

COMBINATORIAL PROOFS OF ZECKENDORF FAMILY IDENTITIES

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ABSTRACT. A general combinatorial approach is presented for proving identities of the form $mf_n = \sum_{i \in I_m} f_{n+i}$, where m is a nonnegative integer constant, $n \geq |\min(I_m)|$ is an integer, I_m is a set of nonconsecutive integers, and f_n is the Fibonacci number F_{n+1} . The approach involves counting phased square-domino tilings, as in the book *Proofs that Really Count* by Benjamin and Quinn. Furthermore, for each proof of an identity of the form $mf_n = \sum_{i \in I_m} f_{n+i}$, there is a corresponding isomorphic proof of the identity $m = \sum_{i \in I_m} \phi^i$, where ϕ is $(1 + \sqrt{5})/2$.

1. INTRODUCTION

Benjamin and Quinn [1] in an exercise ask for a combinatorial approach to proving the following system of identities (the *Zeckendorf family identities* [4]), in which f_{n-1} represents the n th Fibonacci number (this is the combinatorial version of Fibonacci numbers, with $f_0 = f_1 = 1$ as opposed to $F_0 = 0$ and $F_1 = 1$ in the classical definition).

$$1f_n = f_n \quad \text{for } n \geq 0 \quad (1.1)$$

$$2f_n = f_{n+1} + f_{n-2} \quad \text{for } n \geq 2 \quad (1.2)$$

$$3f_n = f_{n+2} + f_{n-2} \quad \text{for } n \geq 2 \quad (1.3)$$

$$4f_n = f_{n+2} + f_n + f_{n-2} \quad \text{for } n \geq 4 \quad (1.4)$$

$$5f_n = f_{n+3} + f_{n-1} + f_{n-4} \quad \text{for } n \geq 4 \quad (1.5)$$

$$6f_n = f_{n+3} + f_{n+1} + f_{n-4} \quad \text{for } n \geq 4 \quad (1.6)$$

$$7f_n = f_{n+4} + f_{n-4} \quad \text{for } n \geq 4 \quad (1.7)$$

$$8f_n = f_{n+4} + f_n + f_{n-4} \quad \text{for } n \geq 4 \quad (1.8)$$

$$9f_n = f_{n+4} + f_{n+1} + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \quad (1.9)$$

$$10f_n = f_{n+4} + f_{n+2} + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \quad (1.10)$$

$$11f_n = f_{n+4} + f_{n+2} + f_n + f_{n-2} + f_{n-4} \quad \text{for } n \geq 4 \quad (1.11)$$

$$12f_n = f_{n+5} + f_{n-1} + f_{n-3} + f_{n-6} \quad \text{for } n \geq 6 \quad (1.12)$$

⋮

Benjamin and Quinn prove several of these identities, and recently Wood [4] has provided proofs of several more. But in both cases, each proof has been handled as a special case. The first goal of this paper is to present an approach based on counting phased square-domino tilings, which leads to an algorithm similar to long division for generating and proving all such identities. We will refer to this algorithm as the Golden Ratio Division (GRD) Algorithm.

Many thanks to Arthur Benjamin, Doron Zeilberger and Philip Matchett Wood for helpful comments. Errors in the paper are my own.

Benjamin and Quinn state (in the same exercise, without proof) moreover that:

The coefficients in the above formulas are the same as in the unique expansion of positive integers in nonconsecutive integer powers of $\phi = (1 + \sqrt{5})/2$. For example, $5 = \phi^3 + \phi^{-1} + \phi^{-4}$ and $6 = \phi^3 + \phi^1 + \phi^{-4}$.

In other words, the coefficients correspond to the positions of the 1-digits in the binary-encoded golden ratio base number system introduced by Bergman [2]. In Section 3.2, we elaborate on this connection, showing how the combinatorially inspired algorithm can be reinterpreted as an algorithm for directly generating these coefficients of ϕ . This leads to a semi-combinatorial proof of the following theorem.

Theorem 1.1. *For nonconsecutive integers $a_1 \dots a_k$, the following two statements are equivalent:*

$$mf_n = f_{n+a_1} + f_{n+a_2} + \dots + f_{n+a_k} \tag{1.13}$$

$$m = \phi^{a_1} + \phi^{a_2} + \dots + \phi^{a_k}. \tag{1.14}$$

Although, the GRD Algorithm was developed for the purpose of establishing Theorem 1.1, it is in fact a practical algorithm. The algorithm is efficient because it is greedy: the optimal value for each a_i is determined before moving on to a_{i+1} . In this way, it differs from the non-deterministic procedure described by Bergman [2], which uses a global optimization step to find optimal values for the coefficients.

