{1}{2} \\ n \in (g') \rightleftarrows \left\{\dfrac{n}{\sqrt{2}}\right\} < \dfrac{1}{2} \end{cases}.$$

Proof. Let $n \in (g)$, so that $a(n) \equiv 1 \pmod 2$. Then

$$[\sqrt{2}\,n] = 2k - 1$$

$$2k - 1 < \sqrt{2}\,n < 2k$$

$$k - \frac{1}{2} < \frac{n}{\sqrt{2}} < k,$$

so that

$$\left\{\frac{n}{\sqrt{2}}\right\} > \frac{1}{2}.$$

Next let $n \in (g')$ so that $a(n) \equiv 0 \pmod 2$. Then

$$[\sqrt{2}\,n] = 2k$$

$$2k < \sqrt{2}\,n < 2k + 1$$

$$k < \frac{n}{\sqrt{2}} < k + \frac{1}{2},$$

so that

$$\left\{\frac{n}{\sqrt{2}}\right\} < \frac{1}{2}.$$

This completes the proof of the theorem.

Theorem 7.8

(7.29) $$g' = a(\tfrac{1}{2}ag') + 1.$$

Proof. This is equivalent to

$$dg' - 1 = b(\tfrac{1}{2}ag')$$

which in turn is equivalent to

(7.30) $$d'ag' = b(\tfrac{1}{2}ag') .$$

Since $d'(2n) = b(n)$, Eq. (7.30) follows at once.

Theorem 7.9

(7.31) $$\begin{cases} d'(2n) = 2d'(n) + 1 & (n \in (g)) \\ d'(2n) = 2d'(n) & (n \in (g')) \end{cases}$$

Theorem 7.10

(7.32) $$ad'(n) = 2d'(n) - n .$$

We show first that Theorems 7.9 and 7.10 are equivalent. Since $d'(2n) = b(n)$, (7.31) may be replaced by

(7.33) $$\begin{cases} bg = 2d'g + 1 \\ bg' = 2d'g' \end{cases}$$

while (7.3) may be replaced by

(7.34) $$\begin{cases} ad'g = 2d'g - g \\ ad'g' = ad'g' - g' \end{cases} .$$

Since, by (7.5),

$$ad'g = d'ag, \qquad ad'g' - d'ag' = 1 ,$$

(7.34) is the same as

(7.35) $$\begin{cases} d'ag = 2d'g - g \\ d'ag' = 2d'g' - g' - 1 \end{cases} .$$

But $d'a = d - 1$, so that (7.35) becomes

(7.36) $$\begin{cases} dg - 1 = 2d'g - g \\ dg' = 2d'g' - g' \end{cases}$$

which is the same as (7.33). This proves the equivalence of (7.31) and (7.32).

We shall now prove (7.32). We have first

$$ad'a = a(d - 1) = b - 2$$

$$2d'a - a = 2(d - 1) - a = b - 2 ,$$

so that

(7.37) $$ad'a = 2d'a - a .$$

Secondly

$$ad'b = a\delta = b + 2d$$

$$2d'b - b = 2\delta - b = b + 2d ,$$

so that

(7.38) $$ad'b = 2d'b - b .$$

Clearly (7.37) and (7.38) imply (7.32).

Theorem 7.11. We have

$$c'(n) + n - 1 = d'(2n - 1) , \tag{7.39}$$

where $c'(n)$ and $c(n)$ are complementary.

Proof. Put

$$\begin{aligned}\bar{c}(n) &= d'(2n - 1) - (n - 1)\\ &= [\tfrac{1}{2}(2 + \sqrt{2})(2n - 1)] - (n - 1)\\ &= \left[n + \frac{1}{\sqrt{2}}(2n - 1)\right] = \left[(1 + \sqrt{2})n - \frac{1}{\sqrt{2}}\right] .\end{aligned}$$

It follows from (7.15) that

$$\bar{c}(n) = \begin{cases} d(n) & (n \in (f')) \\ d(n) - 1 & (n \in (f)) \end{cases} . \tag{7.40}$$

In order to prove that $\bar{c}(n) = c'(n)$, it will suffice to show that c and $\bar{c}$ are complementary. Now, by (7.31),

$$c(n) = \begin{cases} d'(n) + 1 & (n \in (g)) \\ d'(n) & (n \in (g')) \end{cases} .$$

