AN IDENTITY MOTIVATED BY AN AMAZING IDENTITY OF RAMANUJAN

JAMES MC LAUGHLIN

Abstract. Ramanujan stated an identity to the effect that if three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are defined by $r_1(x) =: \sum_{n=0}^{\infty} a_n x^n$, $r_2(x) =: \sum_{n=0}^{\infty} b_n x^n$ and $r_3(x) =: \sum_{n=0}^{\infty} c_n x^n$ (here each $r_i(x)$ is a certain rational function in x), then

$$a_n^3 + b_n^3 - c_n^3 = (-1)^n, \quad \text{for all} \quad n \geq 0.$$

Motivated by this amazing identity, we state and prove a more general identity involving eleven sequences, the new identity being "more general" in the sense that equality holds not just for the power 3 (as in Ramanujan's identity), but for each power j, $1 \leq j \leq 5$.

1. Introduction

In the "lost notebook" [7, page 341], Ramanujan records the following remarkable identity. If the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are defined by

$$\frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3} =: \sum_{n=0}^{\infty} a_n x^n, \tag{1.1}$$

$$\frac{2 - 26x - 12x^2}{1 - 82x - 82x^2 + x^3} =: \sum_{n=0}^{\infty} b_n x^n,$$

$$\frac{2 + 8x - 10x^2}{1 - 82x - 82x^2 + x^3} =: \sum_{n=0}^{\infty} c_n x^n,$$

then

$$a_n^3 + b_n^3 - c_n^3 = (-1)^n, \quad \text{for all} \quad n \geq 0. \tag{1.2}$$

As Hirschhorn remarks in [5], what is amazing about this identity is not only that it is true, but that anyone could come up with it in the first place. As well as giving a proof of the identity, Hirschhorn also gives a plausible explanation of how Ramanujan might have discovered it. A second proof of the identity was given by Hirschhorn in [6], and a third proof was given by Hirschhorn and Han in [4], where the authors also prove that the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ may also be derived from a certain matrix equation.

Motivated by this amazing identity of Ramanujan, and Hirschhorn's explanation of how Ramanujan might have found it, we present a more general identity in the present paper, one where the three sequences in (1.2) are replaced by eleven sequences, and the identity holds not just for a single exponent (3 in the case of (1.2)), but for all integer exponents j, $1 \leq j \leq 5$.

2. A Ramanujan-Type Identity

The identity referred to is described in the following theorem.

Theorem 2.1. *Let the sequences of integers a_k, b_k, c_k, d_k, e_k, f_k, p_k, q_k, r_k, s_k and t_k be defined by*

$$\frac{x^2+164x+3}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} a_k x^k, \qquad \frac{-5x^2+138x+3}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} p_k x^k,$$

$$\frac{-7x^2+134x+1}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} b_k x^k, \qquad \frac{3x^2+244x+1}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} q_k x^k,$$

$$\frac{-x^2+298x-1}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} c_k x^k, \qquad \frac{x^2+254x-7}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} r_k x^k,$$

$$\frac{-5x^2+228x-7}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} d_k x^k, \qquad \frac{-7x^2+148x-5}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} s_k x^k,$$

$$\frac{3x^2+258x-5}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} e_k x^k, \qquad \frac{3}{1-x} =: \sum_{k=0}^{\infty} t_k x^k,$$

$$\frac{-3x^2+94x-3}{x^3-99x^2+99x-1} =: \sum_{k=0}^{\infty} f_k x^k.$$

Then for $1 \le j \le 5$, and each $k \ge 0$,

$$a_k^j + b_k^j + c_k^j + d_k^j + e_k^j + f_k^j - p_k^j - q_k^j - r_k^j - s_k^j - t_k^j = 1. \tag{2.1}$$

We note that (2.1) differs from Ramanujan's identity (1.2), in that (2.1) is true for each integer exponent j, $1 \le j \le 5$, in contrast to (1.2), which is true only for the fixed exponent 3. For example, one can check that

$$\begin{aligned} &\{a_1, b_1, c_1, d_1, e_1, f_1, p_1, q_1, r_1, s_1, t_1\} \\ &\quad = \{-461, -233, -199, 465, 237, 203, -435, -343, 439, 347, 3\} \end{aligned}$$

and that

$$\begin{aligned} &(-461)^j + (-233)^j + (-199)^j + 465^j + 237^j + 203^j \\ &\quad - (-435)^j - (-343)^j - 439^j - 347^j - 3^j = 1, \end{aligned} \tag{2.2}$$

for $1 \le j \le 5$. Like Ramanujan's sequences, the terms in our sequences also grow arbitrarily large (except for t_k which has the constant value 3 for all $k \ge 0$), while the left side of (2.1) maintains the constant value 1.

