ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-650 Proposed by Paul S. Bruckman, Sointula, Canada

Let $D = d/dz$ be the differential operator. Let $f = f(z) = \csc z$, with z any complex number $\neq n\pi$, where n is any integer. Prove the following identity valid for all integers $m \geq 1$

$$f^{2m+2} = \frac{1}{(2m+1)!}\prod_{n=1}^{m}\{D^2 + 4n^2\}(f^2);$$

$$f^{2m+1} = \frac{1}{(2m)!}\prod_{n=1}^{m}\{D^2 + (2n-1)^2\}(f).$$

Show that the same relations hold with the function $f(z) = \sec z$ when $z \neq (n + 1/2)\pi$, where n is any integer.

H-651 Proposed by H.-J. Seiffert, Berlin, Germany

Prove that, for all positive integers n,

$$\sum_{k=0}^{\lfloor (n-3)/5 \rfloor}\binom{4n}{2n-10k-5} = \frac{1}{10}(2^{4n-1} - 5^n L_{2n} + L_{4n})$$

and

$$\sum_{k=0}^{\lfloor (n-3)/5 \rfloor}\binom{4n-2}{2n-10k-6} = \frac{1}{10}(2^{4n-3} - 5^n F_{2n-1} + L_{4n-2}).$$

H-652 **Proposed by José Luis Díaz-Barrero, Barcelona, Spain**

Determine

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{F_kF_{k+1}}{F_nF_{n+1}}\right)^{F_k/F_{k+1}}.$$

H-653 **Proposed by Ovidiu Furdui, Kalamazoo, MI**

Let $n \geq 3$ be a natural number. Prove the following formula

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2(k+1)(k+2)\cdots(k+n)}=\frac{2}{n!}\left(\zeta(3)-\frac{\pi^2}{8}+\frac{1}{4}\right)-\frac{1}{n!}\sum_{k=3}^{n}\frac{1}{k}\left(\frac{\pi^2}{6}-\sum_{j=1}^{n-1}\frac{1}{j^2}\right),$$

where $H_k=\sum_{j=1}^{k}1/j$ is the kth harmonic number and $\zeta(3)=\sum_{j=1}^{\infty}1/j^3$ is the celebrated Apéry constant.

SOLUTIONS

Fibonacci numbers and the Rogers-Ramanujan identities

H-632 **Proposed by Paul S. Bruckman, Sointula, Canada**
(Vol. 43, no. 4, November 2005)

Prove the following identities:

1. $$1+\sum_{n=1}^{\infty}(-1)^n\frac{5^{-n/2}\alpha^{-n(3n-1)/2}}{F_1F_2\ldots F_n}=\prod_{n=0}^{\infty}\left\{1+4(-1)^n\alpha^{-10n-5}-\alpha^{-20n-10}\right\}^{-1}.$$

2. $$1+\sum_{n=1}^{\infty}\frac{5^{-n/2}\alpha^{-n(3n+1)/2}}{F_1F_2\ldots F_n}=\prod_{n=0}^{\infty}\left\{1-(-1)^n\alpha^{-10n-5}-\alpha^{-20n-10}\right\}^{-1}.$$

Here, α is the golden section.

Solution by H.-J. Seiffert, Berlin, Germany

Since, by the Binet form of the Fibonacci numbers

$$F_k=\frac{\alpha^k}{\sqrt{5}}(1-(-\alpha^{-2})^k),\qquad \text{for all } k\geq 1,$$

we have

$$F_1F_2\cdots F_n=5^{-n/2}\alpha^{n(n+1)/2}\prod_{k=1}^{n}(1-(-\alpha^{-2})^k),\qquad \text{for all } n\geq 1.$$

Now, the Rogers-Ramanujan identities (see [1], p. 113)

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_n}=\prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1},$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(q)_n}=\prod_{n=0}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1},$$

valid for all $|q|<1$, where $(q)_n=(1-q)(1-q^2)\cdots(1-q^n)$, $n\geq 1$, with $q=-\alpha^{-2}$ give

$$1+\sum_{n=1}^{\infty}(-1)^n\frac{5^{-n/2}\alpha^{-n(3n-1)/2}}{F_1F_2\cdots F_n}=\prod_{n=0}^{\infty}\{1+4(-1)^n\alpha^{-10n-5}-\alpha^{-20n-10}\}^{-1}$$

and

$$1+\sum_{n=1}^{\infty}\frac{5^{-n/2}\alpha^{-n(3n+1)/2}}{F_1F_2\cdots F_n}=\prod_{n=0}^{\infty}\{1-(-1)^n\alpha^{-10n-5}-\alpha^{-20n-10}\}^{-1},$$

where we have used the relations $\alpha^3-\alpha^{-3}=4$ and $\alpha-\alpha^{-1}=1$.

[1] G.E. Andrews. "The theory of partitions." Encyclopedia of Mathematics and its Applications: (Rota, Editor), Vol. 2, Addison-Wesley, Reading, 1976.

Correction. In the statement of this problem (The Fibonacci Quarterly **43.4** (2005)), the exponents of α in the left hand side of the stated identities were erroneously written as "$n(3n-1)/2$" and "$n(3n+1)/2$" instead of their negatives and there was an extra $(-1)^n$ inside the summation in the second proposed identity.

Also solved by the proposer.

Series Identities and the Regular Heptagon

H-633 Proposed by Kenneth B. Davenport, Dallas, PA (Vol. 43, no. 4, November 2005)

Let A, B and C be $A=\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1}{7n+1}+\frac{1}{7n+6}\Big)$, $B=\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1}{7n+2}+\frac{1}{7n+5}\Big)$ and $C=\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1}{7n+3}+\frac{1}{7n+4}\Big)$. Show that $A=B+C$.

