AN INTERESTING SEQUENCE OF NUMBERS DERIVED FROM VARIOUS GENERATING FUNCTIONS

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The following development, to the best of the author's knowledge, is new. At any rate, it is original and very interesting. We begin by defining the function

(1) $$f(x) = 1/(1 - x)\sqrt{1 + x} \quad .$$

This may be thought of as the generating function of a power series in x, whose coefficients we are to determine. Thus, we seek the values of the coefficients A_k, where

(2) $$f(x) = \sum_{k=0}^{\infty} A_k x^k \quad .$$

That this representation is valid may be seen by observing that $f(x)$ is expressible as the product of the two functions $(1 - x)^{-1}$ and $(1 + x)^{-\frac{1}{2}}$, each of which is of the same form as (2). In fact,

(3) $$(1 - x)^{-1} = \sum_{k=0}^{\infty} x^k , \qquad \text{and} \qquad (1 + x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(-\frac{1}{4}\right)^k x^k \quad .$$

Therefore, it follows that

(4) $$A_k = \sum_{i=0}^{k} \binom{2i}{i} \left(-\frac{1}{4}\right)^i \quad .$$

From the foregoing expression for A_k, it is evident that

(5) $$A_k = A_{k-1} + \binom{2k}{k}\left(-\frac{1}{4}\right)^k , \quad A_0 = 1 .$$

Recursion (5) may be expressed in the form

(6) $$A_k = A_{k-1} - \frac{2k-1}{2k} \cdot \binom{2k-2}{k-1}\left(-\frac{1}{4}\right)^{k-1} .$$

If, in recursion (6), we multiply throughout by $(2k)/2k - 1$, and if, in recursion (5), we replace the subscript k by $k - 1$, we may add the two results, thereby eliminating the factorial term. Upon simplification, this process yields the following recursion, which involves three successive values of A_k:

(7) $$2kA_k = A_{k-1} + (2k - 1)A_{k-2} .$$

This is valid for $k = 2, 3, 4, \cdots$, and if we affix the values $A_0 = 1$ and $A_1 = \frac{1}{2}$, we have fully characterized the coefficients A_k.

We shall now define the sequence of numbers B_k, such that for each non-negative integer k,

(8) $$B_k = 2^k \cdot k! \cdot A_k .$$

Substituting this definition in recursion (7),

$$\frac{2k \cdot B_k}{2^k \cdot k!} = \frac{B_{k-1}}{2^{k-1}(k-1)!} + \frac{(2k-1)B_{k-2}}{2^{k-2}(k-2)!} .$$

If we multiply this result throughout by $2^{k-1} \cdot (k - 1)!$, we obtain:

(9) $$B_k = B_{k-1} + (2k - 1)(2k - 2)B_{k-2} .$$

Recursion (9), plus the initial conditions $B_0 = B_1 = 1$, completely characterize the coefficients B_k. Furthermore, from (9), it is evident that all the B_k's are integers. Upon application of (9), for the first few values of k, we obtain the following values:

$$B_0 \equiv B_1 = 1, \quad B_2 = 7, \quad B_3 = 27,$$

$$B_4 = 321, \quad B_5 = 2{,}265, \quad B_6 = 37{,}575, \quad B_7 = 390{,}915,$$

etc. We may summarize the results thus far derived in the following form:

$$f(2x) = 1/(1 - 2x)\sqrt{1 + 2x} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}, \tag{10}$$

where

$$B_k = 2^k \cdot k! \sum_{i=0}^{k} \binom{2i}{i} \left(-\frac{1}{4}\right)^i .$$

What struck the author as interesting was the fact that the sequence of numbers B_k appears in other power series, derived from generating functions of totally different form from (10).