Many readers will no doubt have recognized that what has been encoded in the various generating functions is a sequence of ideal solutions of size 6 to what has become known as the *Prouhet-Tarry-Escott* Problem (Dickson [3] referred to it as the problem of "equal sums of like powers"). Before coming to the proof of Theorem 2.1, we briefly discuss this problem.

The *Prouhet-Tarry-Escott* Problem, which has a history going back to Goldbach, asks for two distinct multisets of integers $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_m\}$ such that

$$\sum_{i=1}^{m} a_i^e = \sum_{i=1}^{m} b_i^e, \text{ for } e = 1, 2, \dots, k, \tag{2.3}$$

for some integer $k < m$. We call m the *size* of the solution and k the *degree*. If $k = m - 1$, such a solution is called *ideal*. For example, it is easy to check that

$$1^j + 21^j + 36^j + 56^j = 2^j + 18^j + 39^j + 55^j$$

holds for $j = 1, 2$ and 3. Thus, $A = \{1, 21, 36, 56\}$ and $B = \{2, 18, 39, 55\}$ provide an ideal solution of size 4.

We write

$$\{a_1, ..., a_m\} \stackrel{k}{=} \{b_1, ..., b_m\} \tag{2.4}$$

to denote a solution of size m and degree k to the Prouhet-Tarry-Escott Problem. As regards parametric solutions, an early example was given by Euler (see [3, page 705]), who showed that

$$\{a, b, c, a + b + c\} \stackrel{2}{=} \{a + b, a + c, b + c, 0\}.$$

Parametric ideal solutions are known for $m = 1, \ldots, 8$ and particular numerical solutions are known for $m = 9, 10$ and 12. The interested reader may find some of the early history of this interesting problem in Chapter XXIV of [3], and some of the more recent developments at [1] and [8].

The following parametric solution of size 6 is due to Chernick [2]. For any integers m_k and n_k, if

$$\begin{aligned}
a'_k &= -5m_k^2 + 4m_kn_k - 3n_k^2, & p'_k &= -5m_k^2 + 6m_kn_k + 3n_k^2, \\
b'_k &= -3m_k^2 + 6m_kn_k + 5n_k^2, & q'_k &= -3m_k^2 - 4m_kn_k - 5n_k^2, \\
c'_k &= -m_k^2 - 10m_kn_k - n_k^2, & r'_k &= -m_k^2 + 10m_kn_k - n_k^2, \\
d'_k &= 5m_k^2 - 4m_kn_k + 3n_k^2, & s'_k &= 5m_k^2 - 6m_kn_k - 3n_k^2, \\
e'_k &= 3m_k^2 - 6m_kn_k - 5n_k^2, & t'_k &= 3m_k^2 + 4m_kn_k + 5n_k^2, \\
f'_k &= m_k^2 + 10m_kn_k + n_k^2, & u'_k &= m_k^2 - 10m_kn_k + n_k^2.
\end{aligned} \tag{2.5}$$

then

$$\{a', b', c', d', e', f'\} \stackrel{5}{=} \{p', q', r', s', t', u'\}. \tag{2.6}$$

The observant reader will have noticed that the twelve terms actually form 6 pairs, each of the three pairs on each side of (2.6) consisting of a term and its negative ($d'_k = -a'_k$ and so on), so that (2.6) is trivially true for odd powers. To make our generating functions and sequences at least superficially more interesting, we will modify these sequences using the easily-proved fact that if

$$\{a_1, ..., a_m\} \stackrel{k}{=} \{b_1, ..., b_m\},$$

then

$$\{Ma_1 + K, ..., Ma_m + K\} \stackrel{k}{=} \{Mb_1 + K, ..., Mb_m + K\},$$

for constants M and K.

In the present case, we will determine particular sequences $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$, with the sequences $a'_k \ldots u'_k$ being defined by (2.5). We then set $a_k = a'_k + 2u'_k$, $b_k = b'_k + 2u'_k$ and so on. In particular, $r_k = r'_k + 2u'_k = -u'_k + 2u'_k = u'_k$. We will further show that $u'_k = r_k = 1$, so that

$$\{a_k, b_k, c_k, d_k, e_k, f_k\} \stackrel{5}{=} \{p_k, q_k, 1, s_k, t_k, u_k\}$$

will hold automatically for each integer $k \geq 0$, which gives (2.1), after a slight manipulation.

All that will remain will be to show that each of the generating functions has the stated form. We now proceed to the proof.

