Three Lectures on Supergravity

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and Antoine Van Proeyen to appear in
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Public Material

Lectures on supergravity, Amsterdam-Brussels-Paris doctoral school, Paris 2009, October-November 2009

http://itf.fys.kuleuven.be/~toine/SUGRA_DoctSchool.pdf

A. Van Proeyen, Tools for supersymmetry, hep-th 9910030

Lectures on Supergravity. Universidad Catolica. Santiago de Chile.
Novmber-December 2009. Joaquim Gomis, web page Jorge Alfaro

Overview of Supersymmetry and Supergravity

- Super Poincare

Translations P_μ Lorentz transformations $J_{[\nu\sigma]}$

Spinor supercharge (odd) Q_α

Massless multiplets contains spins (s, s-1/2), for s=1/2, 1, 2,

Overview of Supersymmetry and Supergravity

Supergravity

Gauged supersymmetry was expected to be an extension of general Relativity with a superpartner of the gravito call gravitino

$e^a_\mu(x)$ $\psi_{\mu\alpha}(x)$ Multiplet (2,3/2)

S. Ferrara, D. Freedman, P. Van Nieuwenhuizen (1976)
S. Deser, B. Zumino (1976)
D. Volkov, V. Soroka (1973),

Extensions with more supersymmetries and extension has been considered, N=2 supergravity, special geometry. N=1 Supergravity in 11d

Motivation for Supergravity

Supergravity (SUGRA) is an extension of Einstein's general relativity to include supersymmetry (SUSY). General relativity demands extensions since it has shortcomings including at least the following:

Motivation for Supergravity

- Space time singularities. The singularity theorems of Penrose, Hawking and Geroch shows that general relativity is incomplete.
- Failure to unify gravity with the strong and electro weak forces.
- Einstein gravity is not power counting renormalizable. It is renormalizable as an effective theory. It is not a fundamental theory
- If we include supersymmetry in a theory of gravity. The simple example of divergences: zero point energy of the vacuum, can potentially be cancelled by super partners of ordinary particles

The current status of supergravity

- A reliable approximation to M-theory.
- An essential ingrediente for supersymmetric phenomenology (minimal supersymmetric estándar model coupled to N=1 supergravity).
- Applications in cosmology
- An crucial part for the AdS/CFT correspondence

Index

Clifford algebras ans spinors

Free Rarita-Schwinger field

Differential geometry

First and second order formulation of gravity

N=1 Global Supersymmetry in D=4

Index

N=1 pure supergravity in 4 dimensions

D=11 supergravity

Killing spinors equations and BPS solutions

Clifford algebras and spinors

- Clifford algebras in general dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1} \, .$$

Euclidean Clifford algebras

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \ldots \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \ldots \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots \\
\ldots &= \ldots \, .
\end{aligned}$$

Clifford algebras and spinors

These matrices are all hermitian with squares equal to $\mathbb{1}$, and they mutually anticommute. Suppose that $D = 2m$ is even. Then we need m factors in the construction (3.2) to obtain γ^μ, $1 \leq \mu \leq D = 2m$. Thus we obtain a representation of dimension $2^{D/2}$. For odd $D = 2m + 1$ we need one additional matrix, and we take γ^{2m+1} from the list above, but we keep only the first m factors, i.e. deleting a σ_1. Thus there is no increase in the dimension of the representation in going from $D = 2m$ to $D = 2m + 1$, and we can say in general that the construction (3.2) gives a representation of dimension $2^{[D/2]}$, where $[D/2]$ means the integer part of $D/2$.

Clifford algebras and spinors

we need Lorentzian γ's , all we need to do is pick

any single matrix from the Euclidean construction, multiply it by i and label it γ^0 for the time-like direction. This matrix is anti-hermitian and satisfies $(\gamma^0)^2 = -\mathbb{1}$. We then relabel the remaining $D-1$ matrices to obtain the Lorentzian set γ^μ, $0 \leq \mu \leq D-1$. The hermiticity properties of the Lorentzian γ's are summarized by

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

Clifford algebras and spinors

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generating elements γ^μ, plus all independent matrices formed from products of the generators. Since symmetric products reduce to a product containing fewer γ-matrices by (3.1), the new elements must be antisymmetric products. We thus define

$$\gamma^{\mu_1\ldots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}\,, \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \tfrac{1}{2}\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu\,,$$

Non-vanishing tensor components

$$\gamma^{\mu_1\mu_2\ldots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r} \qquad \text{for } \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r\,.$$

The antysymmetrization indicated with […] is always with total weight 1

C^D_r distinc indexes choices

Basis of the algebra for even dimensions

The basis is denoted by the following list $\{\Gamma^A\}$ of matrices

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1,\cdots\mu_D}\} \; \mu_1 < \mu_2 <, \ldots < \mu_r$$

C_r^D index choices at each rank r and a total of 2^D matrices.