Specifically, we will demonstrate that

$$g(x) \equiv e^{x^2/2} \int_0^x e^{-u^2}\, du = \sum_{k=0}^{\infty} B_k \frac{x^{2k+1}}{(2k+1)!}, \tag{11}$$

and

$$h(x) = \tan^{-1} x \Big/ \sqrt{1 - x^2} = \sum_{k=0}^{\infty} (B_k)^2 \frac{x^{2k+1}}{(2k+1)!} . \tag{12}$$

Let $y = g(x)$. If we differentiate y, as defined in (11),

$$y' = e^{x^2/2} \cdot e^{-x^2} + x e^{x^2/2} \int_0^x e^{-u^2}\, du = e^{-x^2/2} + xy .$$

Differentiating again, we obtain

$$y'' = -x e^{-x^2/2} + xy' + y = -x e^{-x^2/2} + x e^{-x^2/2} + x^2 y + y = (1 + x^2)y.$$

Next, we observe that $g(x)$ is an odd function of x. This is demonstrated by replacing x with $-x$ and the dummy variable u with $-u$ in (11), which yields $g(-x) = -g(x)$.

Therefore, $g(x)$ may be expressed in the form

$$\sum_{k=0}^{\infty} r_k x^{2k+1} .$$

Negative powers of x are excluded, for otherwise $g(x)$ would be discontinuous at $x = 0$, along with the first and higher order derivatives. However, it is readily seen that $g(0) = 0$, $g'(0) = 1$, and $g'(0) = 0$.

We will use these conditions to develop a recursion involving the coefficients r_k. If we differentiate the series expression for $g(x)$,

$$(13) \qquad g'(x) = \sum_{k=0}^{\infty} (2k+1) r_k x^{2k} ; \qquad g''(x) = \sum_{k=1}^{\infty} 2k(2k+1) r_k x^{2k-1} .$$

We use the differential equation $y'' = (1 + x^2)y$ derived above, which becomes transformed to the following relationship:

$$(14) \qquad \sum_{k=0}^{\infty} (2k+2)(2k+3) r_{k+1} x^{2k+1} = \sum_{k=0}^{\infty} r_k x^{2k+1} + \sum_{k=1}^{\infty} r_{k-1} x^{2k+1} .$$

If we equate the coefficients of similar powers of x, we obtain:

$$(15) \qquad r_0 = 6 r_1; \qquad (2k+2)(2k+3) r_{k+1} = r_k + r_{k-1}, \text{ if } k = 1, 2, 3, \cdots .$$

Using the condition $g'(0) = 1$, we see that $r_0 = 1$, and therefore,

$$r_1 = \frac{1}{6} .$$

We now define the sequence of numbers R_k such that, for every non-negative integer k, $R_k = (2k+1)!\,r_k$. Substituting this definition in recursion (15), and multiplying throughout by $(2k+1)!$, we obtain:

$$R_{k+1} = R_k + 2k(2k+1)R_{k-1}\ ; \qquad \text{also,} \qquad R_0 = R_1 = 1\ . \tag{16}$$

But if we replace k by $k-1$ in (16), we obtain precisely the same recursion as (9). Since the initial values of R_k are identical to those of B_k, we conclude that $R_k = B_k$ for all values of k, and the validity of (11) is established.

The proof of (12) is similar, though somewhat more complicated. We begin by squaring both sides of (9), and solving for $B_{k-1}B_{k-2}$:

$$B_{k-1}B_{k-2} = \frac{B_k^2 - B_{k-1}^2 - (2k-1)^2(2k-2)^2B_{k-2}^2}{2(2k-1)(2k-2)}\ . \tag{17}$$

Next, we may multiply (9) throughout by B_{k-1}, obtaining

$$B_kB_{k-1} = B_{k-1}^2 + (2k-1)(2k-2)B_{k-1}B_{k-2}\ . \tag{18}$$

If, in (18), we substitute the expression derived in (17) for $B_{k-1}B_{k-2}$, and the corresponding expression for B_kB_{k-1} obtained by increasing the subscript from $k-1$ to k, we arrive at a recursion which involves only the squares of successive B_k's. Upon simplification, this becomes

$$\begin{aligned} B_{k+1}^2 = {} & (4k^2+2k+1)(B_k^2 + 2k(2k+1)B_{k-1}^2) \\ & - (2k-2)^2(2k-1)^2 2k(2k+1)B_{k-2}^2\ . \end{aligned} \tag{19}$$