Solution by Paul S. Bruckman, Sointula, Canada

We begin with the Mittag-Leffler expansion

$$\pi \csc \pi z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{z-n}.$$

This may be also expressed, after some manipulation, as

$$\pi z \csc \pi z = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1 - n^2/z^2}. \tag{1}$$

For example, we refer to [1]. Next, we note that

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+7n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1-7n} = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1-49n^2}.$$

Comparing this with (1), we see that $A = (\pi/7)\csc(\pi/7)$. Similarly,

$$B = \sum_{n=0}^{\infty} \frac{(-1)^n}{2+7n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2-7n} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1-49n^2/4}.$$

Hence, $2B = (2\pi/7)\csc(2\pi/7)$, or $B = (\pi/7)\cos(2\pi/7)$. Likewise,

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{3+7n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{3-7n} = \frac{1}{3} + \frac{2}{3}\sum_{n=1}^{\infty} \frac{(-1)^n}{1-49n^2/9}.$$

Thus, $3C = (3\pi/7)\csc(3\pi/7)$, or $C = (\pi/7)\csc(3\pi/7)$. In order to show that $A = B + C$, it therefore suffices to show that

$$\csc(\pi/7) = \csc(2\pi/7) + \csc(3\pi/7). \tag{2}$$

Write $s = \sin(\pi/7),\ c = \cos(\pi/7)$. Then $\sin(2\pi/7) = 2sc,\ \sin(3\pi/7) = 3s - 4s^3 = s(4c^2 - 1)$. In order to prove (2), it therefore suffices to prove that

$$1 = \frac{1}{2c} + \frac{1}{4c^2 - 1},$$

which is equivalent to $8c^3 - 4c^2 - 4c + 1 = 0$. Let $z = e^{2\pi i/7}$, and observe that

$$0 = \frac{z^7 - 1}{z - 1} = 1 + z + \cdots + z^6.$$

Dividing both sides of the above equation by z^3 and rewriting the resulting equation in terms of $c = (z + z^{-1})/2$, we get the desired equation.

[1] L.I. Pennisi. "Elements of Complex Variables." Holt, Reinhart and Winston, Chicago, 1976:336.

Correction. In the statement of this problem (The Fibonacci Quarterly **43.4** (2005)), the required identity "$A = B + C$" was misstated as "$A + B = C$".

The proposer also sent in a solution due to Piero Filipponi.

Integrals of Fibonacci polynomials

H-634 Proposed by Ovidiu Furdui, Kalamazoo, MI (Vol. 43, no. 4, November 2005)

Prove that $\displaystyle\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{n-2j}\binom{n-j-1}{j} = \frac{\alpha^n - 1}{n} + \frac{\beta^n - (-1)^n}{n}$ holds for all $n \geq 1$,

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

Solution by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = x$, and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ for $n \geq 1$. It is known (see [1], equation (2.15)), that

$$F_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \qquad \text{for all } n \geq 1.$$

Integrating over $x \in [0,1]$ gives

$$\int_0^1 F_n(x)dx = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{n-2j}\binom{n-j-1}{j}, \qquad \text{for all } n \geq 1. \tag{1}$$

From [2], we know that

$$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - 1 - (-1)^n), \qquad \text{for all } n \geq 1. \tag{2}$$

Combining (1) and (2) and using the Binet form $L_n = \alpha^n + \beta^n$ yields the requested identity.

[1] A.F. Horadam and Bro. J.M. Mahon. "Pell and Pell-Lucas polynomials." The Fibonacci Quarterly **23.1** (1985): 7-20.

[2] "Problem H-410." The Fibonacci Quarterly **27.5** (1989): 474.

Also solved by Paul S. Bruckman, G. C. Greubel and the proposer.

A Circulant Determinant

H-635 Proposed by Jayantibhai M. Patel, Ahmedabad, India (Vol. 44, no. 1, February 2006)

For any positive integer n, prove that

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+2} & F_n & F_{n+1} \\ F_{n+1} & F_{n+2} & F_n \end{vmatrix} = 2(F_n^3 + F_{n+1}^3),$$

and that the same holds with the Fibonacci numbers replaced by the corresponding Lucas numbers.

Solution by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Barcelona, Spain

Setting $a = F_n,\ b = F_{n+1}$, we have

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+2} & F_n & F_{n+1} \\ F_{n+1} & F_{n+2} & F_n \end{vmatrix} = \begin{vmatrix} a & b & a+b \\ a+b & a & b \\ b & a+b & a \end{vmatrix} = a^3 + b^3 + (a+b)^3 - 3ab(a+b)$$

$$= 2(a^3 + b^3) = 2(F_n^3 + F_{n+1}^3).$$

Since the identity

$$\begin{vmatrix} a & b & a+b \\ a+b & a & b \\ b & a+b & a \end{vmatrix} = 2(a^3 + b^3)$$

is valid for all complex numbers a and b, the statement is also valid replacing the Fibonacci numbers by the corresponding Lucas numbers.

Also solved by Gökçen Alptekin, Ercan Altinisik, Paul S. Bruckman, Charles Cook, Kenneth B. Davenport, G. C. Greubel, Ralph P. Grimaldi, John F. Morrison, Maitland A. Rose, H.-J. Seiffert and James A. Sellers.

Errata: In problem H-649, the equation to be solved should be "$\sec(F_a) + \sec(F_b) = F_c$" instead of "$\sec(F_a) + \sec(F_b) = \sec(F_c)$".