ORDERINGS OF PRODUCTS OF FIBONACCI NUMBERS

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(Submitted July 2001–Final Revision October 2001)

1. INTRODUCTION

The sequence of all the binary products of Fibonacci numbers arranged in increasing order is not new to the literature (e.g., [1]), but for completeness and ready applicability to the trinary case, we develop the binary case here. Actually, in the binary case, we prove a bit more than is true for the trinary case, namely, that when the binary products of Fibonacci numbers are arranged in increasing order, then the differences between consecutive terms are Fibonacci numbers.

Well known identities (e.g., [4]) to be used are the following:

$$F_{m+r} = F_r F_{m+1} + F_{r-1} F_m \tag{1}$$

$$F_m^2 - F_{m+r} F_{m-r} = (-1)^{m-r} F_r^2 \tag{2}$$

$$F_m F_{r+1} - F_{m+1} F_r = (-1)^r F_{m-r}. \tag{3}$$

These identities hold for all integers m and r; e.g., $F_7F_3 - F_6F_4 = F_{-3} = 2$. We shall also use inequalities connected with the continued fraction (e.g., [2]) of the golden mean, $\phi = (1+\sqrt{5})/2$:

$$\frac{F_2}{F_1} < \frac{F_4}{F_3} < \frac{F_6}{F_5} < \cdots < \phi < \cdots < \frac{F_7}{F_6} < \frac{F_5}{F_4} < \frac{F_3}{F_2}, \tag{4}$$

where

$$\frac{1}{F_{m+3}} < |\frac{F_{m+1}}{\phi} - F_m| < \frac{1}{F_{m+2}},$$

so that

$$\frac{\phi}{F_m F_{m+3}} < |\frac{F_{m+1}}{F_m} - \phi| < \frac{\phi}{F_m F_{m+2}}. \tag{5}$$

We write (5) out according to the parity of m:

$$\phi + \frac{\phi}{F_m F_{m+3}} < \frac{F_{m+1}}{F_m} < \phi + \frac{\phi}{F_m F_{m+2}} \tag{6}$$

for even m,

$$\phi - \frac{\phi}{F_m F_{m+2}} < \frac{F_{m+1}}{F_m} < \phi - \frac{\phi}{F_m F_{m+3}} \tag{7}$$

for odd m.

2. BINARY PRODUCTS OF FIBONACCI NUMBERS

Theorem 1: *Suppose the products F_iF_j, for $2 \le i \le j$, are arranged in nondecreasing order as a sequence $a = (a_1, a_2, a_3, \ldots)$. Then $a_{n+1} - a_n$ is a positive Fibonacci number for $n = 1, 2, 3, \ldots$. The terms of a are determined in three cases:*

i. if $m \equiv 1 \mod 2$, *then*

$$F_m = F_2F_m < F_4F_{m-2} < F_6F_{m-4} < \cdots < F_1F_{m+1} = F_{m+1};$$

ii. if $m \equiv 2 \mod 4$, *then*

$$F_m = F_2F_m < F_4F_{m-2} < \cdots < F_{\frac{m-2}{2}}F_{\frac{m+6}{2}} < F^2_{\frac{m+2}{2}} < F_{\frac{m}{2}}F_{\frac{m+4}{2}} < \cdots < F_1F_{m+1} = F_{m+1};$$

iii. if $m \equiv 0 \mod 4$, *then*

$$F_m = F_2F_m < F_4F_{m-2} < \cdots < F_{\frac{m}{2}}F_{\frac{m+4}{2}} < F^2_{\frac{m+2}{2}} < F_{\frac{m+2}{2}}F_{\frac{m+6}{2}} < \cdots < F_1F_{m+1} = F_{m+1}.$$

Proof: Case i: $m \equiv 1$ mod 2. Write

$$F_{2k+2}F_{m-2k} - F_{2k}F_{m-2k+2} = (F_{2k} + F_{2k+1})F_{m-2k} - F_{2k}(F_{m-2k} + F_{m-2k+1})$$

and apply (3) to obtain

$$F_{2k+2}F_{m-2k} - F_{2k}F_{m-2k+2} = F_{m-4k}. \tag{8}$$

The index $m = 4k$ is odd, so that $F_{m-4k} > 0$ for $m < 4k$ as well as $m > 2k$. Equation (8) establishes the asserted chain of inequalities.

