

# RICCATI MEETS FIBONACCI

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## 1. INTRODUCTION

Consider the *Riccati* differential equation

$$f(x)G'(x) = f_0(x)G^2(x) + f_1(x)G(x) + f_2(x). \quad (1)$$

For  $f_2(x) \equiv 0$  this reduces to a special *Bernoulli* equation (with exponent 2) which will be treated separately. For the history of such eqs. see [9], ch.I, 1.1. If  $f_0(x)$  does not vanish we speak of the non-degenerate case, and

$$G^2(x) = \alpha(x)G'(x) - \beta(x)G(x) - \gamma(x), \quad (2)$$

with  $\alpha(x) = f(x)/f_0(x)$ ,  $\beta(x) = f_1(x)/f_0(x)$ , and  $\gamma(x) = f_2(x)/f_0(x)$ . Let  $G(x)$  generate the number sequence  $\{G_n\}_0^\infty$ , i.e.  $G(x) = \sum_{n=0}^\infty G_n x^n$ . Because  $G^2(x)$  is the generating function for the convolution of the sequence  $\{G_n\}_0^\infty$  with itself, i.e. of  $G_n^{(1)} := \sum_{k=0}^n G_k G_{n-k}$ , one can use eq. 2 in order to express the convolution numbers  $G_n^{(1)}$  in terms of  $\{G_k\}_0^{n+1}$  and the numbers  $\{\alpha_k\}_0^n$ ,  $\{\beta_k\}_0^n$ , and  $\gamma_n$ , which are generated by the functions  $\alpha(x)$ ,  $\beta(x)$ , and  $\gamma(x)$ , respectively, as follows.

$$\begin{aligned} G_n^{(1)} &= \sum_{q=0}^n ((n+1-q)G_{n+1-q}\alpha_q - G_{n-q}\beta_q) - \gamma_n, \\ &= \sum_{q=0}^n ((q+1)G_{q+1}\alpha_{n-q} - G_q\beta_{n-q}) - \gamma_n. \end{aligned} \quad (3)$$

The  $k^{th}$  order convolution sequence  $\{G_n^{(k)}\}_{n=0}^\infty$  is generated by  $G^{k+1}(x)$ , and can be obtained recursively if one first writes  $G^{k+1}(x) = G^{k-1}(x)G^2(x)$  and then employs *Riccati* eq. 2:

$$G^{k+1}(x) = \left( \alpha(x) \frac{1}{k} \frac{d}{dx} - \beta(x) \right) G^k(x) - \gamma(x)G^{k-1}(x) \quad (4)$$

for  $k \in \mathbb{N}$ . This yields in terms of the expansion coefficients, from  $G^{k+1}(x) =: \sum_{n=0}^\infty G_n^{(k)} x^n$ ,

$$G_n^{(k)} = \sum_{q=0}^n \left( \frac{1}{k} (q+1) G_{q+1}^{(k-1)} \alpha_{n-q} - G_q^{(k-1)} \beta_{n-q} - G_q^{(k-2)} \gamma_{n-q} \right) \quad (5)$$

for  $k \in \mathbb{N}$ , with  $G_q^{(-1)} := \delta_{q,0}$  (*Kronecker* symbol) and  $G_q^{(0)} = G_q$ .

As is well-known, *Riccati* eq. 2 can be transformed into a homogeneous second order differential equation of the type (we use  $\alpha(x) \neq 0$ )

$$\alpha(x)H''(x) + (\alpha'(x) - \beta(x))H'(x) + (\gamma(x)/\alpha(x))H(x) = 0. \quad (6)$$

This transformation is accomplished by

$$G(x) = -\alpha(x)(\ln H(x))' \quad \text{or} \quad H(x) = \exp\left(-\int \frac{G(x)}{\alpha(x)} dx\right). \quad (7)$$

Therefore, if a function  $H(x)$  satisfies the differential eq. of type 6 with certain initial conditions for  $H(0)$  and  $H'(0)$  we can use recursion eq. 5 for the  $k^{\text{th}}$  convolution of the sequence  $\{G_n\}_0^\infty$  generated by  $G(x) = -\alpha(x)(\ln H(x))'$ , and  $\alpha(x), \beta(x)$ , and  $\gamma(x)$  generate the coefficients in eq. 5.

In the special *Bernoulli* case, when  $\gamma(x) \equiv 0$ , and  $f(x) \neq 0$  and  $\alpha(x) \neq 0$ , the eq.

$$G'(x) = \frac{f_1(x)}{f(x)}G(x) + \frac{f_0(x)}{f(x)}G^2(x) = (\beta(x)G(x) + G^2(x))/\alpha(x), \quad (8)$$

can be transformed into an inhomogeneous first order linear differential eq. for the inverse of  $G(x)$ ; i.e.  $H(x) := 1/G(x)$  satisfies

$$\alpha(x)H'(x) + \beta(x)H(x) = -1, \quad (9)$$

with the solution

$$H(x) = \frac{1}{G(x)} = e^{F(x)} \left[ C - \int \frac{e^{-F(x)}}{\alpha(x)} dx \right], \quad (10)$$

where  $C$  is an integration constant, and  $F(x) := -\int(\beta(x)/\alpha(x))dx$ .

Therefore, if  $H(x)$  satisfies a differential equation of type 9 with a certain initial condition for  $H(0)$  we can use recursion eq. 5, with  $\gamma_{n-q} \equiv 0$ , for the  $k^{\text{th}}$  convolution of the sequence  $\{G_n\}_0^\infty$  generated by  $G(x) = 1/H(x)$ .  $\alpha(x)$  and  $\beta(x)$  generate the remaining coefficients in eq. 5.

From this set-up we do not gain direct information about convolutions of the sequence of numbers generated by the functions  $H(x)$  in both cases. This method becomes particularly useful if the coefficient functions  $\alpha(x), \beta(x)$  and  $\gamma(x)$  are simple, for example if they are polynomials.