Thus

$$(c) = (d'g + 1) \cup (d'g')$$

$$(c) = (df') \cup (df - 1) .$$

Since

$$d'g + 1 = d'af + 1 = df$$

$$df - 1 = d'af = d'g ,$$

it follows that

$$(c) = (df) \cup (d'g')$$

$$(c) = (dg') \cup (d'g) .$$

Therefore

$$\begin{aligned}(c) \cup (\bar{c}) &= (df) \cup (df') \cup (d'g) \cup (d'g')\\ &= (d) \cup (d') = \underset{\sim}{N}\end{aligned}$$

while $(c) \cap (\bar{c})$ is vacuous. This completes the proof of the Theorem.

Theorem 7.12. We have

$$ac'(n) = c'(n) + n - 1 . \tag{7.41}$$

In view of (7.39), (7.41) is the same as

$$ac'(n) = d'(2n - 1) . \tag{7.42}$$

Proof of (7.41). By (7.40),

$$c'(n) = \begin{cases} d(n) & (n \in (f')) \\ d(n) - 1 & (n \in (f)) \end{cases},$$

so that

$$\begin{cases} c'f' = df' \\ c'f = df - 1 \end{cases}.$$

Thus

$$\begin{cases} ac'f' = adf' = bf' - 1 \\ ac'f = a(df - 1) = bf - 2 \end{cases}.$$

It follows that

$$\begin{cases} ac'f' - c'f' = bf' - 1 - df' = f' - 1 \\ ac'f - c'f = bf - 2 - (df - 1) = f - 1 \end{cases}$$

and therefore

$$ac'(n) - c'(n) = n - 1 .$$

Theorem 7.13. We have

(7.43) $$a^2c'(n) = 2c'(n) - 1 .$$

Proof. By (7.32),

$$ad'(2n - 1) = 2d'(2n - 1) - (2n - 1) .$$

Then by (7.42),

$$a^2c'(n) = ad'(2n - 1) = 2ac'(n) - (2n - 1) .$$

Combining this with (7.41), we get

$$\begin{aligned} a^2c'(n) &= 2(c'(n) + n - 1) - (2n - 1) \\ &= 2c'(n) - 1 . \end{aligned}$$

Theorem 7.14. There exists a strictly monotone function θ such that

(7.44) $$c' = f\theta .$$

Proof. This result is implied by

(7.45) $$f' = cg .$$

To prove (7.45) we take

$$f' = df = acf .$$

Since

$$ac - ca = ab - ba - 1 = s - 1 ,$$

it follows that

$$acf - caf = 0 .$$

Hence

$$f' = caf = cg.$$

Theorem 7.15. There exists a strictly monotone function ψ such that

(7.46) $$f\psi = d' .$$

Proof. This is an immediate consequence of $f' = df$.

Theorem 7.16. There exists a strictly monotone function h such that

$$fh = b \quad (7.47)$$

Proof. Since $f' = df = acf$, it follows that $(f') \subset (a)$ and therefore $(b) \subset (f)$.

Theorem 7.17. We have

$$\psi(2n) = h(n) \quad (7.48)$$

Proof. By (7.46),

$$f\psi(2n) = d'(2n) = b(n)$$

and (7.48) follows at once.

Theorem 7.18. We have

$$c = \epsilon a + 1 \quad (7.49)$$

Proof. We recall that

$$\epsilon(2n) = \epsilon(2n - 1) + 1 = d(n) .$$

Also

$$\begin{cases} a(n) \equiv 1 \pmod 2 \rightleftarrows n \in (g) \\ a(n) \equiv 0 \pmod 2 \rightleftarrows n \in (g') \end{cases}$$