Case ii: $m \equiv 2$ mod 4. By (2),

$$F^2_{\frac{m+2}{2}} - F_{\frac{m-2}{2}}F_{\frac{m+6}{2}} = 1 = F_{\frac{m}{2}}F_{\frac{m+4}{2}} - F^2_{\frac{m+2}{2}}.$$

Also,

$$F_{2k+2}F_{m-2k} - F_{2k}F_{m-2k+2} = -F_{m-4k} = F_{|m-4k|}, \tag{9}$$

so that

$$F_2F_m < F_4F_{m-2} < \cdots < F_{\frac{m-2}{2}}F_{\frac{m+6}{2}}.$$

Write $m = 4M + 2$. Then

$$\begin{aligned} F_{2M+2h+3}F_{2M-2h+1} - F_{2M+2h+1}F_{2M-2h+3} &= F_{2M+2h+2}F_{2M-2h+1} - F_{2M+2h+1}F_{2M-2h+2} \\ &= -F_{-4h} \\ &= F_{4h}, \end{aligned}$$

for $h = 1, 2, \ldots, (m-2)/4$; i.e.,

$$F_{\frac{m}{2}}F_{\frac{m+4}{2}} < \cdots < F_1F_{m+1} = F_{m+1}.$$

Thus, the difference between neighboring terms of the chain in case ii is a positive Fibonacci number.

Case iii: $m \equiv 0 \bmod 4$. By (2),

$$F^2_{\frac{m+2}{2}} - F_{\frac{m}{2}}F_{\frac{m+4}{2}} = 1 = F_{\frac{m-2}{2}}F_{\frac{m+6}{2}} - F^2_{\frac{m+2}{2}}.$$

Also, the inequalities

$$F_2F_m < F_4F_{m-2} < \cdots < F_{\frac{m}{2}}F_{\frac{m+4}{2}}$$

and differences between terms are clearly given by (9). Write $m = 4M$. For $h = 1, 2, \ldots, M$, we have

$$\begin{aligned} F_{2M-2h-1}F_{2M+2h+3} - F_{2M-2h+1}F_{2M+2h+1} &= F_{2M-2h-1}F_{2M+2h+2} - F_{2M-2h}F_{2M+2h+1} \\ &= -F_{4k-2} \\ &= F_{4k+2}, \end{aligned}$$

i.e.

$$F_{\frac{m-2}{2}}F_{\frac{m+6}{2}} < F_{\frac{m-6}{2}}F_{\frac{m+10}{2}} < \cdots < F_1F_{m+1} = F_{m+1},$$

and the difference between neighboring terms of the chain in case iii is a positive Fibonacci number.

Thus, each F_iF_j is a member of a chain of inequalities given in cases i-iii. Conversely, each member stated is such a product. The sequence a therefore results from linking these chains in the obvious manner, corresponding to the half-open intervals $[F_2, F_3), [F_3, F_4), [F_4, F_5)$, and so on.

The first 13 terms of the sequence a (= A049997 in Sloane [3]) and its difference sequence (= A049998) are shown in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	1	2	3	4	5	6	8	9	10	13	15	16	21
$a_{n+1} - a_n$	1	1	1	1	1	2	1	1	3	2	1	5	3

Corollary 1: *If $i + j < i' + j'$, where $2 \le i \le j$ and $2 \le i' \le j'$, then $F_iF_j < F_{i'}F_{j'}$.*

Proof: All products F_iF_j for $2 \le i \le j$ are accounted for in cases i-iii of Theorem 1. For each case, the indices (i, j) in the corresponding chain of inequalities have in common the sum $i + j = m + 2$, and $F_iF_j \in [F_m, F_{m+1})$. If $i + j < i' + j'$, then $F_{i'}F_{j'} \in [F_{m'}, F_{m'+1})$, where $i' + j' = m' + 2$. Since $F_m < F_{m'}$, we have $F_iF_j < F_{i'}F_{j'}$.

