A SMOOTH TIGHT UPPER BOUND FOR THE FIBONACCI REPRESENTATION FUNCTION $R(N)$

PAUL K. STOCKMEYER

ABSTRACT. The function $R(n)$ that counts the number of representations of the integer n as the sum of distinct Fibonacci numbers has been studied for over 40 years, and many fascinating properties have been discovered. In this paper we prove that $R(n) \le \sqrt{n+1}$ for all $n \ge 0$, with equality if and only if $n = F_m^2 - 1$ for some integer $m \ge 2$.

1. INTRODUCTION AND STATEMENT

The observation by Zeckendorf that every positive integer has a unique representation as the sum of distinct nonconsecutive Fibonacci numbers (see [13] and [15]) has lead to many investigations of the function $R(n)$ that counts all representations of the natural number n as a sum of distinct elements from the sequence $\{F_2, F_3, \ldots\}$. Methods for computing $R(n)$ can be found, for example, in [2, 4, 5, 6, 7, 9, 10, 14]. We will use the characterization of $R(n)$ first presented by Klarner in [12]. For all $m \ge 2$ we have

$$R(F_m - 1) = 1, \tag{1}$$

and for $F_m \le n < F_{m+1} - 1,\ m \ge 4$, we have

$$R(n) = R(n - F_m) + R(F_{m+1} - n - 2). \tag{2}$$

We note that by convention we include the case $R(0) = 1$ in equation 1. In equation 2, the first term on the right counts those representations that use the number F_m, while the second term counts those that exclude F_m. Here are the first few values of $R(n)$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$R(n)$	1	1	1	2	1	2	2	1	3	2	2	3	1	3	3	2	4	2	3	3	1

The reader is encouraged to extend this table further, using equations 1 and 2.

The main result of this paper is the presentation of a smooth, tight upper bound for $R(n)$.

Theorem 1. *The representation function satisfies the bound $R(n) \le \sqrt{n+1}$ for all $n \ge 0$, with equality if and only if $n = F_m^2 - 1$ for some integer $m \ge 2$.*

The proof of Theorem 1 is postponed until after we have developed the necessary tools.

2. TOOLS

We will need several combinatorial identities involving Fibonacci and Lucas numbers, in the forms listed below. All can be obtained as minor rewritings or variations of identities found in the book [1] by Benjamin and Quinn.

Identity 1 (Identity 50 in [1]). $F_{m-1} + 2F_{m+2} = L_{m+2}$ for all $m \ge 1$.

Identity 2 (Identity 65 in [1]). $F_{m-1}^2 + 4F_mF_{m+1} = F_{m+2}^2$ for all $m \ge 1$.

Identity 3 (Identity 53 in [1]). $L_{m+2}^2 - 5F_{m+2}^2 = 4(-1)^m$ for all $m \ge 0$.

Identity 4 (Identities 36 and 53 in [1]). $L_{2m+2} = 5F_{m+1}^2 - 2(-1)^m$ for all $m \geq 0$.

Identity 5 (Identity 14 in [1]). $F_m^2 = F_{2m-2} + F_{m-2}^2 > F_{2m-2}$ for all $m \geq 3$.

Identity 6 (Identity 13 in [1]). $F_m^2 = F_{2m-1} - F_{m-1}^2 < F_{2m-1}$ for all $m \geq 2$.

In addition to these identities, we need the following powerful inequality.

Lemma 2. *For all integers n satisfying $F_m - 1 \leq n \leq F_{m+1} - 1$ with $m \geq 3$ we have*

$$\sqrt{(n+1) - F_m} + \sqrt{F_{m+1} - (n+1)} \leq \sqrt{(n+1)}.$$

Equality holds only for $n = F_{k+1}^2 - 1$, where m is even and $k = m/2$.