In this paper we concentrate on examples of *Riccati* equation 8 of the special *Bernoulli* type. It is shown that the generalized *Fibonacci* and corresponding *Lucas* numbers are generated by functions which satisfy such a *Riccati* equation. We discuss the resulting expressions for the  $k^{\text{th}}$  convolution of these number sequences. At the end we extend this method to the so-called generalized *p-Fibonacci* numbers which appeared in a recent paper [6].

## 2. SUMMARY

The generating function for the generalized *Fibonacci* numbers  $\{F_n(a, b)\}_0^\infty$ , defined by the three term recurrence relation

$$F_n(a, b) = aF_{n-1}(a, b) + bF_{n-2}(a, b), \quad F_0(a, b) = 0, \quad F_1(a, b) = 1, \quad (11)$$

with given real  $a \neq 0$  and  $b \neq 0$ , is well-known. For arbitrary  $a$  and  $b$ ,  $F_n(a, b)$  can be considered as a polynomial in two variables. If we introduce the numbers, or polynomials,  $U_n(a, b) := F_{n+1}(a, b)$  we have from the recursion with input  $U_0(a, b) = 1$  and  $U_1(a, b) = a$  (or  $U_{-1}(a, b) = 0$ )

$$U(a, b; x) := \sum_{n=0}^{\infty} U_n(a, b)x^n = \frac{1}{1 - ax - bx^2}. \quad (12)$$

Similarly, for the generalized *Lucas* numbers  $\{L_n(a, b)\}_0^\infty$  which satisfy the same recursion eq. 11 but with inputs  $L_0(a, b) = 2, L_1(a, b) = a$ , we find, with  $V_n(a, b) := L_{n+1}(a, b)/a$ , remembering that  $a \neq 0$ ,

$$V(a, b; x) := \sum_{n=0}^{\infty} V_n(a, b)x^n = \frac{1 + 2bx/a}{1 - ax - bx^2}. \quad (13)$$

The input is now  $V_0(a, b) = 1$  and  $V_1(a, b) = (a^2 + 2b)/a$  (or  $V_{-1}(a, b) = 2/a$ ).

These (ordinary) generating functions can also be written in terms of the characteristic roots corresponding to recursion relation eq. 11

$$\lambda_{\pm} \equiv \lambda_{\pm}(a, b) := \frac{1}{2}(a \pm \sqrt{a^2 + 4b}) \quad (14)$$

as follows.

$$U(a, b; x) = \frac{1}{x(\lambda_+ - \lambda_-)} \left( \frac{1}{1 - \lambda_+x} - \frac{1}{1 - \lambda_-x} \right), \quad (15)$$

$$V(a, b; x) = \frac{1}{(\lambda_+ + \lambda_-)} \left( \frac{\lambda_+}{1 - \lambda_+x} + \frac{\lambda_-}{1 - \lambda_-x} \right). \quad (16)$$

The corresponding *Binet* forms of the generated numbers are, in the non-degenerate case  $\lambda_+ \neq \lambda_-$ , i.e.  $D(a, b) := a^2 + 4b \neq 0$ ,

$$U_n(a, b) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}, \quad (17)$$

$$V_n(a, b) = \frac{\lambda_+^{n+1} + \lambda_-^{n+1}}{\lambda_+ + \lambda_-}. \quad (18)$$

In the degenerate case we have

$$U_n(a) := U_n \left( a, -\frac{a^2}{4} \right) = (n+1) \left( \frac{a}{2} \right)^n, \quad (19)$$

$$V_n(a) := V_n \left( a, -\frac{a^2}{4} \right) = \left( \frac{a}{2} \right)^n. \quad (20)$$

A sum representation of these polynomials is obtained by expanding the generating functions.

$$U_n(a, b) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} a^{n-2l} b^l, \quad (21)$$

$$V_n(a, b) = \sum_{l=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n+1-l} \binom{n+1-l}{l} a^{n-2l} b^l. \quad (22)$$

This result for  $U_n(a, b)$  follows also from a combinatorial interpretation of the recurrence relation, and the one for  $V_n(a, b)$  is also due to the *Girard-Waring* formula in its simplest version (for this cf. [4], [3], also for original refs.).

The generating functions eq. 12 (or eq. 15) and eq. 13 (or eq. 16) are found to be the unique solutions of *Riccati* eqs. (simultaneously a special type of *Bernoulli* eq.) of the type shown in eq. 8. To be precise we have, identically in  $a$  and  $b$ ,

$$(a + 2bx) \frac{\partial}{\partial x} U(a, b; x) + 4bU(a, b; x) - (a^2 + 4b)U^2(a, b; x) = 0, \quad (23)$$

with the initial condition  $U(a, b; 0) = 1$ . Similarly,

$$\left(1 + 2\frac{b}{a}x\right)^2 \frac{\partial}{\partial x} V(a, b; x) + 2\frac{b}{a}\left(1 + 2\frac{b}{a}x\right)V(a, b; x) - \left(a + 4\frac{b}{a}\right)V^2(a, b; x) = 0, \quad (24)$$

with the initial condition  $V(a, b; 0) = 1$ .

Hence the coefficient functions from eq. 9 are at most first degree polynomials, namely  $\alpha(x) \equiv \alpha(a, b; x) = (a + 2bx)/(a^2 + 4b)$  and  $\beta(x) \equiv \beta(a, b; x) = -4b/(a^2 + 4b)$  in the *Fibonacci* case, and  $\alpha(x) \equiv \alpha(a, b; x) = (1 + 2bx/a)^2/(a + 4b/a)$  and  $\beta(x) \equiv \beta(a, b; x) = -2(b/a)(1 + 2bx/a)/(a + 4b/a)$  in the *Lucas* case, provided  $a \neq 0$  and  $a^2 + 4b \neq 0$ .

The degenerate case  $D(a, b) := a^2 + 4b = 0$ , for which the above given differential eqs. become linear, will be considered separately. This case corresponds to vanishing  $f_0(x)$  in *section 1*.