Corollary 2: When the binary products of distinct Fibonacci numbers are arranged in increasing order, the difference between pairs of neighboring terms is a Fibonacci number.

Proof: Omit the squares in cases ii and iii of Theorem 1, and note that each resulting new difference is F_3. The other differences are as in Theorem 1.

3. TRINARY PRODUCTS

For given n, any triple (i,j,k) satisfying $i+j+k=n$ and $2 \leq i < j < k$ will be called an *eligible triple*. It will be expedient to assign an order relation $\prec$ on eligible triples:

$$(i,j,k) \prec (i',j',k') \text{ if and only if } F_iF_jF_k < F_{i'}F_{j'}F_{k'}.$$

Lemma 1: *For $i+j+k \leq 11$, the ordering of eligible triples (i,j,k) is given by*

$$(2,3,4) \prec (2,3,5) \prec (2,4,5) \prec (2,3,6).$$

Proof: $6 < 10 < 15 < 16$.

Lemma 2: *Suppose $n \geq 12$. Let $c = 2\lfloor \frac{n-3}{6} \rfloor$ and $c' = 2\lfloor \frac{n}{6} \rfloor - 1$. For $h = 1, 2, \ldots, c/2$, the least eligible triple having $i = 2h$ is*

$$(2h, 2h+2, n-4h-2), \tag{10}$$

and the greatest is

$$(2h, 2h+1, n-4h-1). \tag{11}$$

For $h = 1, 2, \ldots, (c'-1)/2$, the least eligible triple having $i = 2h+1$ is

$$(2h+1, 2h+2, n-4h-3) \tag{12}$$

and the greatest is

$$(2h+1, 2h+3, n-4h-4). \tag{13}$$

Proof: We prove only (10), as proofs of (11)-(13) are similar. Suppose $2h+1 \leq j \leq n-2h-j-1$. By Theorem 1,

$$F_{2h+2}F_{n-4h-2} \leq F_jF_{n-2h-j}.$$

Consequently,

$$F_{2h}F_{2h+2}F_{n-4h-2} \leq F_{2h}F_jF_{n-2h-j},$$

i.e.,

$$(2h, 2h+2, n-4h-2) \preceq (2h, j, n-2h-j).$$

Lemma 3: *The least and greatest eligible triples* (10)-(13) *listed in Lemma 2 are ordered as follows*:

$$\begin{aligned}
(2,4,n-6) &\prec (2,3,n-5) \\
&\prec (4,6,n-10) \prec (4,5,n-9) \\
&\prec (6,8,n-14) \prec (6,7,n-13) \\
&\quad \ldots \\
&\prec (c,c+2,n-2c-2) \prec (c,c+1,n-2c-1) \\
&\prec (c',c'+1,n-2c'-1) \prec (c',c'+2,n-2c'-2) \\
&\prec (c'-2,c'-1,n-2c'+3) \prec (c'-2,c',n-2c'+2) \\
&\quad \ldots \\
&\prec (5,6,n-11) \prec (5,7,n-12) \\
&\prec (3,4,n-7) \prec (3,5,n-8).
\end{aligned}$$

Proof: Some of these inequalities are covered by Lemma 2. A first grouping that are not begin with $(2,3,n-5) \prec (4,6,n-10)$ and are of the form

$$F_{2h}F_{2h+1}F_{n-4h-1} < F_{2h+2}F_{2h+4}F_{n-4h-6} \tag{14}$$

for $n \geq 6h+11, h = 1,2,3,\ldots$.