1. Let $n = g(k)$. Then

$$\begin{aligned} \epsilon a(n) + 1 &= d\left(\tfrac{1}{2}(a(n) + 1)\right) = d\left(\tfrac{1}{2}(ag(k) + 1)\right) \\ &= d\left(\tfrac{1}{2}(a^2f(k) + 1)\right) = df(k) , \end{aligned}$$

so that

$$\epsilon ag + 1 = df \quad (7.50)$$

2. Let $n \in (g')$ and put

$$a(n) = [\sqrt{2}\, n] = 2k, \qquad k = \left[\frac{n}{\sqrt{2}}\right] .$$

By (7.28)

$$\left\{\frac{n}{\sqrt{2}}\right\} < \frac{1}{2} .$$

We have

$$\begin{aligned} \epsilon a(n) + 1 &= d\left(\tfrac{1}{2}a(n)\right) + 1 = d(k) + 1 \\ &= k + [\sqrt{2}\, k] + 1 \\ &= \left[\frac{n}{\sqrt{2}}\right] + \sqrt{2}\left(\frac{n}{\sqrt{2}} - \left\{\frac{n}{\sqrt{2}}\right\}\right) - \left\{n - \sqrt{2}\left\{\frac{n}{\sqrt{2}}\right\}\right\} + 1 \\ &= n + \left[\frac{n}{\sqrt{2}}\right] = \sqrt{2}\left\{\frac{n}{\sqrt{2}}\right\} - \left(1 - \sqrt{2}\left\{\frac{n}{2}\right\}\right) + 1 \\ &= n + \left[\frac{n}{\sqrt{2}}\right] . \end{aligned}$$

On the other hand

$$d'(n) = [\tfrac{1}{2}(2 + \sqrt{2})n] = n + \left[\frac{n}{\sqrt{2}}\right] ,$$

so that

$$\epsilon a g' + 1 = d'g' . \tag{7.51}$$

Combining (7.50) and (7.51) we get

$$(\epsilon a + 1) = (df) \cup (d'g') = (c) ;$$

the last equality appeared in the proof of Theorem 7.11.

Theorem 7.19. We have

$$e\big(c'(n) + 1\big) = n . \tag{7.52}$$

Proof. By (7.40)

$$\begin{cases} c'f(n) = df(n) - 1 \\ c'f'(n) = df'(n) \end{cases} ,$$

so that

$$\begin{cases} c'f(n) + 1 = df(n) \\ c'f'(n) + 1 = df'(n) + 1 \end{cases} .$$

Since

$$df' + 1 = d^2f + 1 = \delta f ,$$

it follows that

$$\begin{cases} c'f(n) + 1 = df(n) \\ c'f'(n) + 1 = \delta f(n) \end{cases} .$$

Therefore

$$\begin{cases} e\big(c'f(n) + 1\big) = f(n) \\ e\big(c'f'(n) + 1\big) = df(n) = f'(n) \end{cases} .$$

This evidently proves (7.52).

Remark. $c'(n) + 1 \neq d(n)$.

Theorem 7.20. We have

$$\begin{cases} c'f = d'g = d'af \\ c'f' = df' \end{cases} . \tag{7.53}$$

Proof. We have

$$c'(n) = \left[(1 + \sqrt{2})n - \frac{1}{\sqrt{2}}\right] \tag{7.54}$$

and

$$\{\sqrt{2}\, f\} < \frac{1}{2}, \qquad \{\sqrt{2}\, f'\} > \frac{1}{\sqrt{2}} .$$

Hence

$$\begin{cases} c'f = df - 1 \\ c'f = df' \end{cases} .$$

Since

$$d'g = d'af = df - 1,$$

(7.53) follows at once.

Theorem 7.21. We have

(7.55) $$c'(n) \leq d(n) \leq c'(n) + 1 \leq p(n)$$

and

(7.56) $$e(k) = n \quad \text{if and only if} \quad k \in [d(n),\ p(n) + 1].$$

The interval $[d(n), p(n)+1]$ contains exactly three integers if $n \in (d)$ and contains exactly two integers if $n \in (d')$.

Proof. Inequalities (7.55) come from

$$d(n) = [(1 + \sqrt{2})n]$$

together with (4.4) and (7.54). To prove (7.56) we use

$$e(d(n)) = e(p(n) + 1) = n$$

and

$$p(n) + 2 = d(n + 1).$$

The final statement in the theorem follows from

$$d(n + 1) - d(n) = 3 \quad \text{if and only if} \quad n \in (d).$$

8. THEOREMS INVOLVING σ AND τ

Let

(8.1) $$n = f_1P_1 + f_2P_2 + f_3P_3 + \cdots$$

be the first canonical representation of n. Define $\sigma(n)$ by means of

(8.2) $$\sigma(n) \equiv f_1 + f_2 + f_3 + \cdots \pmod 2.$$

If

$$f_1 = \cdots = f_{k-1} = 0, \qquad f_k \neq 0,$$

put

(8.3) $$\tau(n) \equiv k \pmod 2.$$

We may assume that $\sigma(n)$, $\tau(n)$ take on the values 0, 1.

