OPERATIONAL RECURRENCES INVOLVING FIBONACCI NUMBERS

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The Fibonacci numbers may be defined by the linear recurrence relation

(1) $$F_{n+1} = F_n + F_{n-1}$$

together with the initial values $F_0 = 0$, $F_1 = 1$.

There are some unorthodox ways of making up sequences which involve Fibonacci numbers, and we should like to mention a few of these. For want of a better name, we shall call the recurrences below 'operational recurrences.'

Instead of taking the next term in a sequence as the sum of the two preceding terms, let us suppose the terms of a sequence are functions of x, and define

(2) $$u_{n+1}(x) = D_x (u_n u_{n-1}) ,$$

where $D_x = d/dx$. As an example, take $u_0 = 1$, $u_1 = e^x$. Then we find

$$u_2 = D(e^x) = e^x$$

$$u_3 = D(e^{2x}) = 2 e^{2x}$$

$$u_4 = D(2e^{3x}) = (2)(3) e^{3x}$$

$$u_5 = (1)(1)(2)(3)(5) e^{5x}$$

and we can easily show by induction that

(3) $u_n = (F_1 F_2 F_3 \ldots\ldots F_n)\, e^{F_n x}$, $(u_0 = 1,\ u_1 = e^x)$.

Of course, the addition of exponents led to the appearance of the Fibonacci numbers in this case.

Another operation which we may use is differentiation followed by multiplication with x. We define

(4) $$u_{n+1} = (xD_x)(u_n\ u_{n-1}).$$

For an interesting example, let us take

$$u_0 \equiv x^{F_0} = 1,\quad u_1 = x^{F_1} = x\ .$$ Then we claim that

(5) $$u_n = x^{F_n} \prod_{k=1}^{n} F_k^{F_{n+1-k}}, \qquad n \geq 1.$$

Taking x = 1 we obtain the following table of values as a sample:

n	$u_n(1)$
1	$1^1 = 1$
2	$1^1\, 1^1 = 1$
3	$1^2\, 1^1\, 2^1 = 2$
4	$1^3\, 1^2\, 2^1\, 3^1 = 6$
5	$1^5\, 1^3\, 2^2\, 3^1\, 5^1 = 60$
6	$1^8\, 1^5\, 2^3\, 3^2\, 5^1\, 8^1 = 2880$
7	$1^{13}\, 1^8\, 2^5\, 3^3\, 5^2\, 8^1\, 13^1 = 2{,}246{,}400$

For the sake of completeness we give the inductive proof of formula (5). Suppose that

$$u_{n-1} = x^{F_{n-1}} \prod_{k=1}^{n-1} F_k^{F_{n-k}}$$

Then

$$u_n = xD\,(u_{n-1}\, u_{n-2})$$

$$= xD\left\{ x^{F_{n-1}} x^{F_{n-2}} \prod_{k=1}^{n-1} F_k^{F_{n-k}} \prod_{j=1}^{n-2} F_j^{F_{n-1-j}} \right\}$$

$$= xD\left\{ x^{F_n} \right\} F_{n-1}^{F_1} \prod_{k=1}^{n-2} F_k^{F_{n-k}+F_{n-1-k}}$$

$$= F_n\, x^{F_n} F_{n-1}^{F_1} \prod_{k=1}^{n-2} F_k^{F_{n+1-k}}$$

$$= F_n^{F_1}\, x^{F_n} F_{n-1}^{F_2} \prod_{k=1}^{n-2} F_k^{F_{n+1-k}}$$

$$= x^{F_n} \prod_{k=1}^{n} F_k^{F_{n+1-k}}$$

The only 'tricky' part is to recall that $1 = F_1$ and $F_1 = F_2$ so that the factors may be put together at the last step in the desired form.

Suppose that the function $u_n(x)$ has a power series representation of the form

$$(6) \qquad u_n(x) = \sum_{k=0}^{\infty} a_k(n)\, x^k \, .$$

Imposing the operational recurrence (4) we find readily that the coefficients in (6) must obey the convolution recurrence

$$(7) \qquad a_k(n) = k \sum_{j=0}^{k} a_j(n-2)\, a_{k-j}(n-1) .$$

Conversely, if (7) holds then $u_n(x)$ satisfies (4).

As a slight variation of (4) let us next define

$$(8) \qquad u_{n+1} = x^2 D_x (u_n u_{n-1}) ,$$

and take $u_0 = 1$, $u_1 = x$. Then it is easily shown by induction that

$$(9) \qquad u_n = x^{F_{n+2}-1} \prod_{k=4}^{n+2} (F_k - 2)^{F_{n+3-k}} , \quad \text{for } n \geq 2.$$

The reader may find it interesting to derive a formula for the sequence defined by $u_n = u_n(x)$ with

$$(10) \qquad u_{n+1} = x^p D_x (u_n u_{n-1}),\ u_0 = 1,\ u_1 = x,\ p = 3, 4, 5 \ldots$$

As a final example, let us define a sequence by (4) with $u_0 = 1$, $u_1 = e^x$.

Then the first few values of the function sequence are:

$$u_2 = x e^x ,$$

$$u_3 = (2x^2 + x)\, e^{2x}$$

$$u_4 = (6x^4 + 9x^3 + 2x^2)\, e^{3x}$$

$$u_5 = (60x^7 + 192x^6 + 185x^5 + 62x^4 + 6x^3)\, e^{5x}$$

and it is evident that $u_n(x)$ equals $P(x)\, e^{F_n x}$, where $P(x)$ is a polynomial of degree $F_{n+1} - 1$ in x.