Next, we observe that $h(x)$ is an odd function of x, continuous at $x = 0$. Therefore, as before, $h(x)$ may be expressed in the form

$$\sum_{k=0}^{\infty} s_k x^{2k+1}$$

As before, we will develop a recursion involving the s_k's. If we let $z = h(x)$, as defined in (12), we differentiate as follows:

$$z' = \frac{(1 - x^2)^{\frac{1}{2}} \cdot (1 + x^2)^{-1} + x \tan^{-1}x \cdot (1 - x^2)^{-\frac{1}{2}}}{1 - x^2} = \frac{(1 - x^2)^{-\frac{1}{2}}}{1 + x^2} + \frac{xz}{1 - x^2} .$$

Differentiating again,

$$z'' = \frac{x(1 + x^2)(1 - x^2)^{-3/2} - 2x(1 - x^2)^{-\frac{1}{2}}}{(1 + x^2)^2} + \frac{(1 - x^2)(xz' + z) + 2x^2 z}{(1 - x^2)^2} .$$

From the first differentiation,

$$(1 - x^2)^{-\frac{1}{2}} = (1 + x^2)\left(z' - \frac{xz}{1 - x^2}\right) .$$

Substituting this result in the second differentiation, we eliminate all irrational functions of x, and upon simplifying the result:

$$(20) \qquad (1 + x^2)(1 - x^2)^2 z'' + 4x^3(x^2 - 1) z' + (2x^4 - 3x^2 - 1) z = 0 .$$

In the series expression for $h(x)$, there will be no loss in generality if we make the substitution $s_k = S_k + (2k + 1)!$. Then

$$z = \sum_{k=0}^{\infty} S_k \frac{x^{2k+1}}{(2k + 1)!}, \qquad z' = \sum_{k=0}^{\infty} S_k \frac{x^{2k}}{(2k)!}, \qquad z'' = \sum_{k=0}^{\infty} S_{k+1} \frac{x^{2k+1}}{(2k + 1)!} .$$

Each term in differential equation (20) may be expressed in series form by means of the latter expressions. Using the method of equating coefficients (the development is omitted here, in the interest of brevity), we arrive at the following recursion:

$$(21) \qquad S_{k+1} = (4k^2 + 2k + 1)S_k + 2k(2k + 1)(4k^2 + 2k + 1)S_{k-1} - 2k(2k + 1)(2k - 1)^2(2k - 2)^2 S_{k-2}$$

valid for $k = 0, 1, 2, 3, \cdots$. But this recursion is of the same form as (19), and becomes identical to it if $S_k = B_k^2$ for all non-negative values of k. It remains to show that such is the case for the initial values, where $k = 0$ and 1. We observe that $h(0) = 0$, and from the first-order differential equation, $h'(0) = 1$. But we see from the series expression for z' that $h'(0) = S_0 = 1$. From (21), we readily obtain the values $S_1 = 1$, $S_2 = 49$, $S_3 = 729$, etc. This establishes the truth of (12).

We have overlooked the question of convergence in the manipulation of the foregoing infinite series. A more rigorous treatment would only have served to detract interest from the remarkable properties of these series which link them together. It may be demonstrated, however, that $f(x)$ and $h(x)$ are convergent within the interval $(-1,1)$, excluding the end points; $g(x)$ converges for all real values of x.

The purpose of this paper was to demonstrate the validity of (10), (11) and (12). Now that this has been accomplished, it would be desirable to deduce some properties for the coefficients B_k. The remaining portion is devoted to the derivation of several such properties and relationships.