From the general results given in *section 1* the generating functions for the  $k^{\text{th}}$  convolution of these sequences satisfy

$$U^{k+1}(a, b; x) = \frac{1}{(a^2 + 4b)k} \left( (a + 2bx) \frac{\partial}{\partial x} + 4kb \right) U^k(a, b; x), \quad (25)$$

and

$$V^{k+1}(a, b; x) = \frac{a}{(a^2 + 4b)k} \left( \left(1 + 2\frac{b}{a}x\right)^2 \frac{\partial}{\partial x} + 2k\frac{b}{a}\left(1 + 2\frac{b}{a}x\right) \right) V^k(a, b; x). \quad (26)$$

This implies, from eq. 5, that the  $k^{\text{th}}$  convolution  $U_n^{(k)}$ , defined by  $U^{k+1}(a, b; x) =:$

$\sum_{n=0}^{\infty} U_n^{(k)}(a, b)x^n$  can be expressed in terms of the  $k - 1^{\text{st}}$  one according to

$$U_n^{(k)}(a, b) = \frac{1}{k(a^2 + 4b)} \left( a(n + 1)U_{n+1}^{(k-1)}(a, b) + 2b(n + 2k)U_n^{(k-1)}(a, b) \right), \quad (27)$$

with input  $U_n^{(0)}(a, b) = U_n(a, b)$ , and similarly

$$\begin{aligned} V_n^{(k)}(a, b) = \frac{1}{ka(a^2 + 4b)} & \left( (n + 1)a^2V_{n+1}^{(k-1)}(a, b) + 2ab(2n + k)V_n^{(k-1)}(a, b) \right. \\ & \left. + 4b^2(n + k - 1)V_{n-1}^{(k-1)}(a, b) \right), \end{aligned} \quad (28)$$

with input  $V_n^{(0)}(a, b) = V_n(a, b)$ . The formula given in eq. 27 has been found earlier in [1] (p. 202, III and p. 213, eq. (30)) without using the defining *Riccati* eq. for  $U(a, b; x)$ . The notations have to be translated with the help of  $F_n^{(k)} \hat{=} U_n^{(k-1)}$ ,  $a_1 \hat{=} a$ , and  $a_2 \hat{=} b$ .

For example, the convolution of  $\{V_n(a, b)\}_0^\infty$  with itself ( $k = 1$ ) becomes, after use of recursion eq. 11

$$V_n^{(1)}(a, b) = \frac{1}{a(a^2 + 4b)} ([a^2(n+1) + 4bn] V_{n+1}(a, b) + 2baV_n(a, b)). \quad (29)$$

For  $a = b = 1$  one recovers well-known formulae for the first convolutions of ordinary *Fibonacci*, *resp. Lucas* numbers (e.g. [7], p. 183, eqs. (98) and (99) (with corrected  $L_{n-1} \rightarrow L_{n-i}$ )). To see this, observe that  $U_n^{(1)}(1, 1) = F_{n+2}^{(1)}$  and  $V_n^{(1)}(1, 1) = L_{n+2}^{(1)} - 4L_{n+2}$ .

$$F_n^{(1)} = U_{n-2}^{(1)}(1, 1) = \frac{1}{5} ((n-1)F_n + 2nF_{n-1}) = \frac{1}{5}(nL_n - F_n), \quad (30)$$

$$L_n^{(1)} = V_{n-2}^{(1)}(1, 1) + 4L_n = \frac{1}{5} ((5n-9)L_n + 2L_{n-1}) + 4L_n = (n+2)L_n + F_n. \quad (31)$$

We note, in passing, a sum representation of these convolutions obtained from the expansion of the generating functions which is valid for  $k \in \mathbb{N}_0$ .

$$U_n^{(k)}(a, b) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k+n-l}{k} \binom{n-l}{l} a^{n-2l} b^l, \quad (32)$$

$$V_n^{(k)}(a, b) = \sum_{p=0}^{\min(n, k+1)} 2^p \binom{k+1}{p} \sum_{l=p}^n \binom{n-l+k}{k} \binom{n-l}{l-p} a^{n-2l} b^l. \quad (33)$$

Before discussing iteration of recursion relations 27 and 28 we state results for the degenerate case  $D(a, b) := a^2 + 4b = 0$ . *Riccati* eqs. 23 and 24 collapse to linear differential eqs. for  $U(a; x) := U(a, -a^2/4; x)$  and  $V(a; x) := V(a, -a^2/4; x)$

$$\left(1 - \frac{a}{2}x\right) \frac{\partial}{\partial x} U(a; x) = aU(a; x), \quad U(a; 0) = 1, \quad (34)$$

$$\left(1 - \frac{a}{2}x\right) \frac{\partial}{\partial x} V(a; x) = \frac{a}{2}V(a; x), \quad V(a; 0) = 1. \quad (35)$$

For the last eq.  $x \neq 2/a$  was assumed. Because the solutions to these eqs. imply

$$\frac{\partial^2}{\partial x^2} U(a; x) = \frac{3}{2}a^2U^2(a; x), \quad \frac{\partial}{\partial x} V(a; x) = \frac{a}{2}V^2(a; x), \quad (36)$$

the corresponding first ( $k = 1$ ) convolutions of these numbers  $U_n(a) := U_n(a, -a^2/4)$  and  $V_n(a) := V_n(a, -a^2/4)$  are given by

$$U_n^{(1)}(a) = \frac{2}{3a^2}(n+2)(n+1)U_{n+2}(a), V_n^{(1)}(a) = \frac{2}{a}(n+1)V_{n+1}(a), \quad (37)$$

with eqs. 19 and 20.