In Section 2, we present the algorithm as an uninterpreted procedure, saving the two interpretations of the algorithm for Section 3. The paper concludes with an appendix with examples.

2. GOLDEN RATIO DIVISION ALGORITHM

We present the algorithm here by way of a couple of examples. First consider the case of 4. The algorithm begins with the following two steps, which we will refer to as the *init- α* rule and the *reduce- β* rule. In this case, α is determined by the initial problem to be 4.

$$4 \overline{)4 \quad 4}$$

$$\begin{array}{r} a \\ 4 \overline{)4 \quad 4} \\ \underline{f_a \quad f_{a-1}} \\ 4 - f_a \quad 4 - f_{a-1} \end{array}$$

The goal now is to find the largest value for a such that the pair of numbers $4 - f_a$ and $4 - f_{a-1}$ are nonnegative. The condition on nonnegativity will be refined shortly, but is sufficient for now. Clearly the maximal value for a is 3, so we can apply the *reduce-3* rule:

$$\begin{array}{r} 3 \\ 4 \overline{)4 \quad 4} \\ \underline{3 \quad 2} \\ 1 \quad 2 \end{array}$$

Now, as in normal long division, we move to the next column and repeat the process. Before we can move to the next column, however, we need to bring the bottom pair of numbers (1, 2) into columns 2 and 3. To do this we use the *shift* rule:

$$\begin{array}{r} 3 \\ 4 \overline{) 4 \ 4} \\ \underline{3 \ 2} \\ 1 \ 2 \\ \underline{ \ 1} \\ 3 \ 1 \end{array}$$

The general form of the shift rule is:

$$\begin{array}{r} a \quad b \\ \hline a + b \quad a \end{array}$$

After a few more steps, we terminate with zeros at the bottom:

$$\begin{array}{r} 3 \ 2 \ 1 \\ 4 \overline{) 4 \ 4} \\ \underline{3 \ 2} \\ 1 \ 2 \\ \underline{ \ 1} \\ 3 \ 1 \\ \underline{2 \ 1} \\ \ 1 \ 0 \\ \underline{ \ 1} \\ \ 1 \\ \underline{1 \ 1} \\ \ 0 \ 0 \end{array}$$

The output (3, 2, 1) is now interpreted by using a procedure which we will refer to as the *evaluation* rule. 3 is in column 1, 2 is in column 2 and 1 is in column 3. Subtracting the column numbers from the output, we get (3, 2, 1) - (1, 2, 3) = (2, 0, -2), which corresponds to the right hand side in the identities $4f_n = f_{n+2} + f_n + f_{n-2}$ and $4 = \phi^2 + \phi^0 + \phi^{-2}$.

As a second example, let's try 7, starting off in the following way:

$$\begin{array}{r} 5 \\ 7 \overline{) 7 \ 7} \\ \underline{8 \ 5} \\ -1 \ 2 \end{array}$$

This, of course, violates the nonnegativity condition. We will see however that by repeated application of the shift rule, the numbers at the bottom will eventually become positive. After one shift, we get:

$$\begin{array}{r} 5 \\ 7 \overline{) 7 \ 7} \\ \underline{8 \ 5} \\ -1 \ 2 \\ \underline{ \ 1} \\ 1 \ -1 \end{array}$$

THE FIBONACCI QUARTERLY

Since the numbers at the bottom are still not nonnegative, we add a placeholder “x” to the output, and try shifting one more time.

$$\begin{array}{r}
 5 \quad x \\
 \hline
 7 \) \ 7 \ 7 \\
 \quad 8 \ 5 \\
 \quad \hline
 \quad -1 \ 2 \\
 \quad \hline
 \quad \quad 1 \ -1 \\
 \quad \quad \hline
 \quad \quad \quad 0 \ 1
 \end{array}$$

At this point, both of the numbers at the bottom are nonnegative, but there is still no possible output value that can be added without going irretrievably negative. For example, try the output value 0:

$$\begin{array}{r}
 5 \quad x \quad 0 \\
 \hline
 7 \) \ 7 \ 7 \\
 \quad 8 \ 5 \\
 \quad \hline
 \quad -1 \ 2 \\
 \quad \hline
 \quad \quad 1 \ -1 \\
 \quad \quad \hline
 \quad \quad \quad 0 \ 1 \\
 \quad \quad \quad \quad 1 \ 0 \\
 \quad \quad \quad \quad \hline
 \quad \quad \quad \quad -1 \ 1
 \end{array}$$