Other possible basis $\{\Gamma_A = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \cdots, \gamma_{\mu_D,\cdots\mu_1}\}$

$$\Gamma^A\Gamma^B = \pm\Gamma^C,$$

$$\mathrm{Tr}(\Gamma^A\Gamma_B) = 2^m\,\delta^A_B\,.$$

$$M = \sum_A m_A\Gamma^A, \qquad m_A = \frac{1}{2^m}\mathrm{Tr}(M\Gamma_A)$$

The highest rank Clifford algebra element

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}$$

Provides the link bewteen even and odd dimensions

$\gamma_*^2 = \mathbb{1}$ in every even dimension and is hermitian

$D = 2m$, the matrix γ_* is frequently called γ_{D+1}

Properties

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\,\gamma_*$$

$$\begin{aligned} \{\gamma_*, \gamma^\mu\} &= 0\,, \\ [\gamma_*, \gamma^{\mu\nu}] &= 0\,. \end{aligned}$$

Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\,\gamma_* = 0$

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Weyl spinors

chiral projectors

$$P_L = \tfrac{1}{2}(\mathbb{1} + \gamma_*)\,, \qquad P_R = \tfrac{1}{2}(\mathbb{1} - \gamma_*)$$

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L \Psi\,, \qquad \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} \equiv P_R \Psi$$

$$P_L P_L = P_L,\; P_R P_R = P_R \;\, and \;\, P_L P_R = 0$$

No explicity Weyl representation will be used in these lectures

Odd space dimension D=2m+1

we already have enough matrices if we consider the matrices up to $\gamma_{\mu_1 \dots \mu_{(D-1)/2}}$

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \quad \cdots, \gamma_{\mu_1 \dots \mu_{(D-1)/2}}\}$$

Symmetries of gamma matrices

the $2^m \times 2^m$ matrices, for both $D = 2m$ and $D = 2m + 1$

distinguish between symmetric and the antisymmetric

exists a unitary matrix charge conjugation matrix, such that

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)}\,, \qquad t_r = \pm 1$$

implies

$$C^T = -t_0 C\,, \qquad \gamma^{\mu T} = t_0 t_1\, C\gamma^\mu C^{-1}$$

Explicit forms conjugation matrix

$$\begin{aligned} C_+ &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots \qquad & t_0 t_1 = 1\,, \\ C_- &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots \qquad & t_0 t_1 = -1 \end{aligned}$$

For odd dimension C is unique (up to phase factor)

The possible sign factors depend on the spacetime dimension D modulo 8
And on r modulo 4

Symmetries of gamma matrices

D (mod 8)	$t_r = -1$	$t_r = +1$
0	$0, \quad 3$	$2, \quad 1$
	$\mathbf{0, \quad 1}$	$\mathbf{2, \quad 3}$
1	$0, \quad 1$	$2, \quad 3$
2	$0, \quad 1$	$2, \quad 3$
	$\mathbf{1, \quad 2}$	$\mathbf{0, \quad 3}$
3	$1, \quad 2$	$0, \quad 3$
4	$\mathbf{1, \quad 2}$	$\mathbf{0, \quad 3}$
	$2, \quad 3$	$0, \quad 1$
5	$2, \quad 3$	$0, \quad 1$
6	$2, \quad 3$	$0, \quad 1$
	$\mathbf{0, \quad 3}$	$\mathbf{1, \quad 2}$
7	$0, \quad 3$	$1, \quad 2$

Symmetries of gamma matrices

- Since we use hermitian representations, the symmetry properties of gamma matrices determines also its complex conjugation

$$\gamma^{\mu *} = -t_0 t_1 \, B \gamma^\mu B^{-1} \qquad B = \mathrm{i} t_0 C \gamma^0$$

$$B^* B = -t_1 \mathbb{1}$$

Adjoint spinor

- We have defined the Dirac adjoint, which involves the complex conjugate. Here we define the conjugate of “any” spinor using the transpose and the charge conjugation matrix

$$\bar{\lambda} \equiv \lambda^T C$$

Symmetry properties for bilinears

$$\bar{\lambda}\gamma_{\mu_1 \ldots \mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1 \ldots \mu_r}\lambda$$