In order to derive the result for the  $k^{\text{th}}$  convolution we start with identities which follow from the solutions of eqs. 34 and 35, namely

$$U^{k+1}(a; x) = \frac{2}{a^2 k(2k+1)} \frac{\partial^2}{\partial x^2} (U^k(a; x)), \quad (38)$$

$$V^{k+1}(a; x) = \frac{2}{ak} \frac{\partial}{\partial x} (V^k(a; x)). \quad (39)$$

These identities imply for the  $k^{\text{th}}$  convolutions

$$U_n^{(k)}(a) = \frac{2}{a^2 k(2k+1)} (n+2)(n+1)U_{n+2}^{(k-1)}(a), \quad (40)$$

$$V_n^{(k)}(a) = \frac{2}{ak} (n+1)V_{n+1}^{(k-1)}(a), \quad (41)$$

with inputs  $U_n^{(0)}(a) = U_n(a) = (n+1)(a/2)^n$  and  $V_n^{(0)}(a) = V_n(a) = (a/2)^n$ . See eqs. 19 and 20. The iteration of these eqs. yields the final result, which for  $k \in \mathbb{N}_0$ , and in the degenerate case  $b = -a^2/4$ , is

$$U_n^{(k)}(a) = \binom{n+2k+1}{2k+1} \left(\frac{a}{2}\right)^n, \quad (42)$$

$$V_n^{(k)}(a) = \binom{n+k}{k} \left(\frac{a}{2}\right)^n. \quad (43)$$

Thus  $V_n^{(2l+1)}(a) = U_n^{(l)}(a)$ , and it suffices to treat  $V_n^{(k)}(a)$ . For even  $a$  these are non-negative integer sequences. For  $n, k \in \mathbb{N}_0$ ,  $V_{n+k}^{(k)}(2l)$  constitutes a convolution triangle of numbers based on the  $k = 0$  column sequence  $V_n^{(0)}(2l) = l^n$  (powers of  $l$ ). See [5] for these triangles of numbers.

In the non-degenerate case recursion eq. 27 can be iterated in order to express the  $k^{\text{th}}$  convolution of  $U_n(a, b)$  as linear combination of these numbers according to

$$U_n^{(k)}(a, b) = \frac{1}{k!(a^2 + 4b)^k} (AU_{k-1}(a, b; n)(n+1)aU_{n+1}(a, b) + BU_{k-1}(a, b; n)(n+2)bU_n(a, b)), \quad (44)$$

with certain polynomials  $AU_{k-1}(a, b; n)$  and  $BU_{k-1}(a, b; n)$  of degree  $k - 1$  in the variable  $n$ , for arbitrary, but fixed,  $a \neq 0, b \neq 0$ , and  $b \neq -a^2/4$ .

The (mixed) recursion relations for these polynomials are deduced from eq. 27, and for  $k = 1, 2, \dots$ , they are

$$AU_k(a, b; n) = a^2(n+2)AU_{k-1}(a, b; n+1) + 2b(n+2(k+1))AU_{k-1}(a, b; n) + b(n+3)BU_{k-1}(a, b; n+1), \quad (45)$$

$$BU_k(a, b; n) = a^2(n+1)AU_{k-1}(a, b; n+1) + 2b(n+2(k+1))BU_{k-1}(a, b; n), \quad (46)$$

with inputs  $AU_0(a, b; n) = 1$  and  $BU_0(a, b; n) = 2$ .

In eqs. (26), *resp.* (27), of [1] one can find explicit results for  $U_n^{(k)}(a, b)$  for the instances  $k = 2$ , *resp.*  $k = 3$  (in eq. (26) of this ref. one has to multiply the *lhs* with  $2!$ , and in the second line of  $N$  of eq. (27) it should read  $B(2, n+1)$ ).

For the case  $a = 1 = b$  the triangles of the coefficients of these polynomials can be viewed under the nrs. A057995 and A057280 in [5]. For  $a = 2, b = 1$  see A058402 and A058403, and for  $a = 1, b = 2$  A073401 and A073402.

Similarly, iteration of recursion eq. 28 results, with the help of recursion eq. 11, in

$$V_n^{(k)}(a, b) = \frac{1}{k!a(a^2+4b)^k} (AV_k(a, b; n)V_{n+1}(a, b) + BV_k(a, b; n)V_n(a, b)), \quad (47)$$

with certain polynomials  $AV_k(a, b; n)$  and  $BV_k(a, b; n)$  of generic degree  $k$  in the variable  $n$ , for fixed  $a \neq 0, b \neq 0$ , with  $b \neq -a^2/4$ .

The (mixed) recursion relations for these polynomials are found from eq. 28, and for  $k = 1, 2, \dots$ , they are

$$AV_k(a, b; n) = a^2(n+1)AV_{k-1}(a, b; n+1) + 2b(2n+k)AV_{k-1}(a, b; n) + a(n+1)BV_{k-1}(a, b; n+1) + 4\frac{b}{a}(n+k-1)BV_{k-1}(a, b; n-1), \quad (48)$$

$$BV_k(a, b; n) = 2b(2n+k)BV_{k-1}(a, b; n) - 4b(n+k-1)BV_{k-1}(a, b; n-1) + ab(n+1)AV_{k-1}(a, b; n+1) + 4\frac{b^2}{a}(n+k-1)AV_{k-1}(a, b; n-1), \quad (49)$$

with inputs  $AV_0(a, b; n) = 0$  and  $BV_0(a, b; n) = a$ .

For  $a = 1 = b$  the triangles of coefficients of these polynomials in  $n$  can be found under the nrs. A061188 and A061189 in [5]. Observe that  $BV_1(1, 1; n)$  is accidentally of degree 0. For  $a = 2, b = 1$  see nrs. A062133 and A062134.

Motivated by a recent paper [6] we consider also the following generalized *p-Fibonacci* numbers  $U_n(p; a, b)$  defined for  $p \in \mathbb{N}_0$  by the generating function

$$U(p; a, b; x) := \frac{1}{1 - ax - bx^{p+1}} = \sum_{n=0}^{\infty} U_n(p; a, b)x^n. \quad (50)$$

Of course, we assume  $b \neq 0$  and also take  $a \neq 0$ . For  $p = 1$  these numbers reduce to the  $U_n(a, b)$  treated above, and for  $p = 0$  they become the powers  $(a + b)^n$ .  $U(p; 1, 1; x)$  appears in eq. 71 of [2]. The recursion relations are

$$U_n(p; a, b) = aU_{n-1}(p; a, b) + bU_{n-(p+1)}(p; a, b), \quad (51)$$

with inputs  $U_j(p; a, b) = a^j$  for  $j = 0, 1, \dots, p$ . In order to derive expressions for the  $k^{\text{th}}$  convolution of these  $p$ -Fibonacci numbers consider first the following *Riccati* eq. of type 8 satisfied by  $U(p; a, b; x)$  written for the non-degenerate case  $D(p; a, b) := (p+1)^{p+1}b + a(ap)^p \neq 0$  if  $p \in \mathbb{N}$ , and  $a + b \neq 0$  if  $p = 0$  (*i.e.* one puts  $(ap)^p = 1$  if  $p = 0$ ).