We begin a proof of (14) with a technical result. Let

$$\theta_i = F_{2h+i+1}/F_{2h+i} \quad \text{and} \quad \Theta = (3+5\theta_3)(9+24\theta_0-10\theta_1).$$

We omit proving that $\Theta > 5$ for $h \geq 1$, as this is easily proved and unsurprising. This inequality can be written as

$$3(5+8\theta_0)(3+5\theta_2) - 2(3+5\theta_1)(3+5\theta_3) > 5,$$

which, via (1), is equivalent to

$$3\frac{F_{2h+6}}{F_{2h}}\frac{F_{2h+8}}{F_{2h+3}} - 2\frac{F_{2h+6}}{F_{2h+1}}\frac{F_{2h+8}}{F_{2h+3}} > 5,$$

hence to

$$\frac{3\phi}{F_{2h}F_{2h+3}} - \frac{2\phi}{F_{2h+1}F_{2h+3}} > 5\frac{\phi}{F_{2h+6}F_{2h+8}}.$$

By (6),

$$\frac{3\phi}{F_{2h}F_{2h+3}} - \frac{2\phi}{F_{2h+1}F_{2h+3}} > 5\left(\frac{F_{2h+7}}{F_{2h+6}} - \phi\right) \geq 5\left(\frac{F_{n-4h-5}}{F_{n-4h-6}} - \phi\right), \tag{15}$$

this last $\geq$ being given by (4) for $n = 6h+12$. By (6) and (7),

$$\frac{3F_{2h+1}}{F_{2h}} > 3\phi + \frac{3\phi}{F_{2h}F_{2h+3}}$$

and

$$\frac{2F_{2h+2}}{F_{2h+1}} > 2\phi - \frac{2\phi}{F_{2h+1}F_{2h+3}},$$

so that

$$\frac{2F_{2h+2}}{F_{2h+1}} + \frac{3F_{2h+1}}{F_{2h}} > 5\phi + \frac{3\phi}{F_{2h}F_{2h+3}} - \frac{2\phi}{F_{2h+1}F_{2h+3}}.$$

This inequality and (15) imply

$$5\frac{F_{n-4h-5}}{F_{n-4h-6}} < 2\frac{F_{2h+2}}{F_{2h+1}} + 3\frac{F_{2h+1}}{F_{2h}},$$

so that

$$3 + 5\frac{F_{n-4h-5}}{F_{n-4h-6}} < \left(F_2 + F_3\frac{F_{2h+2}}{F_{2h+1}}\right) + \left(F_3 + F_4\frac{F_{2h+1}}{F_{2h}}\right),$$

which by (1) yields

$$\frac{F_{n-4h-1}}{F_{n-4h-6}} < \frac{F_{2h+4}}{F_{2h}} + \frac{F_{2h+4}}{F_{2h+1}} = \frac{(F_{2h} + F_{2h+1})F_{2h+4}}{F_{2h}F_{2h+1}} = \frac{F_{2h+2}F_{2h+4}}{F_{2h}F_{2h+1}},$$

and (15) is proved.

The next inequality in the statement of Lemma 3 that is not covered by Lemma 2 is

$$(c, c+1, n-2c-1) \prec (c', c'+1, n-2c'-1). \tag{16}$$

If the residue of n mod 6 is 3,4, or 5 then $c' = c-1$, and (16) is then equivalent to

$$(c, c+1, n-2c-1) \prec (c-1, c, n-2c-1),$$

which is true since $F_{c+1}F_{n-2c-1} < F_{c-1}F_{n-2c+1}$, by Theorem 1. Otherwise, $c' = c+1$, and (16) follows from $F_cF_{n-2c-1} < F_{c+2}F_{n-2c-3}$.

The remaining inequalities to be proved are of the form

$$F_{2h+3}F_{2h+5}F_{n-4h-8} < F_{2h+1}F_{2h+2}F_{n-4h-3} \tag{17}$$

for $n \geq 6h+14, h = 1, 2, 3, \ldots$. A proof of (17) is analogous to that of (15). Using the same notation, we start with $\theta_4(3\theta_1 - 2\theta_2) > 5$ for $h \geq 1$, a simple proof of which is omitted. By (1), we then obtain

$$\frac{3}{F_{2h+1}F_{2h+4}} - \frac{2}{F_{2h+2}F_{2h+4}} > \frac{5}{F_{2h+3}F_{2h+5}}. \tag{18}$$

By (7),

$$\frac{F_{2h+4}}{F_{2h+3}} - \phi > -\frac{\phi}{F_{2h+3}F_{2h+5}},$$

and this with (18) gives

$$5\left(\frac{F_{2h+4}}{F_{2h+3}} - \phi\right) > \frac{2\phi}{F_{2h+2}F_{2h+4}} - \frac{3\phi}{F_{2h+1}F_{2h+4}}.$$