Majorana flip

Spinor indexes

The components of the basic spinor λ are indicated as λ_α.

barred spinor are indicated with upper indices: λ^α

raising matrix $\mathcal{C}^{\alpha\beta}$

$$\bar{\lambda}^\alpha = \lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta$$

$\mathcal{C}^{\alpha\beta}$ are the components of the matrix C^T Note NW-SE line

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}\,.$$

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha\,, \qquad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

$\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}

Spinor indexes

The gamma matrices have components $(\gamma_\mu)_\alpha{}^\beta$

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta\,,$$

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma\mathcal{C}_{\gamma\beta} \;= (\gamma_\mu C^{-1})_{\alpha\beta}$$

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r(\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha}$$

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha$$

Fierz rearrangement

- In supergravity we will need changing the pairing of spinors in products of bilinears, which is called Fierz rearrangement

Basic Fierz identity from

$$M = \sum_A m_A \Gamma^A, \qquad m_A = \frac{1}{2^m} \mathrm{Tr}(M\Gamma_A)$$

Expanding any A as

$$A_\alpha{}^\beta = \sum_A (\Gamma^A)_\alpha{}^\beta \, c_A, \quad \rightarrow \quad c_A = 2^{-\frac{d}{2}} tr(\Gamma_A \, A)$$

$$A_\alpha{}^\beta = \sum_A (\Gamma^A)_\alpha{}^\beta \, 2^{-\frac{d}{2}} (\Gamma_A)_\gamma{}^\delta A_\delta{}^\gamma$$

Fierz rearrangement

Completeness relation $$\delta_\alpha{}^\beta \delta_\gamma{}^\delta = \frac{1}{2^m} \sum_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta$$

Note that the 'column indices' on the left and right sides have been exchanged

We get

$$(\gamma^\mu)_\alpha{}^\beta (\gamma_\mu)_\gamma{}^\delta = \frac{1}{2^m} \sum_A v_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta$$

$v_A = (-)^{r_A}(D - 2r_A)$ Where r_A Is the rank of Γ_A

Charge conjugate spinor

Complex conjugation is necessary to verify that the lagrangian involving spinor bilinears is hermitian.

Charge conjugate of any spinor

$$\lambda^C \equiv B^{-1}\lambda^* \qquad\qquad B = \mathrm{i}t_0 C\gamma^0$$

Barred charge conjugate spinor

$$\overline{\lambda^C} = (-t_0 t_1)\mathrm{i}\lambda^\dagger\gamma^0$$

It coincides withe Dirac conjugate except for the numerical factor $(-t_0 t_1)$

[3] We use the convention that we interchange fermion fields in the process of complex conjugation.

Majorana spinors

- Majorana fields are Dirac fields that satisfy and addtional "reality" condition, whic reduces the number degrees of freedom by two. More fundamental like Weyl fields

  Particles described by a Majorana field are such that particles and antiparticles are identical

Majorana field

$$\psi = \psi^C = B^{-1}\psi^* \,, \qquad \text{i.e.} \qquad \psi^* = B\psi$$

We have $\psi = B^* B\, \psi$ which implies $B^* B = \mathbb{1}$

Recall

$$B^* B = -t_1 \mathbb{1}$$

which implies $t_1 = -1$

Majorana spinors

Two cases $t_0 = \pm 1$

$t_0 = +1$ holds for spacetime dimension $D = 2,\ 3,\ 4,\ \mathrm{mod}\ 8.$

In this case we have Majorana spinors. We have that the barred conjugated spinor and Dirac adjoint spinor coincide

In the Majorana case we can have real representations for the gamma Matrices . For D=4

$$
\begin{aligned}
\gamma_0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1} \\
\gamma_1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1}\,, \\
\gamma_2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1 \\
\gamma_3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3
\end{aligned}
$$

Majorana spinors

Note that the γ_i are symmetric, while γ_0 is antisymmetric

$$\gamma^{\mu *} = -t_0 t_1 \, B \gamma^\mu B^{-1}$$

We have B=1, then Implies $\Psi^* = \Psi$

Dimensions of minimal spinors

Dim	Spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	1,2
4	M	4	1,2
5	S	8	2,3
6	SW	8	3
7	S	16	0,3
8	M	16	0,1
9	M	16	0,1
10	MW	16	1
11	M	32	1,2