$$U^2(p; a, b; x) = \frac{1}{(p+1)^{p+1}b + a(ap)^p} \left\{ A_p(a, b; x) \frac{\partial}{\partial x} + b(p+1)^2 B_{p-1}(a; x) \right\} U(p; a, b; x), \quad (52)$$

with

$$A_p(a, b; x) = (ap)^p + b(p+1)x B_{p-1}(a; x), \quad (53)$$

$$B_{p-1}(a; x) = (p+1)^{p-1} \sum_{i=0}^{p-1} \left( \frac{ap}{p+1} \right)^i x^i = \frac{(p+1)^p - (apx)^p}{p+1 - apx}. \quad (54)$$

Hence, the coefficient functions  $\alpha(x)$ , *resp.*  $\beta(x)$  from the general set-up in *section 1* are polynomials of degree  $p$  *resp.*  $p-1$ , namely  $\alpha(x) \equiv \alpha_p(a, b; x) = A_p(a, b; x)/D(p; a, b)$  *resp.*  $\beta(x) \equiv \beta_p(a, b; x) = -b(p+1)^2 B_{p-1}(a; x)/D(p; a, b)$ , and  $\gamma(x) \equiv 0$ . For  $p = 0$  one has to use  $A_0(a, b; x) = 1$  and  $B_{-1}(a; x) = 0$ . For given non-vanishing  $a$  and  $b$  these polynomials  $A_p(a, b; x)$ , *resp.*  $B_{p-1}(a; x)$ , in the variable  $x$  of degree  $p$ , *resp.*  $p-1$ , have therefore the following explicit form.

$$A_p(a, b; x) = \sum_{m=0}^p A(a, b; p, m) x^m, \quad B_{p-1}(a; x) = \sum_{m=0}^{p-1} B(a; p-1, m) x^m, \quad (55)$$

with the coefficients

$$A(a, b; p, m) = \begin{cases} 0 & \text{if } m > p, \\ 1 & \text{if } m = 0 \text{ and } p = 0, \\ (ap)^p & \text{if } m = 0 \text{ and } p \geq 1, \\ b(p+1)^p \left( \frac{ap}{p+1} \right)^{m-1} & \text{if } m \geq 1, \end{cases} \quad (56)$$

$$B(a; p, m) = \begin{cases} 0 & \text{if } p < m \text{ or } p = -1, \\ (p+2)^p \left( \frac{a(p+1)}{p+2} \right)^m & \text{if } p \geq m \geq 0. \end{cases} \quad (57)$$



For  $a = 1 = b$  these triangles of coefficients can be viewed under the numbers A055858 and A055864 in [5] where further details may be found.

Even though we cannot compute the integral in the solution eq. 10 of the linear differential eq. 9, which is equivalent to *Riccati* eq. 52 for  $p \neq 0, 1$ ,  $H(x) = 1 - ax - bx^{p+1}$  is the unique solution due to the existence and uniqueness theorem for the linear first order differential eq. 9 with initial value  $H(p; a, b; 0) = 1$ .

The result for the first ( $k = 1$ ) convolution of the numbers  $U_n(p; a, b)$  which flows from *Riccati* eq. 52 is

$$U_n^{(1)}(p; a, b) = \frac{1}{b(p+1)^{p+1} + a(ap)^p} \sum_{j=0}^p C_j(n; p; a, b) U_{n+1-j}(p; a, b), \quad (58)$$

with

$$C_j(n; p; a, b) = \begin{cases} n+1 & \text{if } p=0=j, \\ (n+1)(ap)^p & \text{if } p \geq 1 \text{ and } j=0, \\ b(p+1)^p(n+p+2-j) \left(\frac{ap}{p+1}\right)^{j-1} & \text{if } p \geq 1 \text{ and } j=1, \dots, p. \end{cases} \quad (59)$$

The  $U_n(p; a, b)$  recursion cannot be used to simplify the sum in eq. 58.

This result can now be compared, after putting  $a = 1 = b$ , with a different formula for the same convolution found in [6], eq. (14). For given  $p \in \mathbb{N}_0$  and  $k = 2, 3, \dots$ , the recursion for  $F_p^{(2)}(k) \hat{=} U_{k-2}^{(1)}(p; 1, 1)$  in [6] involves all  $k-1$  terms  $F_p(n) \hat{=} U_{n-1}(p; 1, 1)$ , for  $n = 1, \dots, k-1$ , whereas our result needs only  $p+1$  terms for all  $k$ . For example,  $F_3^{(2)}(7) \hat{=} U_5^{(1)}(3; 1, 1)$  is reduced to six terms involving  $F_3(1) \hat{=} U_0(3; 1, 1), \dots, F_3(6) \hat{=} U_5(3; 1, 1)$  in [6], but only to four terms, involving  $U_8(3; 1, 1) \hat{=} F_3(9), U_7(3; 1, 1) \hat{=} F_3(8), \dots, U_5(3; 1, 1) \hat{=} F_3(6)$  in eq. 58.