We begin by noting that $g(x)$ and $h(x)$ are expressible as the products of two functions, as is the case with $f(x)$. By application of Maclaurin's formula,

$$e^{x^2/2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}; \qquad \int_0^x e^{-u^2}\,du = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!} .$$

Multiplying these two series term-by-term, we obtain:

$$g(x) = \sum_{k=0}^{\infty} c_k x^{2k+1} ,$$

where

$$c_k = \sum_{i=0}^{k} \frac{(-1)^i}{2^{k-i}(k-i)!\,i!\,(2i+1)} .$$

But, as we have already shown, $c_k = B_k + (2k+1)!$. Therefore, we are led to an alternate expression for B_k:

(22) $$B_k = \frac{(2k+1)!}{2^k \cdot k!} \sum_{k=0}^{k} \binom{k}{i} \frac{(-2)^i}{(2i+1)} .$$

In a similar fashion, we may derive an expression for B_k^2 by using the component functions of $h(x)$:

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad ;$$

$$(1 - x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{2k}{k} (x/2)^{2k} \quad .$$

Therefore,

$$h(x) = \sum_{k=0}^{\infty} d_k x^{2k+1} \quad ,$$

where

$$d_k = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{2k - 2i + 1} \frac{\binom{2i}{i}}{2^{2i}} \quad ,$$

But, since $d_k = B_k^2 + (2k+1)!$, we are led to the expression:

(23) $$B_k^2 = (-1)^k (2k+1)! \sum_{i=0}^{k} \binom{2i}{i} \frac{\left(-\frac{1}{4}\right)^i}{2k - 2i + 1} \quad .$$

We may also express each B_k in the form of a definite integral as follows:

First, we define the polynomial $P_k(x)$ by the following summation:

(24)
$$P_k(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{x^{2i+1}}{2i + 1} .$$

If we differentiate,

$$P'_k(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^{2i} .$$

But the latter expression is equivalent to the binomial expansion for $(1 - x^2)^k$. Noting that $P_k(0) = 0$, we may integrate and obtain:

(25)
$$P_k(x) = \int_0^x (1 - u^2)^k \, du$$

Next, we observe that

$$P_k(\sqrt{2}) = \sqrt{2} \sum_{i=0}^{k} \binom{k}{i} \frac{(-2)^i}{2i + 1} .$$

Comparing this with the expression for B_k in (22), we obtain:

(26)
$$B_k = \frac{(2k + 1)!}{2^{k+\frac{1}{2}} k!} \int_0^{\sqrt{2}} (1 - u^2)^k \, du .$$

Next, we prove the following property:

(27) B_k is divisible by $\frac{(2m)!}{2^m m!}$, where m is the greatest integer in $\frac{1}{2}(k+1)$.

If we multiply (5) throughout by $2^k k!$ and apply relation (8), we obtain the recursion

$$(28) \quad B_k = 2k B_{k-1} + (-1)^k \frac{(2k)!}{2^k k!} = 2k B_{k-1} + (-1)^k (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)) .$$

Recursion (28) may be expressed in the following alternative forms, depending on whether k is even or odd:

$$(28a) \qquad B_{2m} = 4m B_{2m-1} + 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (4m-1)$$

$$(28b) \qquad B_{2m+1} = (4m+2) B_{2m} - 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (4m+1) .$$

We may now prove (27) by induction. Let us first assume that (27) is true for $k = 2m$, i.e., B_{2m} is divisible by $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)$. Then, by (28b), B_{2m+1} is divisible by $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m+1)$. But this is equivalent to the assertion of (27), where $k = 2m+1$. Now, if we replace m by m + 1 in (28a), we see that B_{2m+2} is also divisible by $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m+1)$. This, in turn, is equivalent to the assertion of (27), where $k = 2m+2$. This establishes the inductive chain. Since (27) is true for $k = 0$, it is therefore true for all values of k.

The readers are invited to discover any other properties of the sequence B_k which they feel might be of interest. It is the belief of the author that a deeper analysis of this series of numbers, though perhaps not of any lasting value, might be a source of recreation for those who derive pleasure from such studies.