Now

$$\frac{F_{n-4h-7}}{F_{n-4h-8}} \geq \frac{F_{2h+4}}{F_{2h+3}}$$

for $n \geq 6h+11$, by (4), so that

$$5\frac{F_{n-4h-7}}{F_{n-4h-8}} > 5\phi + \frac{2\phi}{F_{2h+2}F_{2h+4}} - \frac{3\phi}{F_{2h+1}F_{2h+4}}. \tag{19}$$

By (6) and (7),

$$2\phi + \frac{2\phi}{F_{2h+2}F_{2h+4}} > \frac{2F_{2h+3}}{F_{2h+2}} \text{ and } 3\phi - \frac{3\phi}{F_{2h+1}F_{2h+4}} > \frac{3F_{2h+2}}{F_{2h+1}},$$

so that

$$5\phi + \frac{2\phi}{F_{2h+2}F_{2h+4}} - \frac{3\phi}{F_{2h+1}F_{2h+4}} > \frac{2F_{2h+3}}{F_{2h+2}} + \frac{3F_{2h+2}}{F_{2h+1}}. \tag{20}$$

Using (19), (20), and (1), we obtain

$$5\frac{F_{n-4h-7}}{F_{n-4h-8}} > \frac{2F_{2h+3}}{F_{2h+2}} + \frac{3F_{2h+2}}{F_{2h+1}},$$

$$3 + 5\frac{F_{n-4h-7}}{F_{n-4h-8}} > F_2 + F_3\frac{F_{2h+3}}{F_{2h+2}} + F_3 + F_4\frac{F_{2h+2}}{F_{2h+1}} = \frac{F_{2h+5}}{F_{2h+2}} + \frac{F_{2h+5}}{F_{2h+1}} = \frac{F_{2h+3}F_{2h+5}}{F_{2h+1}F_{2h+2}},$$

and since

$$\frac{F_{n-4h-3}}{F_{n-4h-8}} = 3 + 5\frac{F_{n-4h-7}}{F_{n-4h-8}},$$

a proof of (17) is finished.

Theorem 2: *If $2 \le i < j < k$ and $k \ge 5$, then*

$$F_{i+j+k-4} < F_iF_jF_k < F_{i+j+k-3}. \tag{21}$$

Proof: Either the number $n = i + j + k$ satisfies $10 \le n \le 11$ and (i, j, k) is one of the last three triples listed in Lemma 1, or else $n \ge 12$. Inequalities (21) are easily checked for $n = 10$ and $n = 11$, so assume $n \ge 12$. By Lemma 3, we have $(2, 4, n-6) \preceq (i, j, k)$. Since $F_2F_4F_{n-6} = F_4F_{n-6}$, Theorem 1 applies, and

$$F_{4+(n-6)-2} < F_2F_4F_{n-6}.$$

Consequently, $F_{n-4} < F_iF_jF_k$.

On the other side, by Lemma 3 we have

$$F_iF_jF_k < F_3F_5F_{n-8} = 2 \cdot 5 \cdot F_{n-8},$$

so it suffices to prove $10F_{n-8} < F_{n-3}$. By (6) and (7), $F_{n-7}/F_{n-8} > 7/5$, so that, using (1), we have

$$10F_{n-8} < 5F_{n-7} + 3F_{n-8} = F_{n-3}.$$

Theorems 1 and 2 show that binary and trinary products with fixed sum of indices lie in an interval whose endpoints are consecutive Fibonacci numbers. This property does not extend to products of more than three Fibonacci numbers; for example,

$$F_{12} < F_2F_4F_5F_7 < F_{13} < F_2F_3F_5F_8 < F_{14}.$$

One may ask instead for the greatest L and least U such that for given s and k all products $\prod_{i=1}^{k} F_{m_i}$ having $\sum_{i=1}^{k} m_i = s$ lie in the interval $[F_L, F_U]$.

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AMS Classification Numbers: 11B39

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