For the  $k^{\text{th}}$  convolution we use eq. 4 with  $\gamma(x) \equiv 0$  and the above given functions  $\alpha_p(a, b; x)$  and  $\beta_p(a, b; x)$ . For the non-degenerate case, and for  $k \in \mathbb{N}$ , we have

$$U^{k+1}(p; a, b; x) = \frac{1}{k(b(p+1)^{p+1} + a(ap)^p} \left\{ A_p(a, b; x) \frac{\partial}{\partial x} + kb(p+1)^2 B_{p-1}(a; x) \right\} U^k(p; a, b; x). \quad (60)$$

For  $p \in \mathbb{N}_0$  the corresponding recursion relation for the  $k^{\text{th}}$  convolution is (remember that we put  $(ap)^p = 1$  if  $p = 0$ )

$$U_n^{(k)}(p; a, b) = \frac{1}{k(b(p+1)^{p+1} + a(ap)^p) \sum_{j=0}^p C_j^{(k)}(n; p; a, b) U_{n+1-j}^{(k-1)}(p; a, b). \quad (61)$$

with

$$C_j^{(k)}(n; p; a, b) = \begin{cases} n+1 & \text{if } p=0=j, \\ (n+1)(ap)^p & \text{if } p \geq 1 \text{ and } j=0, \\ b(p+1)^p(n+1+k(p+1)-j) \left(\frac{ap}{p+1}\right)^{j-1} & \text{if } p \geq 1 \text{ and } j=1, \dots, p. \end{cases} \quad (62)$$

Instead of showing the rather unwieldy formula for the iteration of this recursion relation, after employing the fundamental recursion eq. 51, we prefer to state the result for the instance  $p=2, k=2, a=1=b$ , with the notation  $U_n^{(1)}(2; 1, 1) \equiv U_n^{(1)}(2)$  and  $U_n(2; 1, 1) \equiv U_n(2)$ :

$$\begin{aligned} U_n^{(2)}(2) &= \frac{1}{2 \cdot 31} \left( 4(n+1)U_{n+1}^{(1)}(2) + 9(n+6)U_n^{(1)}(2) + 6(n+5)U_{n-1}^{(1)}(2) \right) \\ &= \frac{1}{2 \cdot 31^2} \left( (217n^2 + 1425n + 1922)U_n(2) + 2(n+2)(62n+305)U_{n-1}(2) + \right. \\ &\quad \left. 4(n+1)(31n+143)U_{n-2}(2) \right). \end{aligned} \quad (63)$$

The recursion relation eq. 51 has been used twice.

In the degenerate case  $D(p; a, b) := (p+1)^{p+1}b + a(ap)^p = 0$  (where we put  $(ap)^p \equiv 1$  if  $p=0$ ) we find for  $U(p; a, b = b(p; a); x) =: U(p; a; x)$ , where  $b(p; a) := -p^p(a/(p+1))^{p+1}$ , the linear differential eq.

$$\left\{ A_p(a, b(p; a); x) \frac{\partial}{\partial x} + b(p; a)(p+1)^2 B_{p-1}(a; x) \right\} U(p; a; x) = 0 \quad (64)$$

with  $B_{p-1}$  and  $A_p$  taken in explicit form known from eqs. 55 with 57 and 56. For general  $p$  and  $D(p; a, b) = 0$  we cannot say anything about convolutions because we have no suitable expression for  $U^2(p; a, b; x)$ . Recurrence eq. 51 with depth  $p+1$  can be replaced by one with only depth  $p$ . See eq. 65.

In the non-degenerate case we could also consider the other  $p$  linear independent (*Lucas*-type) sequences defined by recurrence eq. 51 with appropriate inputs, but we will not do this here.

The remainder of this paper provides proofs for the above given statements.

### 3. RICCATI EQUATIONS FOR FIBONACCI AND LUCAS GENERATING FUNCTIONS

**Proposition 1:**  $U(a, b; x)$  defined in eq. 12 is for  $a^2 + 4b \neq 0$  equivalent to *Riccati* eq. 23 with initial condition  $U(a, b; 0) = 1$ .

**Proof:** a)  $H(a, b; x) = 1/U(a, b; x) = 1 - ax - bx^2$  satisfies eq. 9 with  $\alpha(x) \equiv \alpha(a, b; x) = (a + 2bx)/(a^2 + 4b)$  and  $\beta(x) \equiv \beta(a, b; x) = -4b/(a^2 + 4b)$ . Therefore,  $U(a, b; x)$  obeys eq. 8 with coincides with eq. 23.

b) With  $\alpha(a, b; x)$  and  $\beta(a, b; x)$  from eq. 23, as given in part a) we can compute the integral in eq. 10 and determine the constant  $C$  from the initial condition. This produces  $1/U(a, b; x)$ .  $\square$

**Lemma 1:** In the degenerate case  $U(a; x) := U(a, -a^2/4; x)$  yields the first order linear differential eq. 34 as well as the second order non-linear differential eq. given as the first of eqs. 36.

**Proof:** Elementary.  $\square$

**Note 1:** The degenerate case is equivalent to  $\lambda_+(a, b) = \lambda_-(a, b)$  with the definition of the characteristic roots of the recursion relation eq. 11 given in eq. 14. We may assume that not both,  $a$  and  $b$ , vanish and  $x \neq 1/\lambda_{\pm}(a, b) = -\lambda_{\mp}/b$ . In each case  $U(a, b; 0) = 1$ .

**Proposition 2:**  $V(a, b; x)$ , defined in eq. 13 for  $a \neq 0$ , is for  $a^2 + 4b \neq 0$  equivalent to *Riccati* eq. 24 with initial condition  $V(a, b; 0) = 1$ .

**Proof:** Analogous to the proof of *Proposition 1*.  $\square$

**Lemma 2:** In the degenerate case  $V(a; x) := V(a, -a^2/4; x)$  satisfies the first order linear differential eq. 35 as well as the first order non-linear differential eq. given as the second of eqs. 36. In each case  $V(a, b; 0) = 1$ .

**Proof:** Elementary.  $\square$

#### 4. CONVOLUTIONS OF GENERALIZED FIBONACCI AND LUCAS SEQUENCES

Because the  $k+1$ st power of the (ordinary) generating functions of a sequence generates  $k$ -fold convolutions of this sequence we obtain in the non-degenerate case,  $a^2 + 4b \neq 0$ , according to the general set-up of *section 1*, for the generalized *Fibonacci* resp. *Lucas* case, expression eq. 27, resp. eq. 28. For the definition of the  $k^{\text{th}}$  convolutions  $U_n^{(k)}(a, b)$  (similarly of  $V_n^{(k)}(a, b)$ ) see the line after eq. 26. The first convolutions ( $k = 1$ ) can be determined in each case from linear combinations of the two independent original sequences. See eq. 29 for the *Lucas* case. For  $a = 1 = b$  these formulae are well-known (see *section 2* after eq. 29).