APPENDIX

DERIVATION OF EQUATION (21)

In addition to the series expressions for the derivatives of h(x), we will need the following expressions:

$$x^2z = \sum_{k=1}^{\infty} S_{k-1}(2k + 1)^{(2)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^4z = \sum_{k=2}^{\infty} S_{k-2}(2k + 1)^{(4)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^3z' = \sum_{k=1}^{\infty} S_{k-1}(2k + 1)^{(3)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^5z' = \sum_{k=2}^{\infty} S_{k-2}(2k + 1)^{(5)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^2z'' = \sum_{k=1}^{\infty} S_{k}(2k + 1)^{(2)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^4z'' = \sum_{k=2}^{\infty} S_{k-1}(2k + 1)^{(4)} \frac{x^{2k+1}}{(2k + 1)!}$$

$$x^6z'' = \sum_{k=3}^{\infty} S_{k-2}(2k + 1)^{(6)} \frac{x^{2k+1}}{(2k + 1)!} \quad .$$

In the foregoing, the symbol $(2k + 1)^{(r)}$ represents

$$(2k + 1)(2k)(2k - 1)(2k - 2) \cdots (2k + 1 - (r - 1)) = \frac{(2k + 1)!}{(2k + 1 - r)!} \quad .$$

Equation (20) may be expressed in the following manner:

$$(1 - x^2 - x^4 - x^6)z'' + (4x^5 - 4x^3)z' + (2x^4 - 3x^2 - 1)z = 0 \, .$$

Substituting the previous expressions in the latter equation, we obtain:

$$\sum_{k=0}^{\infty} S_{k+1} \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=1}^{\infty} S_k (2k+1)^{(2)} \frac{x^{2k+1}}{(2k+1)!}$$

$$- \sum_{k=2}^{\infty} S_{k-1}(2k+1)^{(4)} \frac{x^{2k+1}}{(2k+1)!} + \sum_{k=3}^{\infty} S_{k-2}(2k+1)^{(6)} \frac{x^{2k+1}}{(2k+1)!}$$

$$+ \sum_{k=2}^{\infty} 4S_{k-2}(2k+1)^{(5)} \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=1}^{\infty} 4S_{k-1}(2k+1)^{(3)} \frac{x^{2k+1}}{(2k+1)!}$$

$$+ \sum_{k=2}^{\infty} 2S_{k-2}(2k+1)^{(4)} \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=1}^{\infty} 3S_{k-1}(2k+1)^{(2)} \frac{x^{2k+1}}{(2k+1)!}$$

$$- \sum_{k=0}^{\infty} S_k \frac{x^{2k+1}}{(2k+1)!} = 0 .$$

If we equate like coefficients, we obtain the following recursions:

$$S_1 - S_0 = 0; \qquad S_2 - 6S_1 - 24S_0 - 18S_0 - S_1 = 0;$$

$$S_3 - 20S_2 - 120S_1 + 480S_0 - 240S_1 + 240S_0 - 60S_1 - S_2 = 0 ;$$

if $k = 3, 4, 5, \cdots,$

$$S_{k+1} - (2k(2k+1) + 1)S_k - 2k(2k+1)Q_k S_{k-1}$$
$$+ (2k+1)^{(4)}((2k-3)(2k-4) + 4(2k-3) + 2)S_{k-2} = 0 ,$$

where

$$Q_k = (2k-1)(2k-2) + 4(2k-1) + 3 .$$

Upon simplification, these results become:

$$(21)\quad S_{k+1} = (4k^2 + 2k + 1)S_k + 2k(2k + 1)(4k^2 + 2k + 1)S_{k-1} - 2k(2k + 1)(2k - 1)^2(2k - 2)^2S_{k-2} ,$$

balid for $k = 0, 1, 2, \cdots$.

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etc. Of course, that is (abstractly) the same thing we are doing in (2), (3).

In [7], Emma Lehmer examines the quadratic character of

$$\theta = (1 + \sqrt{5})/2 \qquad (\text{mod } p) .$$

If θ is a quadratic residue of p, but not a higher power residue, then all quadratic residues can be generated by addition. In our construction, θ is a primitive root and generates the quadratic nonresidues also.

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