Proof. Setting $x = n + 1$, we consider the real function

$$g(x) = \sqrt{x} - \sqrt{x - F_m} - \sqrt{F_{m+1} - x}.$$

At the extremes of the domain of g we have $g(F_m) = \sqrt{F_m} - \sqrt{F_{m-1}} > 0$ and $g(F_{m+1}) = \sqrt{F_{m+1}} - \sqrt{F_{m-1}} > 0$.

We seek places where $g(x) = 0$. Setting

$$\sqrt{x} = \sqrt{x - F_m} + \sqrt{F_{m+1} - x},$$

then squaring both sides and simplifying, we get

$$x - F_{m-1} = 2\sqrt{(x - F_m)(F_{m+1} - x)}.$$

Squaring and simplifying again yields

$$5x^2 - 2x(F_{m-1} + 2F_{m+2}) + \left(F_{m-1}^2 + 4F_mF_{m+1}\right) = 0,$$

and Identities 1 and 2 simplify this equation to

$$5x^2 - 2xL_{m+2} + F_{m+2}^2 = 0.$$

The solutions are

$$x = \frac{L_{m+2} \pm \sqrt{L_{m+2}^2 - 5F_{m+2}^2}}{5},$$

or, using Identity 3,

$$x = \frac{L_{m+2} \pm \sqrt{4(-1)^m}}{5}.$$

Clearly there are no real solutions for m odd, so the function $g(x)$ is strictly positive over its entire domain, proving the lemma for that case. For m even we set $k = m/2$ and use Identity 4 to obtain the two roots $x_1 = F_{k+1}^2$ and $x_2 = F_{k+1}^2 - (\frac{4}{5})(-1)^k$. It follows that $g(x)$ is negative only strictly between these two roots. But as there are no integers in this open interval, we have that $g(x)$ is non-negative for all integers in its domain. Thus the lemma is true in this case as well, with equality if and only if $n = x_1 - 1 = F_{k+1}^2 - 1$. ☐

It is perhaps interesting to observe what happens if we replace F_m and F_{m+1} with their approximations $\phi^m/\sqrt{5}$ and $\phi^{m+1}/\sqrt{5}$ in the definition of $g(x)$, where $\phi = \left(1 + \sqrt{5}\right)/2$. Instead of $g(x)$ having two nearby complex zeros when m is odd and two nearby real zeros when m is even, in this case $g(x)$ is always non-negative with a double zero at $x = \phi^{m+2}/5$. The proof is left to the reader.

3. The Proof

Proof of Theorem 1. For $n = F_m - 1$ with $m \geq 2$, equation 1 gives $R(n) = 1 \leq \sqrt{n+1}$, with equality exactly at $n = 0 = F_2^2 - 1$. Thus the theorem is true for integers of this form.

For other values of n we proceed by induction. Suppose n satisfies $F_m \leq n < F_{m+1} - 1$ for some $m \geq 4$, and that the theorem is true for all integers less than n. Then

$$\begin{aligned} R(n) &= R(n - F_m) + R(F_{m+1} - n - 2) \\ &\leq \sqrt{(n+1) - F_m} + \sqrt{F_{m+1} - (n+1)} \\ &\leq \sqrt{n+1}, \end{aligned}$$

where we have used equation 2, the induction hypothesis, and Lemma 1.

A necessary condition for equality here is equality in Lemma 1, i.e., n must be of the form $n = F_m^2 - 1$. To see that this condition is also sufficient, we suppose that $n = F_m^2 - 1$ for some $m \geq 3$, and that equality holds in the theorem for all smaller integers of this form. Now Identities 5 and 6 imply that $F_{2m-2} \leq n < F_{2m-1} - 1$, so we have

$$\begin{aligned} R(n) &= R(n - F_{2m-2}) + R(F_{2m-1} - n - 2) \\ &= R(F_m^2 - 1 - F_{2m-2}) + R(F_{2m-1} - F_m^2 - 1) \\ &= R(F_{m-2}^2 - 1) + R(F_{m-1}^2 - 1) \\ &= F_{m-2} + F_{m-1} \\ &= F_m \\ &= \sqrt{n+1}, \end{aligned}$$

where we have used equation 1, the definition of n, Identities 5 and 6, and the induction hypothesis. Thus equality holds exactly when n is of the designated form. □

Several papers (see, for example, [3, 8, 11]) have studied the function $A(n) = \min\{k : R(k) = n\}$, a minimal inverse of $R(n)$. Here are the first few values of $A(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$A(n)$	1	3	8	16	24	37	58	63	97	105	152	160	168	249	257	270	406	401

The following result is an immediate consequence of Theorem 1.