**Lemma 3:** (Recurrence for  $k$ -fold convolution, degenerate case):

For  $b = -\frac{a^2}{4} \neq 0$  the recurrence formulae for the  $k$ -fold convolution of the generalized *Fibonacci*, resp. *Lucas*, sequences are those stated in eqs. 40, resp. 41.

**Proof:** This statement is equivalent to eq. 25, resp., eq. 26 for the powers of the corresponding generating functions. They are deduced from the second, resp. first, order differential eq., given in eqs. 36, which coincides with the  $k = 1$  assertion. To verify the general  $k$  case,

eq. 38 resp. eq. 39, one may use  $U(a; x) = U(a, -\frac{a^2}{4}; x) = 1/(1 - ax/2)^2$  from eq. 12, resp.  $V(a; x) = V(a, -\frac{a^2}{4}; x) = 1/(1 - ax/2)$  from eq. 13.  $\square$

**Lemma 4:** The explicit form for the  $k$ -fold convolution in the degenerate case is given by eq. 42, resp. 43, for the generalized *Fibonacci*, resp. *Lucas*, case.

**Proof:** Iteration of the recurrence eq. 40, resp. eq. 41, with input  $U_n^{(0)}(a) = U_n(a) = (n+1) \cdot (a/2)^n$ , resp.  $V_n^{(0)}(a) = V_n(a) = (a/2)^n$ , which originates from the generating functions  $U(a, -\frac{a^2}{4}; x)$ , resp.  $V(a, -\frac{a^2}{4}; x)$ .  $\square$

**Proposition 3:** (Iteration of recurrence for  $k$ -fold convolutions; non-degenerate *Fibonacci* case):

For  $a^2 + 4b \neq 0$  the  $k$ -fold convolution of the generalized *Fibonacci* sequence  $\{U_n(a, b)\}$  is expressed as linear combinations of the two independent solutions of recurrence eq. 11 as given in eq. 44. The coefficient polynomials  $AU_k(a, b; n)$  and  $BU_k(a, b; n)$  satisfy the mixed recurrence relations eqs. 45 and 46.

**Proof:** If one considers eq. 44 as *ansatz* and puts it into recurrence eq. 27 we find, after elimination of  $U_{n+2}(a, b)$  via its recursion relation and a comparison of the coefficients of the linear independent  $U_n(a, b)$  and  $U_{n-1}(a, b)$  sequences, the mixed recurrence relations for  $AU_k(a, b; n)$  and  $BU_k(a, b; n)$ . The inputs  $AU_0(a, b; n) = 1$  and  $BU_0(a, b; n) = 2$  are necessary in order that for  $k = 1$  eq. 44 coincides with eq. 27. With these inputs and the mixed recurrence one proves, by induction over  $k$ , that  $AU_k(a, b; n)$  and  $BU_k(a, b; n)$  are polynomials in  $n$  of degree  $k$ , provided  $a$  and  $b$  are fixed with  $b \neq -a^2/4, b \neq 0$  and  $a \neq 0$ .  $\square$

**Note 2:** For fixed integers  $a$  and  $b \neq -a^2/4$  the coefficients of the polynomials  $AU_n(a, b; x)$  and  $BU_n(a, b; x)$  furnish two lower triangular (infinite) integer matrices. For the ordinary *Fibonacci* case  $a = 1 = b$  these positive integer triangles can be found in [5] under the nrs. A057995 and A057280. For the *Pell* case  $a = 2, b = 1$  see nrs. A058402 and A058403, and for the case  $a = 1, b = 2$  see nrs. A073401 and A073402.

**Proposition 4:** (Iteration of recurrence for  $k$ -fold convolutions; non-degenerate *Lucas* case):

For  $a^2 + 4b \neq 0$  the  $k$ -fold convolution of the generalized *Lucas* sequence  $\{V_n(a, b)\}$  is expressed as linear combination of the two independent solutions of recurrence eq. 11 as given in eq. 47. The coefficient polynomials  $AV_k(a, b; n)$  and  $BV_k(a, b; n)$  satisfy the mixed recurrence relations eq. 48 and eq. 49.

**Proof:** Analogous to the proof of *Proposition 3*.  $\square$

**Note 3:** For fixed integers  $a$  and  $b \neq -a^2/4$  the coefficients of the polynomials  $AV_n(a, b; x)$  and  $BV_n(a, b; x)$  furnish two lower triangular (infinite) integer matrices. For the ordinary *Lucas* case  $a = 1 = b$  these positive integer triangles can be found in [5] under the nrs. A061188 and A061189. For the *Pell* case  $a = 2, b = 1$  see nrs. A062133 and A062134.

## 5. CONVOLUTIONS OF GENERALIZED $p$ -FIBONACCI SEQUENCES

Generalized  $p$ -*Fibonacci* numbers  $U_n(p; a, b)$  are defined by eq. 51 for  $p \in \mathbb{N}_0, b \neq 0$  and  $a \neq 0$ , together with the inputs  $U_j(p; a, b) := a^j$  for  $j = 0, \dots, p$ . For  $p = 1$  we recover the generalized *Fibonacci* numbers  $U_n(a, b)$  treated above.

**Lemma 5:** The generating function  $U(p; a, b; x)$  for the generalized  $p$ -*Fibonacci* numbers is given by eq. 50.

**Proof:** From the recurrence with inputs given in eq. 51.  $\square$

**Lemma 6:** (*Riccati* eq. for the generalized  $p$ -*Fibonacci* case):

If  $D(p; a, b) := (p + 1)^{p+1}b + a(ap)^p \neq 0$  (non-degenerate case) then  $U(p; a, b; x)$  satisfies *Riccati* eq. 52 with the polynomials  $A_p(a, b; x)$  and  $B_{p-1}(a; x)$  defined in eqs. 53 and 54.