No amount of shifting will ever save this dead end, so we back up and put another placeholder in the third column.¹

$$\begin{array}{r}
 5 \quad x \quad x \\
 \hline
 7 \) \ 7 \ 7 \\
 \quad 8 \ 5 \\
 \quad \hline
 \quad -1 \ 2 \\
 \quad \hline
 \quad \quad 1 \ -1 \\
 \quad \quad \hline
 \quad \quad \quad 0 \ 1 \\
 \quad \quad \quad \hline
 \quad \quad \quad \quad 1 \ 0
 \end{array}$$

Now we can terminate with the value 0.

$$\begin{array}{r}
 5 \quad x \quad x \quad 0 \\
 \hline
 7 \) \ 7 \ 7 \\
 \quad 8 \ 5 \\
 \quad \hline
 \quad -1 \ 2 \\
 \quad \hline
 \quad \quad 1 \ -1 \\
 \quad \quad \hline
 \quad \quad \quad 0 \ 1 \\
 \quad \quad \quad \hline
 \quad \quad \quad \quad 1 \ 0 \\
 \quad \quad \quad \quad \hline
 \quad \quad \quad \quad \quad 1 \ 0 \\
 \quad \quad \quad \quad \quad \hline
 \quad \quad \quad \quad \quad \quad 0 \ 0
 \end{array}$$

¹Despite this backing up, the algorithm is still greedy. Once the maximum value for a particular column is found, it is never again changed.

Subtracting the column numbers, we get $(5, 0) - (1, 4) = (4, -4)$, corresponding to the identities $7f_n = f_{n+4} + f_{n-4}$ and $7 = \phi^4 + \phi^{-4}$.

2.1. Termination. Before turning to the interpretation of the algorithm, it is worth saying a few words about termination. The algorithm alternates between making a sequence of shifts and outputting the next term. Bergman [2] shows that numbers can be finitely represented in the golden ratio base, so this alternating pattern can only repeat a finite number of times. Shifting, however, is problematic, since shifting continues an indefinite number of times until the two numbers involved are both positive. There is, however, an optimization that can be used to determine whether a sequence of shifts will terminate with a positive output. Consider starting with two numbers a and b . After one shift, this is $a + b$ and a . After two shifts, it is $2a + b$ and $a + b$, and after n shifts, it is $f_n a + f_{n-1} b$ and $f_{n-1} a + f_{n-2} b$. Since, in the limit, the ratio of f_n and f_{n-1} is ϕ , one can simply test whether $\phi a + b$ is positive, and if it is positive, the shifting will eventually terminate with two positive numbers, otherwise shifting will lead to a negative result.²

3. TWO INTERPRETATIONS OF THE ALGORITHM

We turn now to the interpretation of the algorithm. Not surprisingly, identities as in (1.13) can be understood combinatorially. However, since an irrational number is involved, identities as in (1.14) are more easily understood algebraically. Possibly the identities involving ϕ could be interpreted as counting probabilities as in [1], but it seems unlikely that it would lead to any extra insight.

3.1. Combinatorial Explanation of the Algorithm. There are numerous ways in which the algorithm can be understood. We present here an explanation based on *phased* square-domino tilings. A phased tiling is a tiling in which the initial tile is assigned a certain number of phases (or colors) depending on its type. The combinatorial explanation for the GRD algorithm is based on such phased tilings, starting from three combinatorial theorems.

Combinatorial Theorem 3.1. *mf_n counts the phased square domino tilings of length n with m phases for an initial square, and m phases for an initial domino.*

Proof. Every unphased tiling begins with a square or domino, so there is a one-to- m correspondence between the phased and unphased tilings. □

Combinatorial Theorem 3.2. *The number of phased tilings of length n with a phases for an initial square and b phases for an initial domino is equal to the number of phased tilings of length $n - 1$ with $a + b$ phases for an initial square and a phases for an initial domino.*

Proof. Let the phases for the length n tilings be labeled $s_1 \dots s_a$ for the squares and $d_1 \dots d_b$ for the dominoes. For the length $n - 1$ tilings, let the phases for squares be labeled $s_1 \dots s_{a+b}$ and let the phases for the dominoes be labeled $d_1 \dots d_a$. Then the initial tile sequences in