Corollary 1. *The function $A(n)$ satisfies the bound $A(n) \geq n^2 - 1$ for all $n \geq 1$, with equality if and only if n is a Fibonacci number.*

References

[1] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count–The Art of Combinatorial Proof*, Dolciani Mathematical Expositions, no. 27, Mathematical Association of America, Washington, DC, 2003.

[2] J. Berstel, *An Exercise on Fibonacci Representations*, Theor. Inform. Appl., **35.6** (2001), 491–498.

[3] M. Bicknell-Johnson, *The Smallest Positive Integer Having F_i Representations as a Sum of Distinct Fibonacci Numbers*, Applications of Fibonacci Numbers, **8** (Rochester, NY, 1998) (Dordrecht), Kluwer Acad. Publ., 1999, 47–52.

[4] M. Bicknell-Johnson, *The Zeckendorf-Wythoff Array Applied to Counting the Number of Representations of N as Sums of Distinct Fibonacci Numbers*, Applications of Fibonacci Numbers, **8** (Rochester, NY, 1998) (Dordrecht), Kluwer Acad. Publ., 1999, 53–60.

[5] M. Bicknell-Johnson, *The Least Integer Having p Fibonacci Representations, p Prime*, The Fibonacci Quarterly, **40.3** (2002), 260–265.

[6] M. Bicknell-Johnson, *Stern's Diatomic Array Applied to Fibonacci Representations*, The Fibonacci Quarterly, **41.2** (2003), 169–180.
[7] M. Bicknell-Johnson and D. C. Fielder, *The Number of Representations of N Using Distinct Fibonacci Numbers, Counted by Recursive Formulas*, The Fibonacci Quarterly, **37.1** (1999), 47–60.
[8] M. Bicknell-Johnson and D. C. Fielder, *The Least Number Having 331 Representations as a Sum of Distinct Fibonacci Numbers*, The Fibonacci Quarterly, **39.5** (2001), 455–461.
[9] L. Carlitz, *Fibonacci Representations*, The Fibonacci Quarterly, **6.4** (1968), 193–220.
[10] D. A. Englund, *An Algorithm for Determining $R(N)$ from the Subscripts of the Zeckendorf Representation of N*, The Fibonacci Quarterly, **39.3** (2001), 250–252.
[11] D. C. Fielder and M. Bicknell-Johnson, *The First 330 Terms of Sequence A013583*, The Fibonacci Quarterly, **39.1** (2001), 75–84.
[12] D. A. Klarner, *Representations of N as a Sum of Distinct Elements from Special Sequences.*, The Fibonacci Quarterly, **4.4** (1966), 289–306, 322.
[13] C. G. Lekkerkerker, *Representation of Natural Numbers as a Sum of Fibonacci Numbers*, Simon Stevin, **29** (1952), 190–195.
[14] N. Robbins, *Fibonacci Partitions*, The Fibonacci Quarterly, **34.4** (1996), 306–313.
[15] E. Zeckendorf, *Représentation des Nombres Naturels par une Somme de Nombres de Fibonacci ou de Nombres de Lucas*, Bull. Soc. Roy. Sci Liege, **41** (1972), 179–182.

MSC2000: 11B39, 11B34, 11D85

Department of Computer Science, The College of William and Mary, PO Box 8795, Williamsburg, Virginia 23187-8795
E-mail address: stockmeyer@cs.wm.edu