**Proof:**  $H(p; a, b; x) = 1/U(p; a, b; x) = 1 - ax - bx^{p+1}$  satisfies eq. 9 with  $\alpha(x) \equiv \alpha_p(a, b; x) = A_p(a, b; x)/D(p; a, b)$  and  $\beta(x) \equiv \beta_p(a, b; x) = -b(p + 1)^2 B_{p-1}(a; x)/D(p; a, b)$  with  $A_p(a, b; x)$  and  $B_{p-1}(a; x)$  given by eq. 53 and 54. This is shown by comparing coefficients of powers  $x^i$  for  $i = 0, 1, \dots, 2p$ . According to *section 1 Riccati* eq. 8 ensues which becomes eq. 52.  $\square$

**Note 4:** i) If  $p = 0$ ,  $U(0; a, b; x) = 1/(1 - (a + b)x)$  generates powers of  $a + b$ , and one has to put  $A_0(a, b; x) \equiv 1$  and  $B_{-1}(a; x) \equiv 0$ . This means that one puts  $(ap)^p = 1$  for  $p = 0$ .

ii) For given non-vanishing  $a$  and  $b$   $A_p(a, b; x)$  is a polynomial in  $x$  of degree  $p$ , and  $B_{p-1}(a; x)$  is one of degree  $p - 1$ . The sum in  $B_{p-1}(a; x)$  can be evaluated to yield the second of eqs. 54 provided  $p \neq 0$ .

**Lemma 7:** (Coefficient triangles of numbers for polynomials  $A_p(a, b; x)$  and  $B_p(a; x)$ ):

The coefficients of the polynomials defined in eqs. 55 are given by eqs. 56 and 57.

**Proof:**  $B_p(a; x)$  from eq. 54 leads immediately to eq. 57, remembering that  $B_{-1}(a; x) \equiv 0$ . Then eq. 56 follows from eq. 53 and  $A_0(a, b; x) \equiv 1$ .  $\square$

**Proposition 5:** (Uniqueness of *Riccati* solution; non-degenerate case):

If  $D(a, b) \neq 0$  then  $y \equiv U(p; a, b; x) = 1/(1 - ax - bx^{p+1})$  is the unique solution of *Riccati* eq. 52 with eqs. 53, 54 and initial value  $U(p; a, b; 0) = 1$ .

**Proof:** From *section 1* we know that the *Riccati* eq. is equivalent to the inhomogeneous linear differential eq. for the inverse  $H = 1/U : H' \equiv F(x, H) = (-\beta(x)/\alpha(x))H - 1/\alpha(x)$ . Because  $F(x, H)$  is continuous in the strip  $0 \leq x \leq A < \infty, |H| < \infty$  and is there ( $K = K(p; a, b; A)$ )-*Lipschitz*, the existence and uniqueness theorem for linear differential eqs. proves the assertion (see *e.g.* [8], §6,I,p.62ff). In order to find  $K$  we use the summed expression for  $B_{p-1}$  from eq. 54 and apply the triangle inequality repeatedly.  $\square$

**Proposition 6:** (Recursion for  $k^{\text{th}}$  convolution of  $\{U_n(p; a, b)\}$ ; non-degenerate case):

The  $k^{\text{th}}$  convolution of the sequence  $\{U_n(p; a, b)\}$  is given in the non-degenerate case  $D(p; a, b) := b(p + 1)^{p+1} + a(ap)^p \neq 0$  recursively by eq. 61 with eq. 62.

**Proof:** This follows from the general set-up of *section 1*, eq. 5 with  $\gamma_{n-q} \equiv 0$  and the appropriate coefficient functions  $\alpha(x) = \alpha_p(a, b; x)$  and  $\beta(x) = \beta_p(a, b; x)$  given after eq. 54. See the corresponding eq. 60 for the  $k + 1$ -st power of the generating function.  $\square$

**Lemma 8:** (Degenerate case  $D(p; a, b) = 0$ ):

If  $D(p; a, b) := (p + 1)^{p+1}b + a(ap)^p = 0$  then  $U(p; a; x) = 1/(1 - ax - b(p; a)x^{p+1}) = 1/(1 - ax + (((apx)/(p + 1))^{p+1})/p)$  satisfies the first order linear differential eq. 64.

**Proof:** We prove  $(a + (p + 1)bx^p)A_p(a, b; x) + b(p + 1)^2(1 - ax - bx^{p+1})B_{p-1}(a; x) = 0$  with eqs. 54 and 53 in the version where the sum has been evaluated (the case  $p = 0$  is treated separately). If we factor out  $b/(p + 1 - apx)$  we see that all terms cancel provided we replace  $a(ap)^p$  by  $-b(p + 1)^{p+1}$ .  $\square$

**Note 5:** The solution  $1/(1 - ax - bx^{p+1})$  of this linear differential eq. 64 with input  $U(p; a, b; 0) = 1$  is unique. The proof is analogous to the one of *Proposition 5*.

**Note 6:** If  $U(p; a; x) = 1/(1 - ax + (((apx)/(p + 1))^{p+1})/p)$  we do not have a formula for  $U^2(p; a; x)$ , valid for all  $p$ , like in the non-degenerate case. Therefore, we cannot derive results for convolutions along the line shown above.

**Lemma 9:** (Recurrence in the degenerate case):

If  $D(p; a, b) := (p + 1)^{p+1}b + a(ap)^p = 0$  (and  $b \neq 0$ ) then one can replace recurrence eq. 51 which has depth  $p + 1$ , by the following one with depth  $p \in \mathbb{N}$ .

$$U_{n+1}(p; a) = \frac{a}{(p + 1)(n + 1)} \sum_{j=1}^p \left( \frac{ap}{p + 1} \right)^{j-1} (n + p + 2 - j)U_{n+1-j}(p; a), \quad (65)$$

where one uses the inputs  $U_j(p; a) = a^j$  for  $j = 0, 1, \dots, p - 1$ .

**Proof:** This derives from the sum on the *rhs* of eq. 58 which now vanishes. If the coefficients  $C_j$  from eq. 59 are used with the replacement of  $a(ap)^p$  by  $-b(p+1)^{p+1}$  one arrives at the desired recurrence, after the common factor  $b$  has been dropped. The inputs are adopted from the original recurrence except that  $U_p$  can now be computed to be  $a^p$ .  $\square$

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Note added in proof: See also the work by Hong Feng and Zhizheng Zhang. "Computational Formulas for Convolutated Generalized Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **41** (2003): 144-151.

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