It follows from (8.1) that

$$p(n) = \cdot 0\, f_1f_2f_3 \cdots.$$

Since

$$P_k \equiv k \pmod 2$$

it follows that

(8.4) $$n + p(n) \equiv \sigma(n) \pmod 2.$$

Since

$$b(n+1) = n + p(n) + 3$$

we get

(8.5) $$a(n+1) \equiv b(n+1) \equiv \sigma(n) + 1 \pmod 2 .$$

In the next place, by Theorem 5.4,

(8.6) $$(d) = \{n \mid \tau(n) = 0\}$$

so that

(8.7) $$(d') = \{n \mid \tau(n) = 1\}$$

Since $(b) \subset (d')$ it follows that

(8.8) $$\tau(b(n)) = 1 \qquad (n = 1, 2, 3, \cdots) .$$

By (8.5)

(8.9) $$\sigma(b(n)) \equiv a(b(n) + 1) \equiv ab(n) \equiv 0 \pmod 2 .$$

On the other hand, for n such that $a(n) \in (d')$,

$$\sigma(a(n)) + 1 \equiv a(a(n) + 1) \equiv a^2(n) + 1 .$$

Since $(d') \subset (f)$,

$$a^2(n) = 2n - 1 \equiv 1 \pmod 2$$

and therefore

(8.10) $$\sigma(a(n)) = 1 \qquad (a(n) \in (d')) .$$

Combining (8.8), (8.9) and (8.10), we get the following.

Theorem 8.1. The set (b) is characterized by

(8.11) $$(b) = \{n \mid \sigma(n) = 0, \quad \tau(n) = 1\} .$$

Put

(8.12) $$A_{i,j} = \{n \mid \tau(n) = i, \quad \sigma(n) = j\} \qquad (i, j = 0, 1)$$

Thus by (8.11)

(8.13) $$(b) = A_{1,0}, \qquad (a) = A_{0,0} \cup A_{0,1} \cup A_{1,1} .$$

Theorem 8.2. We have

(8.14) $$A_{0,0} = (ad'g')$$

(8.15) $$A_{0,1} = (af') = (adf)$$

(8.16) $$A_{1,1} = (ac') = (adf') \cup (ad'g) .$$

Proof.

1. Let $n \in (a) \cap (d')$. By (8.10), $\sigma(n) = 1$; also by (8.7), $\tau(n) = 1$. Therefore

(8.17) $$(a) \cap (d') \subset A_{1,1} .$$

2. Next let $n \in (d)$, so that $\tau(n) = 0$. Since $d = ac$ and $(c) = (df) \cup (d'g')$, we have

$$(d) = (adf) \cup (ad'g') . \tag{8.18}$$

Since $n \in (d)$,

$$\sigma(n) \equiv a(n+1) + 1 \equiv a(n) + 1 .$$

Let $n = a(k)$, $k \in (df)$. Then

$$\sigma\bigl(a(k)\bigr) \equiv a^2(k) + 1 \equiv 1 .$$

Hence

$$(adf) \subset A_{0,1} . \tag{8.19}$$

Now let $n = a(k)$, $k \in (d'g')$. Then

$$\sigma\bigl(a(k)\bigr) \equiv a^2(k) + 1 \equiv 0 ,$$

so that

$$(ad'f') \subset A_{0,0} . \tag{8.20}$$

Since

$$\begin{aligned}(a) &= \bigl((a) \cap (d')\bigr) \cup (ac) \\ &= \bigl((a) \cap (d')\bigr) \cup (adf) \cup (ad'g') ,\end{aligned}$$

it follows that the inclusion sign $\subset$ in (8.17), (8.19) and (8.20) may be replaced by equality. This completes the proof of the theorem.

Theorem 8.3. We have

$$\begin{cases} \sigma(n) = \tau(n) & \bigl(n \in (g)\bigr) \\ \sigma(n) + \tau(n) = 1 & \bigl(n \in (g')\bigr) \end{cases} . \tag{8.21}$$

Proof. Since $g = af$, $(g) \subset (a)$ but $(g) \not\subset (af')$. Consequently, by the last theorem,

$$\begin{cases} (g) = A_{0,0} \cup A_{1,1} \\ (g') = A_{0,1} \cup A_{1,0} \end{cases} \tag{8.22}$$

and (8.21) follows at once.