Majorana spinors in physical theories

we consider a prototype action for a Majorana spinor field

for D=2,3, 4 mod 8 . Majorana and Dirac fields transform in the same way under Lorentz transformations, but half degrees of freedom

$$S[\Psi] = -\tfrac{1}{2} \int \mathrm{d}^D x\, \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

For commuting spinors $\Psi^T C \Psi'$ vanishes

$\Psi^T C \gamma^\mu \partial_\mu \Psi$ Is a total derivative, we need anticommuting Majorana spinors

$$\delta S[\Psi] = - \int \mathrm{d}^D x\, \delta\bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

The Majorana field satisfies the conventional Dirac equation

Majorana spinors in physical theories

Majorna action in terms of “Weyl” fields, D=4

$$
\begin{aligned}
S[\psi] &= -\tfrac{1}{2}\int \mathrm{d}^4x\,\left[\bar{\Psi}\gamma^\mu\partial_\mu - m\right](P_L + P_R)\Psi \\
&= -\int \mathrm{d}^4x\,\left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_R\Psi\right]
\end{aligned}
$$

equations of motion

$$
\not\partial P_L\Psi = mP_R\Psi\,, \qquad \not\partial P_R\Psi = mP_L\Psi
$$

The free Rarita-Schwinger field

Consider now a free spinor abelian gauge field

$\Psi_\mu(x)$ we omit the spinor indexes

Gauge transformation $\Psi_\mu(x) \rightarrow \Psi_\mu(x) + \partial_\mu \epsilon(x)$

Ψ_μ and ϵ are complex spinors with $2^{[D/2]}$ spinor components

This is fine for a free theory, but interacting supergravity theories are more restrictive .
We will need to use Majorana and/or Weyl spinors

Field strenght $\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu$ gauge invariant

The free Rarita-Schwinger field

- Action

Properties: a) Lorentz invariant, b) first order in space-time derivatives
c) gauge invariant, d) hermitean

$$S = -\int \mathrm{d}^D x \, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho$$

contains the third rank Clifford algebra element $\gamma^{\mu\nu\rho}$

$\bar{\Psi}_\mu$ is the Dirac conjugate $\quad \bar{\Psi}_\mu = \Psi^\dagger \mathrm{i}\gamma^0$

The lagrangian is invariant up to a total derivative

$$\delta\mathcal{L} = -\partial_\mu (\bar{\epsilon}\gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho)$$

The free Rarita-Schwinger field

- Equation of motion

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0$$

Noether identities $\quad \gamma^{\mu\nu\rho}\partial_\mu\partial_\nu\Psi_\rho = 0$

Using $\quad \gamma_\mu\gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$

$$\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^{\nu\rho} - \eta^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\mu$$

We can write the equations of motion as

$$\gamma^\mu(\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu) = 0$$

Differential geometry

- The metric and the frame field

Line element $\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$ Non-degenerate metric

$$g^{\mu\rho}g_{\rho\nu} = g_{\nu\rho}g^{\rho\mu} = \delta^\mu_\nu$$

Frame field $g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x)$ $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$

Inverse frame field $e^\mu_a(x)$ $e^a_\mu e^\mu_b = \delta^a_b$ and $e^\mu_a e^a_\nu = \delta^\mu_\nu$

Given the metric $g_{\mu\nu}(x)$, the frame field $e^a_\mu(x)$ is not uniquely determined. Any local Lorentz transformation $\Lambda^a{}_b(x)$, which leaves η_{ab} invariant, produces an equally good frame field

$$e'^a_\mu(x) = \Lambda^{-1\,a}{}_b(x)e^b_\mu(x)\,. \tag{6.27}$$

Differential geometry

- Frame field

$$e'^{a}_{\mu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} e^{a}_{\rho}(x) \,, \qquad e'^{\mu}_{a}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} e^{\rho}_{a}(x) \,.$$

$$V^{\mu}(x) = V^{a}(x) e^{\mu}_{a}(x) \text{ with } V^{a}(x) = V^{\mu}(x) e^{a}_{\mu}(x).$$

Vector under Lorentz transformations

$$V'^{a}(x) = \Lambda^{-1\,a}{}_{b}(x) V^{b}(x).$$

Vector field

$$E_a \equiv e^{\mu}_{a}(x) \frac{\partial}{\partial x^{\mu}} \,.$$

Dual form

$$e^{a} \equiv e^{a}_{\mu}(x) \mathrm{d}x^{\mu} \,. \qquad \langle e^{a} | E_b \rangle = \delta^{a}_{b}.$$