²This argument is only intended to show that the algorithm terminates, and not to show a practical optimization. In fact, the ϕ test makes no apparent difference with the author's computer implementation, and for paper and pencil calculation, the ϕ test is clearly undesirable.

the length n tilings can be mapped to initial tiles in the $n - 1$ tilings in the following way:

$$\begin{array}{ll}
 s_1s \rightarrow s_1 & s_1d \rightarrow d_1 \\
 \vdots & \vdots \\
 s_as \rightarrow s_a & s_ad \rightarrow d_a \\
 d_1 \rightarrow s_{a+1} & \\
 \vdots & \\
 d_b \rightarrow s_{a+b} . &
 \end{array}$$

The mapping in each case reduces the length by one, and is clearly reversible. □

Combinatorial Theorem 3.3. *The number of unphased square-domino tilings of length $n + m$ is equal to the number of phased tilings of length n with f_{m+1} phases for an initial square and f_m phases for an initial domino.*

Proof. Consider the case whether or not the unphased $n + m$ -tiling is breakable at cell $m + 1$. If it is breakable, then the tiling consists of a prefix of length $m + 1$ followed by a suffix of length $n - 1$. Since there are f_{m+1} possible tilings for the prefix, these may be mapped one-to-one onto a set of squares with f_{m+1} phases. Replacing the prefix of length $m + 1$ by the corresponding phased square reduces the length of the tiling to n .

If the unphased $n + m$ -tiling is not breakable at cell $m + 1$, then it consists of a prefix of length m , followed by a domino covering cells $m + 1$ and $m + 2$, and ending with a suffix of length $n - 2$. In this case there are f_m possible tilings for the prefix, and these may be mapped one-to-one onto a set of phases for the following domino. Replacing the prefix by the corresponding phase to be placed on the domino again reduces the length of the tiling to n . □

Combinatorial Theorem 3.2 is clearly the basis of the shift rule in the GRD algorithm, and Combinatorial Theorem 3.1 is the basis of the initialization step. So, when we write

$$4 \overline{)4 \quad 4}$$

the leftmost 4 represents the 4 sets of length n square-domino tilings. By Combinatorial Theorem 3.1, the number of such tilings is equal to the number of tilings of length n beginning with a four-phased square, as indicated by the middle 4, plus number of tilings of length n beginning with a four-phased domino, as indicated by the rightmost 4.

Now that we've represented the problem in terms of phased tilings, the issue is how to count these tilings. Clearly the idea is to use Combinatorial Theorem 3.3, but a problem arises here when the needed number of phases for squares and dominoes does not exactly match f_{i+1} and f_i , respectively, for some i . In this case, an *error term* or *remainder* needs to be introduced. To illustrate the problem, consider the following identity relating Lucas numbers to Fibonacci numbers.

Identity 3.4.

$$L_n = f_{n+1} - f_{n-1} + f_{n-2}.$$

Proof. L_n counts the phased tilings of length n with 1 phase for an initial square and 2 phases for an initial domino. By Combinatorial Theorem 3.3, f_{n+1} counts the number of phased tilings of length n with 2 phases for an initial square and 1 phase for an initial domino. So in this case, we need 2 error terms: $-f_{n-1}$ to compensate for overcounting the phases for squares, and f_{n-2} to compensate for undercounting the phases for dominoes. \square

The reader should be able to supply proofs for the following identities by following the same pattern:

Identity 3.5.

$$L_n = f_n + f_{n-2}.$$

Identity 3.6.

$$L_n = f_{n+2} - 2f_{n-1}.$$

Returning to our example with 4 phases for initial squares and 4 phases for initial dominoes, we may simulate these phases with a tiling of length $n + 2$. This is represented in the following way, where the number 3 on the top line represents the length of prefix of the tiling that is used to represent the various phases of an initial square:

$$\begin{array}{r} 3 \\ 4 \overline{)4 \ 4} \\ \underline{3 \ 2} \\ 1 \ 2 \end{array}$$

The remainder (1, 2) indicates that we have failed to count one of the phases for initial squares and two of the phases for initial dominoes.