Theorem 8.4. We have

$$\sigma(n-1) = 0 \rightleftarrows n \in (g) . \tag{8.23}$$

Proof. By (7.6),

$$a(n) \equiv 1 \pmod 2 \rightleftarrows n \in (g) .$$

Since

$$\sigma(n-1) \equiv a(n) + 1 \pmod 2 ,$$

(8.23) follows at once.

Theorem 8.5. We have

$$\begin{cases} (dg) = \{n \mid n \in (d),\ \ \sigma(n) = 1\} \\ (dg') = \{n \mid n \in (d),\ \ \sigma(n) = 0\} \end{cases} . \tag{8.24}$$

Proof. Since $(d) \subset (a)$ and

$$\tau(n) = 0 \qquad (n \in (d)),$$

it follows from Theorem 8.3 that

$$(d) = A_{0,0} \cup A_{0,1} = (ad'g') \cup (af') .$$

Thus

$$(dg) \cup (dg') = (ad'g') \cup (af') . \tag{8.25}$$

Now assume that

$$n \in (af'), \qquad n \in (dg') = (acg') .$$

It follows that there exists an integer k such that

$$k \in (f'), \qquad k \in (cg') .$$

But

$$f' = df = acf = caf = cg ,$$

so that

$$k \in (cg), \qquad k \in (cg') ,$$

which is impossible.

Next assume that

$$n \in (dg), \qquad n \in (ad'g') .$$

Then there is a k such that

$$k \in (cg), \qquad k \in (d'g') .$$

But

$$cg = caf = acf = df ,$$

so that

$$k \in (dg), \qquad k \in (d'g') ,$$

which is impossible. It therefore follows from (8.25) that

$$(dg) = (af'), \qquad (dg') = (ad'g') .$$

This completes the proof of the theorem.

Theorem 8.6. We have

$$\begin{cases} (\delta g) = \{n \mid n \in (\delta), \quad \sigma(n) = 1\} \\ (\delta g') = \{n \mid n \in (\delta), \quad \sigma(n) = 0\} \end{cases} . \tag{8.26}$$

Proof. Since

$$(\delta) = \bigcup_{1}^{\infty} A_{2k+1}$$

and $e\delta = d$, Theorem 8.6 is an immediate corollary of Theorem 8.5.

SUMMARY OF FORMULAS

1. $p(n) = 2n + e(n)$
2. $e(p(n)) = e(p(n) + 1) = n$
3. $e(p(n) + 2) = n + 1$
4. $a(n + 1) = e(n) + n + 1$
5. $b(n + 1) = p(n) + n + 3$
6. $d(n + 1) = p(n) + 2$
7. $ad(n) = b(n) - 1, \quad a(d(n) + 1) = b(n) + 1, \quad a(d(n) - 1) = b(n) - 2$
8. $ed(n) = n$
9. $eb(n) = a(n)$
10. $d^2(n) = \delta(n) - 1$
11. $e\delta(n) = d(n)$
12. $e^2\delta(n) = n$
13. $e(\delta(n) - 1) = d(n)$
14. $e^2(\delta(n) - 1) = n$
15. $ab(n) = a(n) + b(n) = 2d(n)$
16. $db(n) = bd(n) + 1$
17. $ad - da + 1 = ab - ba$
18. $a\delta(n) = d(n) + \delta(n)$
19. $a(n) = e(b(n) - 1) = ead(n)$
20. $ebd(n) = b(n) - 1$
21. $d'a(n) = d(n) - 1$
22. $d'(a(n) + 1) = d(n) + 1$
23. $\epsilon(2d(n)) = \delta(n) - 1$
24. $\epsilon(2d(n) + 1) = \delta(n) + 1$
25. $e(d(n) - 1) = n - 1$
26. $e(a^2(n) + a(n)) = a(n)$
27. $e(b(n) - 1) = a(n)$
28. $a(d(n) - 1) = b(n) - 2$
29. $\epsilon(2n) = d(n), \quad \epsilon(2n - 1) = d(n) - 1$
30. $e(\epsilon(n)) = [n/2]$