Volume forms and integration

any top degree D-form $\omega^{(D)}$ can be integrated

$$\begin{aligned} I &= \int \omega^{(D)} \\ &= \frac{1}{D!} \int \omega_{\mu_1 \cdots \mu_D}(x) \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_D} \\ &= \int \omega_{01 \cdots D-1} \mathrm{d}x^0 \mathrm{d}x^1 \ldots \mathrm{d}x^{D-1} \,. \end{aligned}$$

Canonical volume form depends of the metric or frame field

$$\begin{aligned} \mathrm{d}V &\equiv e^0 \wedge e^1 \wedge \ldots \wedge e^{D-1} \\ &= \frac{1}{D!} \varepsilon_{a_1 \cdots a_D} e^{a_1} \wedge \cdots \wedge e^{a_D} \\ &= \frac{1}{D!} e\, \varepsilon_{\mu_1 \cdots \mu_D} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_D} \end{aligned}$$

Hodge duality of forms

$$^*e^{a_1} \wedge \ldots e^{a_p} = \frac{1}{q!} e^{b_1} \wedge \ldots e^{b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p}$$

$$\begin{aligned} \Omega^{(q)} =^* \omega^{(p)} &= {}^*(\frac{1}{p!} \omega_{a_1 \cdots a_p} e^{a_1} \wedge \ldots e^{a_p}) \\ &= \frac{1}{p!} \omega_{a_1 \cdots a_p} {}^*e^{a_1} \wedge \ldots e^{a_p} \,. \end{aligned}$$

Lorentzian signature $\quad {}^*({}^*\omega^{(p)}) = -(-)^{pq} \omega^{(p)}$

Euclidean signature $\quad {}^*({}^*\omega^{(p)}) = (-)^{p\breve{q}} \omega^{(p)}$

p-forms gauge fields

$$S_0 = -\tfrac{1}{2}\int {}^*F^{(1)} \wedge F^{(1)}\,, \qquad F^{(1)} = \mathrm{d}\phi\,,$$

$$S_1 = -\tfrac{1}{2}\int {}^*F^{(2)} \wedge F^{(2)}\,, \qquad F^{(2)} = \mathrm{d}A^{(1)}$$

Bianchi identity $\quad \mathrm{d}F^{(1)} = 0 \text{ and } \mathrm{d}F^{(2)} = 0.$

$$S_p = -\tfrac{1}{2}\int {}^*F^{(p+1)} \wedge F^{(p+1)}\,, \qquad F^{(p+1)} = \mathrm{d}A^{(p)}$$

$$S_p = -\frac{1}{2\,(p+1)!}\int \mathrm{d}^D x\,\sqrt{-g}\,F^{\mu_1\cdots\mu_{p+1}}F_{\mu_1\cdots\mu_{p+1}}$$

$$F_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}$$

p-forms gauge fields

$$d^*F^{(p+1)} = 0$$ equations of motion, useful relation

$$A \wedge B^* = (-1)^{p(D-p)} A^* \wedge B$$

$$dF^{(p+1)} = 0$$ Bianchi identity

A p-form and D-p-2 form are dual

$$S_p = -\int \left[\tfrac{1}{2}{}^*F^{(p+1)} \wedge F^{(p+1)} + b^{(D-p-2)} \wedge dF^{(p+1)} \right]$$

can consider $F^{(p+1)}$ and $b^{(D-p-2)}$ as the independent fields.

Algebraic equation of motion $${}^*F^{(p+1)} = (-)^{D-p}\, db^{(D-p-2)}$$

$b^{(D-p-2)}$ takes the role of $A^{(p)}$

First structure equation

- Spin connection

Let us consider the differential of the vielbvein

$$\mathrm{d}e^a = \frac{1}{2}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

it is not a Lorentz vector. Introduce the spin connection connection one form

$$\omega'^a{}_b = \Lambda^{-1\,a}{}_c \mathrm{d}\Lambda^c{}_b + \Lambda^{-1\,a}{}_c\,\omega^c{}_d\,\Lambda^d{}_b$$