Proof of Theorem 2.1. Set $h_0 = 0$, $h_1 = 1$, and for $k > 1$, set

$$h_k = 10h_{k-1} - h_{k-2}. \tag{2.7}$$

Upon solving the characteristic equation $x^2 - 10x + 1 = 0$ and applying the stated initial conditions, we find that

$$\begin{aligned} h_k &= -\frac{\left(5-2\sqrt{6}\right)^k}{4\sqrt{6}} + \frac{\left(5+2\sqrt{6}\right)^k}{4\sqrt{6}}, \\ h_k^2 &= \frac{-2 + \left(49-20\sqrt{6}\right)^k + \left(49+20\sqrt{6}\right)^k}{96}, \\ h_{k+1}h_k &= \frac{-10 + \left(5-2\sqrt{6}\right)\left(49-20\sqrt{6}\right)^k + \left(5+2\sqrt{6}\right)\left(49+20\sqrt{6}\right)^k}{96}. \end{aligned}$$

In (2.5), we set $m_k = h_{k+1}$ and $n_k = h_k$, noting that (2.7) implies that

$$h_{k+1}^2 - 10h_{k+1}h_k + h_k^2 = h_k^2 - 10h_k h_{k-1} + h_{k-1}^2 = \cdots = h_1^2 - 10h_1h_0 + h_0^2 = 1,$$

so that $r_k = u_k' = 1$. Thus all that remains is to show, with these choices for m_k and n_k, that the various generating functions have the stated forms. We do this for $\sum_{k=0}^{\infty} a_k x^k$ only, since the proofs for the other generating functions are virtually identical.

Define

$$\begin{aligned} H_1(x) &:= \sum_{k=0}^{\infty} h_k^2 x^k = \sum_{k=0}^{\infty} \frac{-2 + \left(49-20\sqrt{6}\right)^k + \left(49+20\sqrt{6}\right)^k}{96} x^k \\ &= \frac{1}{96}\left(\frac{-2}{1-x} + \frac{1}{1-\left(49-20\sqrt{6}\right)x} + \frac{1}{1-\left(49+20\sqrt{6}\right)x}\right) \\ &= \frac{-x(x+1)}{x^3 - 99x^2 + 99x - 1}, \\ H_2(x) &:= \sum_{k=0}^{\infty} h_{k+1}h_k x^k \\ &= \frac{1}{96}\left(\frac{-10}{1-x} + \frac{5-2\sqrt{6}}{1-\left(49-20\sqrt{6}\right)x} + \frac{5+2\sqrt{6}}{1-\left(49+20\sqrt{6}\right)x}\right) \\ &= \frac{-10x}{x^3 - 99x^2 + 99x - 1}, \\ H_3(x) &:= \sum_{k=0}^{\infty} h_{k+1}^2 x^k \\ &= \frac{H_1(x)}{x} = \frac{-x-1}{x^3 - 99x^2 + 99x - 1}. \end{aligned}$$

These formulas for $H_1(x)$, $H_2(x)$ and $H_3(x)$ follow after using the summation formula for an infinite geometric series a number of times, and then using a little algebra to combine the resulting rational expressions.

Next,

$$a_k = a_k' + 2u_k' = -5h_{k+1}^2 + 4h_{k+1}h_k - 3h_k^2 + 2,$$

so that

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty}(-5h_{k+1}^2 + 4h_{k+1}h_k - 3h_k^2 + 2)x^k$$
$$= -5H_3(x) + 4H_2(x) - 3H_1(x) + \frac{2}{1-x}$$
$$= \frac{x^2 + 164x + 3}{x^3 - 99x^2 + 99x - 1},$$

as claimed in Theorem 2.1. The claimed formulas for the other generating functions follow similarly, giving the result. □

References

[1] P. Borwein and C. Ingalls, *The Prouhet-Tarry-Escott problem revisited*, Enseign. Math., **40** (1994), 3–27.

[2] J. Chernick, *Ideal solutions of the Tarry-Escott problem*, Amer. Math. Monthly, **44.10** (1937), 626–633.

[3] L. E. Dickson, *History of the Theory of Numbers*, Vol. II: Diophantine Analysis, Chelsea Publishing Co., New York, 1966.

[4] J. H. Han and M. D. Hirschhorn, *Another look at an amazing identity of Ramanujan*, Math. Mag., **79** (2006), 302–304.

[5] M. D. Hirschhorn, *An amazing identity of Ramanujan*, Math. Mag., **68.3** (1995), 199–201.

[6] M. D. Hirschhorn, *A proof in the spirit of Zeilberger of an amazing identity of Ramanujan*, Math. Mag., **69.4** (1996), 267–269.

[7] S. Ramanujan, *The Lost Notebook and other Unpublished Papers*, With an introduction by George E. Andrews. Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988.

[8] C. Shuwen, *The Prouhet-Tarry-Escott problem*, http://euler.free.fr/eslp/TarryPrb.htm.

MSC2000: 11B75

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383

E-mail address: jmclaughl@wcupa.edu