31. $e(n) - e(n-1) = 1 \rightleftarrows n \in (d)$

32. $a(n+1) = a(n) + 2 \rightleftarrows n \in (d)$

33. $d'(n+1) = d'(n) + 2 \rightleftarrows n \in (a)$

34. $d(n) = ac(n), \quad c(n) = b(n) - d'(n)$

35. $A_1 = d(\underset{\sim}{N}) - 1$

36. $A_{2k} = d\delta^{k-1}\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

37. $A_{2k+1} = \delta^k\epsilon(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

38. $B_{2k} = d\delta^{k-1}d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

39. $B_{2k+1} = \delta^k d(\underset{\sim}{N}) \qquad (k = 1, 2, 3, \cdots)$

40. $d(\underset{\sim}{N}) = \bigcup_1^\infty A_{2k}$

41. $\delta(\underset{\sim}{N}) = \bigcup_1^\infty A_{2k+1}$

42. $\epsilon(\underset{\sim}{N}) = d(\underset{\sim}{N}) \cup \left(d(\underset{\sim}{N}) - 1\right)$

43. $a^2b = 2b - 1$

44. $n \in (d) \rightleftarrows \left\{\frac{n}{1+\sqrt{2}}\right\} > 2 - \sqrt{2}$

45. $b^k = u_k a + v_k b,$

where

$$u_k = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}, \quad v_k = u_{k+1} - u_k, \quad \alpha = 2 + \sqrt{2}, \quad \beta = 2 - \sqrt{2}.$$

46. $ab^k = u_k n + v_k b - t_k,$

where

$$u_k = 2^{k-1}Q_{k-1}, \quad v_k = 2^{k-1}Q_k, \quad t_k = \frac{1}{7}(2^{k+1}P_{k+1} - 3\cdot 2^k P_k - 2),$$

and

$$Q_k = P_k + P_{k-1}.$$

47. $s = ab - ba$

48. $af(n) = 1, \quad af'(n) = 2$

49. $a^2f(n) \equiv 1, \quad a^2f'(n) \equiv 0 \pmod 2$

50. $t = ad' - d'a$

51. $tg(n) = 0, \quad tg'(n) = 1$

52. $ag(n) \equiv 1, \quad ag'(n) \equiv 0 \pmod 2$

53. $g = af$

54. $df = f'$

55. $df - dg = 1$

56. $n \in (f) \rightleftarrows \{\sqrt{2}\, n\} < \frac{1}{\sqrt{2}}$

57. $bf - af' = 1$

58. $f' = acf = caf$

59. $n \in (g) \rightleftarrows \left\{\frac{n}{\sqrt{2}}\right\} < \frac{1}{2}$

60. $g' = a(\tfrac{1}{2}ag') + 1$

61. $\begin{cases} d'(2n) = 2d'(n) + 1 & \bigl(n \in (g)\bigr) \\ d'(2n) = 2d'(n) & \bigl(n \in (g')\bigr) \end{cases}$

62. $ad'(n) = 2d'(n) - n$

63. $ac'(n) = c'(n) + n - 1 = d'(2n - 1)$

64. $c'(n) = \begin{cases} d(n) & \bigl(n \in (f')\bigr) \\ d(n) + 1 & \bigl(n \in (f)\bigr) \end{cases}$

65. $\begin{cases} (c) = (df) \cup (d'g') \\ (c') = (df') \cup (d'g') \end{cases}$

66. $a^2c'(n) = 2c'(n) - 1$

67. $c' = f\theta$

68. $d' = f\psi$

69. $fh = b$

70. $\psi(2n) = h(n)$

71. $c = \epsilon a + 1$

72. $e\bigl(c'(n) + 1\bigr) = n$

73. $\begin{cases} c'f = d'g = d'af \\ c'f = df' \end{cases}$

74. $(b) = \{n \mid \sigma(n) = 0, \quad \tau(n) = 1\}$

75. $A_{i,j} = \{n \mid \tau(n) = i, \quad \sigma(n) = j\} \quad (i,j = 0,1)$

76. $A_{0,0} = (ad'g')$

77. $A_{0,1} = (af') = (adf)$

78. $A_{1,1} = (ac') = (adf') \cup (ad'g)$

79. $$\begin{cases} \sigma(n) = \tau(n) & (n \in (g)) \\ \sigma(n) = \tau(n) = 1 & (n \in (g')) \end{cases}$$