The quantity

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a\,.$$

transforms as a vector

$$T'^a = \Lambda^{-1\,a}{}_b T^b.$$

$$\text{torsion tensor } T^a_{\mu\nu} = -T^a_{\nu\mu}$$

$$\omega'_\mu{}^a{}_b = \Lambda^{-1\,a}{}_c\,\partial_\mu \Lambda^c{}_b + \Lambda^{-1\,a}{}_c\,\omega_\mu{}^c{}_d\,\Lambda^d{}_b\,.$$

same transformation properties that YM potential for the group O(D-1,1)

First structure equation

- Lorentz Covariant derivatives

$$
\begin{aligned}
V'^{a}(x) &= \Lambda^{-1\,a}{}_{b}(x)V^{b}(x)\,, \\
U'_{a}(x) &= U_{b}(x)\Lambda^{b}{}_{a}(x)\,, \\
T'_{ab}(x) &= T_{cd}(x)\Lambda^{c}{}_{a}(x)\Lambda^{d}{}_{b}(x)\,,
\end{aligned}
$$

$$
\begin{aligned}
D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b\,, \\
D_\mu U_a &= \partial_\mu U_a - U_b\omega_\mu{}^b{}_a = \partial_\mu U_a + \omega_{\mu a}{}^b U_b\,, \\
D_\mu T_{ab} &= \partial_\mu T_{ab} - T_{cb}\omega_\mu{}^c{}_a - T_{ac}\omega_\mu{}^c{}_b\,.
\end{aligned}
$$

$$
D_\mu \eta_{ab} = -\eta_{cb}\omega_\mu{}^c{}_a - \eta_{ac}\omega_\mu{}^c{}_b = -\omega_{\mu ba} - \omega_{\mu ab} = 0\,.
$$

The metric has vanishing covarint derivative.

First structure equation

scalar products $V^a U_a$ are preserved under parallel transport

$$\tilde{V}^a(x) = V^a(x) + \omega^a_{\mu b} V^a(x) \Delta x^\mu$$

The geometrical effect of torsion is seen in the properties of an infinitesimal parallelogram constructed by the parallel transport of two vector fields.

For the Levi-Civita connection the torsion vanishes

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$$

Non-vanishing torsion appears in supergravity

First structure equation

- Covariant derivatives

$$\omega_{\mu[\rho\nu]} = \omega_{\mu ab} e^a_\rho e^b_\nu \text{ in terms of}$$

$$\Omega_{[\mu\nu]\rho} = (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)\ e_{a\rho}\,, \qquad T_{[\mu\nu]\rho} = T_{\mu\nu}{}^a e_{a\rho}$$

The structure equation $$T_{[\mu\nu]\rho} = \Omega_{[\mu\nu]\rho} + \omega_{\mu[\rho\nu]} - \omega_{\nu[\rho\mu]}\,.$$

implies

$$\begin{aligned}
\omega_{\mu[\nu\rho]} &= \omega_{\mu[\nu\rho]}(e) + K_{\mu[\nu\rho]}\,,\\
\omega_{\mu[\nu\rho]}(e) &= \tfrac{1}{2}(\Omega_{[\mu\nu]\rho} - \Omega_{[\nu\rho]\mu} + \Omega_{[\rho\mu]\nu}) = \omega_{\mu ab}(e) e^a_\nu e^b_\rho\,,\\
\omega_\mu{}^{ab}(e) &= 2e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c\,,\\
K_{\mu[\nu\rho]} &= -\tfrac{1}{2}(T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu})\,.
\end{aligned}$$

$K_{\mu[\nu\rho]}$ is called contorsion

First structure equation

- The affine connection

Our next task is to transform Lorentz covariant derivatives to covariant derivatives with respect to general conformal transformations

$$\begin{aligned}\nabla_\mu V^\rho &\equiv e_a^\rho D_\mu V^a \\ &= e_a^\rho D_\mu (e_\nu^a V^\nu) \\ &= \partial_\mu V^\rho + e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b) V^\nu \,.\end{aligned}$$

Affine connection $\quad \Gamma_{\mu\nu}^\rho = e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b)\,.$

relates affine connection with spin connection

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma_{\mu\nu}^\rho V^\nu$$

First structure equation

- The affine connection

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}(g) - K_{\mu\nu}{}^{\rho} \,,$$

$$\Gamma^{\rho}_{\mu\nu}(g) = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) .$$

$$\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = -K_{\mu\nu}{}^{\rho} + K_{\nu\mu}{}^{\rho} = T_{\mu\nu}{}^{\rho} \,.$$