So at this point, we have the identity

$$4f_n = f_{n+2} + remainder \tag{3.1}$$

where *remainder* is the number of tilings of length n with one phase for an initial square and two phases for an initial domino. By Combinatorial Theorem 3.2, we can equivalently count the tilings of length $n - 1$ with three phases for an initial square and one phase for an initial domino.

$$\begin{array}{r} 3 \\ 4 \overline{)4 \ 4} \\ \underline{3 \ 2} \\ \underline{1 \ 2} \\ \hline 3 \ 1 \end{array}$$

And then the process repeats with tilings now of length $n - 1$ instead of n .

3.2. Bergman-style Interpretation of the Algorithm. As indicated in Section 1, the algorithm can also be interpreted as calculating the coefficients of ϕ in (1.14). In this case, it is unclear how to provide a combinatorial proof, and so we rely instead upon known identities involving ϕ . In particular, we need the following identity.

Identity 3.7.

$$\phi^n = f_{n-1}\phi + f_{n-2}.$$

For example, $\phi^6 = 8\phi + 5$ is used below in the transition from line 3.3 to line 3.4, right column, and $\phi^4 = 3\phi + 2$ is used for the same pair of lines in the left column. As can be seen, this identity plays the role of Combinatorial Theorem 3.3 in the phased tiling interpretation.

It is possible to extend this identity for negative values of n [2], however for our purposes this extension will not be needed. One special case of this identity is particularly useful, as it will correspond to the shift rule, used below in the transition from 3.5 to 3.6 in both columns.

Identity 3.8.

$$\phi^2 = \phi + 1.$$

And now we have all we need to prove particular examples such as $4 = \phi^2 + \phi^0 + \phi^{-2}$ or $7 = \phi^4 + \phi^{-4}$.

$$4 = \frac{4\phi^2}{\phi^2} \qquad 7 = \frac{7\phi^2}{\phi^2} \qquad (3.2)$$

$$= \frac{4\phi + 4}{\phi^2} \qquad = \frac{7\phi + 7}{\phi^2} \qquad (3.3)$$

$$= \frac{\phi^4 + (\phi + 2)}{\phi^2} \qquad = \frac{\phi^6 + (-\phi + 2)}{\phi^2} \qquad (3.4)$$

$$= \frac{\phi^4 + \frac{\phi^2+2\phi}{\phi}}{\phi^2} \qquad = \frac{\phi^6 + \frac{-\phi^2+2\phi}{\phi}}{\phi^2} \qquad (3.5)$$

$$= \frac{\phi^4 + \frac{3\phi+1}{\phi}}{\phi^2} \qquad = \frac{\phi^6 + \frac{\phi-1}{\phi}}{\phi^2} \qquad (3.6)$$

$$= \frac{\phi^4 + \frac{\phi^3+\phi}{\phi}}{\phi^2} \qquad = \frac{\phi^6 + \frac{\phi^2-\phi}{\phi}}{\phi^2} \qquad (3.7)$$

$$= \phi^2 + \phi^0 + \phi^{-2}. \qquad = \frac{\phi^6 + \frac{1}{\phi}}{\phi^2} \qquad (3.8)$$

$$= \frac{\phi^6 + \frac{\phi^0}{\phi}}{\phi^2} \qquad (3.9)$$

$$= \phi^4 + \phi^{-4}. \qquad (3.10)$$

Clearly these derivations correspond point by point to the combinatorial arguments used to prove identities as in (1.13). To be precise, we represent an application of the GRD algorithm as a sequence of steps: *init- α* , *reduce- β_1* , *shift- k_1* , ..., *reduce- β_n* , *shift- k_n* , *evaluation*, where

$shift_{k_i}$ refers to a sequence of k_i shifts. The *init- α* step then corresponds to the following equation:

$$\alpha = \frac{\alpha\phi^2}{\phi^2} = \frac{\alpha\phi + \alpha}{\phi^2}. \quad (3.11)$$

We will refer to the numerator $\alpha\phi + \alpha$ in (3.11) as the *focus term*. In general, the focus term is the upper, rightmost term, as indicated in (3.12).

$$\frac{\phi^{\beta_1+1} + \frac{\phi^{\beta_2+1} + \frac{\dots}{\phi}}{\phi}}{\phi^2}. \quad (3.12)$$

Both the *reduce- β* rule and the *shift* rule apply just to the focus term. The following equivalence is used for *reduce- β_i*

$$a\phi + b = \phi^{\beta_i+1} + ((a - f_{\beta_i})\phi + (b - f_{\beta_i-1})). \quad (3.13)$$

And the *shift* rule applies the equivalence in (3.14) to the focus term.

$$\phi^k + (a\phi + b) = \phi^k + \frac{a\phi^2 + b\phi}{\phi} = \phi^k + \frac{(a+b)\phi + a}{\phi}. \quad (3.14)$$

Finally, the *evaluation* rule simply corresponds to using the law of exponents to reduce the fractions. There is a slight complication here in that the ϕ at the very bottom is raised to the power 2 instead of 1, but this is compensated for by the fact that each exponent β_i is increased by 1.