80. $\sigma(n - 1) = 0 \rightleftarrows n \in (g)$

81. $$\begin{cases} (dg) = \{n \mid n \in (d), \quad \sigma(n) = 1\} \\ (dg') = \{n \mid n \in (d), \quad \sigma(n) = 0\} \end{cases}$$

82. $$\begin{cases} (\delta g) = \{n \mid n \in (\delta), \quad \sigma(n) = 1\} \\ (\delta g') = \{n \mid n \in (\delta), \quad \sigma(n) = 0\} \end{cases}$$

Table 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	1	2	4	5	7	8	9	11	12	14	15	16	18	19	21	22	24	25	26	28
b	3	6	10	13	17	20	23	27	30	34	37	40	44	47	51	54	58	61	64	68
d	2	4	7	9	12	14	16	19	21	24	26	28	31	33	36	38	41	43	45	48
d'	1	3	5	6	8	10	11	13	15	17	18	20	22	23	25	27	29	30	32	34
e	0	1	1	2	2	2	3	3	4	4	4	5	5	6	6	7	7	7	8	8
p	2	5	7	10	12	14	17	19	22	24	26	29	31	34	36	39	41	43	46	48
	1	2	3	4	6	7	8	9	11	12	13	14	15	16	18	19	20	21	23	24
	5	10	17	22	29	34	39	46	51	58	63	68	75	80	87	92	99	104	109	116

Table 2

n	1	2	3	4	5	6	7	8	9	10	11	12
a	1	2	4	5	7	8	9	11	12	14	15	16
ab	4	8	14	18	24	28	32	38	42	48	52	56
ba	3	6	13	17	23	27	30	37	40	47	51	54
s	1	2	1	1	1	1	2	1	2	1	1	2
f	1	3	4	5	6	8	10	11	13	15	16	17
f'	2	7	9	12	14	19	24	26	31	36	38	41

Table 3

n	1	2	3	4	5	6	7	8	9	10	11	12
a	1	2	4	5	7	8	9	11	12	14	15	16
d'	1	3	5	6	8	10	11	13	15	17	18	20
ad'	1	4	7	8	11	14	15	18	21	24	25	28
d'a	1	3	6	8	11	13	15	18	20	23	25	27
t	0	1	1	0	0	1	0	0	1	1	0	1
g	1	4	5	7	8	11	14	15	18	21	22	24
g'	2	3	6	9	10	12	13	16	17	19	20	23

Table 4

n	1	2	3	4	5	6	7	8	9	10	11	12
c'	1	4	6	8	11	13	16	18	21	23	25	28
c	2	3	5	7	9	10	12	14	15	17	19	20
θ	1	3	5	6	8	9	11	13	15	17	18	20
d'	1	3	5	6	8	10	11	13	15	17	18	20
ψ	1	2	4	5	6	7	8	9	10	12	13	14
ϵ	1	2	3	4	6	7	8	9	11	12	13	14
h	2	5	7	9	12	14	17	19	22	25	27	29
c'+1	2	5	7	9	12	14	17	19	22	24	26	28

REFERENCES

1. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Fibonacci Representations," Fibonacci Quarterly, Vol. 10, No. 1 (1972), pp. 1-28.
2. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Lucas Representations," Fibonacci Quarterly, Vol. 10, No. 1 (1972), pp. 29-42.
3. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Fibonacci Representations of Higher Order," Fibonacci Quarterly, Vol. 10, No. 1 (1972), pp. 43-69.
4. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Fibonacci Representations of Higher Order — II," Fibonacci Quarterly, Vol. 10, No. 1 (1972), pp. 71-80.
5. Problem 5254, American Math. Monthly, Vol. 71 (1964), p. 1138; Solution, Vol. 72 (1965), pp. 1144-1145.
6. Theodore Skolem, "On Certain Distributions of Integers in Pairs with Given Differences," Mathematica Scandinavica, Vol. 5 (1957), pp. 55-68.
7. T. J. Keller, "Generalizations of Zeckendorf's Theorem," Fibonacci Quarterly, Vol. 10 (1972), pp. 95-102.