For mixed quantities with both coordinate ans frame indexes, it is useful to distinguish among local Lorentz and coordinate covariant derivatives

$$D_\mu \Psi_\nu \equiv \left(\partial_\mu + \tfrac{1}{4} \omega_{\mu ab} \gamma^{ab} \right) \Psi_\nu \,,$$

$$\nabla_\mu \Psi_\nu = D_\mu \Psi_\nu - \Gamma^{\rho}_{\mu\nu} \Psi_\rho \,.$$

Vielbein postulate equivalent to $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^{\sigma}_{\mu\nu} e^a_\sigma = 0$

First structure equation

- Partial integration We have

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\,\Gamma^{\rho}_{\rho\mu}(g)$$ from which

$$\int \mathrm{d}^D x \sqrt{-g}\,\nabla_\mu V^\mu = \int \mathrm{d}^D x\, \partial_\mu \left(\sqrt{-g}\, V^\mu\right) - \int \mathrm{d}^D x \sqrt{-g}\, K_{\nu\mu}{}^{\nu} V^\mu$$

The second term shows the violation of the manipulations of the integration by Parts in the case of torsion

$$K_{\nu\mu}{}^{\nu} = -T_{\nu\mu}{}^{\nu}$$

Second structure equation

- Curvature tensor

spin connection $\omega_{\mu ab}$ transforms as a YM gauge potential for the Group O(D-1,1)

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b$$

YM field strength. We define the curvature two form

$$\rho^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,.$$

Second structure equation

$$\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab} \,.$$

Bianchi identities

$$\rho^{ab} \wedge e_b = \mathrm{d}T^a + \omega^{ab} \wedge T_b \,,$$
$$\mathrm{d}\rho^{ab} + \omega^a{}_c \wedge \rho^{cb} - \rho^{ac} \wedge \omega_c{}^b = 0$$

$using\ R_{\mu\nu\rho}{}^a = R_{\mu\nu b}{}^a e^b_\rho$ we have

$$R_{\mu\nu\rho}{}^a + R_{\nu\rho\mu}{}^a + R_{\rho\mu\nu}{}^a = -D_\mu T_{\nu\rho}{}^a - D_\nu T_{\rho\mu}{}^a - D_\rho T_{\mu\nu}{}^a$$

First Bianchi identity, it has no analogue in YM

$$D_\mu R_{\nu\rho}{}^{ab} + D_\nu R_{\rho\mu}{}^{ab} + D_\rho R_{\mu\nu}{}^{ab} = 0$$

usual Bianchi identity for YM

useful relation $\delta R_{\mu\nu ab} = D_\mu \delta\omega_{\nu ab} - D_\nu \delta\omega_{\mu ab}$

Ricci identities and curvature tensor

Commutator of covariant derivatives

$$
\begin{aligned}
[D_\mu, D_\nu]\Phi &= \tfrac{1}{2} R_{\mu\nu ab} M^{ab} \Phi \\
[D_\mu, D_\nu]V^a &= R_{\mu\nu}{}^a{}_b V^b \,, \\
[D_\mu, D_\nu]\Psi &= \tfrac{1}{4} R_{\mu\nu ab} \gamma^{ab} \Psi \,.
\end{aligned}
$$

$$
[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma - T_{\mu\nu}{}^\sigma \nabla_\sigma V^\rho
$$

Curvature tensor

$$
R_{\mu\nu}{}^\rho{}_\sigma \;=\; \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau}\Gamma^\tau_{\mu\sigma}
$$

$$
R_{\mu\nu}{}^\rho{}_\sigma \quad = \quad R_{\mu\nu ab} e^{a\rho} e^b_\sigma
$$

Second Bianchi identity

$$
\nabla_\mu R_{\nu\rho}{}^{\sigma\tau} + \nabla_\nu R_{\rho\mu}{}^{\sigma\tau} + \nabla_\rho R_{\mu\nu}{}^{\sigma\tau} = T_{\mu\nu}{}^\xi R_{\xi\rho}{}^{\sigma\tau} + T_{\nu\rho}{}^\xi R_{\xi\mu}{}^{\sigma\tau} + T_{\rho\mu}{}^\xi R_{\xi\nu}{}^{\sigma\tau}
$$

Ricci tensor

Ricci tensor $$R_{\mu\nu} \; = \; R_{\mu}{}^{\sigma}{}_{\nu\sigma}$$

Scalar curvature R= $g^{\mu\nu} R_{\mu\nu}$

If there is no torsion $$R_{\mu\nu} \; = \; R_{\nu\mu} \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