We now have all the pieces in place to prove Theorem 1.1.

Proof. We have seen that each application of the GRD algorithm corresponds to a pair of proofs: a tiling proof of an identity of the form $mf_n = f_{n+a_1} + f_{n+a_2} + \dots + f_{n+a_k}$ and an algebraic proof of $m = \phi^{a_1} + \phi^{a_2} + \dots + \phi^{a_k}$. Both of the correspondences are clearly reversible, so that there is a one-to-one correspondence between proofs of $mf_n = f_{n+a_1} + f_{n+a_2} + \dots + f_{n+a_k}$ and proofs of $m = \phi^{a_1} + \phi^{a_2} + \dots + \phi^{a_k}$. \square

4. DISCUSSION

The primary result in this paper is the proof of Theorem 1.1. Thus, it is shown that there is a one-to-one correspondence between proofs of $mf_n = f_{n+a_1} + f_{n+a_2} + \dots + f_{n+a_k}$ and proofs of $m = \phi^{a_1} + \phi^{a_2} + \dots + \phi^{a_k}$. There are, moreover, two bonus results. First, instances of these equations are efficiently generated using the GRD algorithm. And second, the proofs of the first identity type (the *Zeckendorf family identities*), are combinatorial, being based on counting phased tilings.

A recent paper covering some of the same territory is that of Wood, who applies the techniques described in [5] for automatically translating inductive proofs into enumerative combinatorial proofs. This approach was successfully applied to generate proofs for the identities (1-12). The approach is certainly ingenious, but also has the disadvantage that

the machine-generated combinatorial proofs are exceedingly complex and unintuitive.³

This paper ends with an appendix with a small number of examples. It is, however, trivial to generate a couple million more examples (interested readers should contact the author). Such a list of examples is quite useful for exploring subfamilies of the Zeckendorf family identities. There are quite a few identities that can be discovered and proved where m in mf_n is, to mention a few striking special cases, a Fibonacci number, a Lucas number, five times a Fibonacci number or two times a Lucas number. As one example of an empirically discoverable pattern, we leave the reader with the following identities (closely related to A056854 in the Online Encyclopedia of Integer Sequences).

$$\begin{aligned}
 8f_n &= f_{n+4} + f_n + f_{n-4} \\
 48f_n &= f_{n+8} + f_n + f_{n-8} \\
 323f_n &= f_{n+12} + f_n + f_{n-12} \\
 2208f_n &= f_{n+16} + f_n + f_{n-16} \\
 15128f_n &= f_{n+20} + f_n + f_{n-20} \\
 103683f_n &= f_{n+24} + f_n + f_{n-24}.
 \end{aligned}$$

APPENDIX A. EXAMPLES

$$1f_n = f_n$$

$$\begin{array}{r}
 1 \\
 1 \overline{)1 \ 1} \\
 \underline{1 \ 1} \\
 0 \ 0
 \end{array}$$

$$2f_n = f_{n+1} + f_{n-2}$$

$$\begin{array}{r}
 2 \ 0 \\
 2 \overline{)2 \ 2} \\
 \underline{2 \ 1} \\
 1 \\
 \underline{1} \\
 0
 \end{array}$$

$$3f_n = f_{n+2} + f_{n-2}$$

$$\begin{array}{r}
 3 \ 0 \\
 3 \overline{)3 \ 3} \\
 \underline{3 \ 2} \\
 1 \\
 \underline{1} \\
 0
 \end{array}$$

³Part of the extra complexity of Wood's approach is due to the fact that his proofs specify a bijection, whereas the proofs in this paper only show that sets of tilings are of equal size without specifying which tiling is mapped to which other tiling. To turn the proofs in this paper into bijections would require imposing an ordering on tilings, for example using the approach specified in [3].

- [5] P. M. Wood and D. Zeilberger, *A Translation Method for Finding Combinatorial Bijections*, Annals of Combinatorics (to appear).

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