Useful relation $$\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} = \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}$$

Hilbert action

$$\frac{1}{(D-2)!}\int \varepsilon_{abc_1\ldots c_{(D-2)}} e^{c_1} \wedge \cdots \wedge e^{c_{(D-2)}} \wedge \rho^{ab} = \int \mathrm{d}^D x \sqrt{-g} R$$

Dimensional analysis and Planck units

$$\frac{Gm_p}{c^2} = \frac{\hbar}{m_p c}$$

$$m_p = \sqrt{\frac{\hbar c}{G}} \qquad \simeq 10^{-5}\,\mathrm{grams} \simeq 10^{19}\,\mathrm{GeV}$$

$$l_p = \sqrt{\frac{G\hbar}{c^3}} = \sqrt{\frac{\kappa^2}{8\pi}} \qquad \simeq 10^{-33}\,\mathrm{cm}$$

$$t_p = \sqrt{\frac{G\hbar}{c^5}} = \sqrt{\frac{\kappa^2}{8\pi c^2}} \quad \simeq 10^{-44}\,\mathrm{seconds}$$

General relativity

- Einstein-Hilbert action in first order formalism

$$S = \frac{1}{2\kappa^2}\int \epsilon_{abcd} R^{ab}(\omega)\wedge e^c \wedge e^d = \frac{1}{2\kappa^2}\int d^4 x e R(e,\omega)$$

GR can be view as a “gauge” theory of the Poincare group?

$$\{M_{ab}, P_a\} \longrightarrow \mathcal{A}_\mu = \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a$$

$$F_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] = R_{\mu\nu}{}^{ab} M_{ab} + T_{\mu\nu}{}^a P_a$$

$R_{\mu\nu}{}^{ab}$ $T_{\mu\nu}{}^a$ are the components of the Lorentz curvature and torsion

General relativity

The Poincare gauge transformations are

$$\delta \mathcal{A}_\mu = \partial_\mu \epsilon + [\epsilon, \mathcal{A}_\mu] \qquad \epsilon = \epsilon^a P_a + \epsilon^{ab} M_{ab}$$

Local translations $\quad \delta e^a = D\epsilon^a, \qquad \delta\omega^{ab} = 0$

Local Lorentz $\quad \delta e^a = \epsilon^a{}_b e^a, \qquad \delta\omega^{ab} = -D\epsilon^{ab}$

Up to a total derivate we EH action is not invariant

$$\delta S = \frac{1}{\kappa^2} \int \epsilon_{abcd} R^{ab} \wedge T^c \epsilon^d$$

If we impose by hand the vanishing of the torsion we have invariance. Notice the vanishing is the equation of motion of EH action with respect to the spin connection. In this way we get the second order formulation

The first order formalism for gravity and fermions

Field content frame field e^a_μ and spin connection $\omega_{\mu ab}$

Fermion field $\Psi(x)$

Action

$$S = S_2 + S_{1/2} = \int \mathrm{d}^D x\, e \left[\frac{1}{2\kappa^2} e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}(e) - \frac{1}{2}\bar{\Psi}\gamma^\mu \nabla_\mu \Psi + \frac{1}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu \gamma^\mu \Psi \right]$$

$$\begin{aligned} \nabla_\mu \Psi &= D_\mu \Psi = (\partial_\mu + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab})\Psi \\ \bar{\Psi}\overleftarrow{\nabla}_\mu &= \bar{\Psi}\overleftarrow{D}_\mu = \bar{\Psi}(\overleftarrow{\partial}_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}) \end{aligned}$$

The total covariant derivative and the Lorentz covariant derivative coincide for spinor field but not for the gravitino

The first and second order formulations of general relativity

- Curved space gamma matrices

Constant gamma matrices verify $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$

Frame fields are used to transform frame *vector* indices to a coordinate basis.

$$\gamma^\mu = e^\mu_a \gamma^a = g^{\mu\nu}\gamma_\nu \qquad \{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$$

The curved gamma matrices transforms a vector under coordinate transformations
But they have also spinor indexes

$$\begin{aligned}\nabla_\mu \gamma_\nu &= \partial_\mu \gamma_\nu + \frac{1}{4}\omega_\mu^{ab}[\gamma_{ab}, \gamma_\nu] - \Gamma^\rho_{\mu\nu}\gamma_\rho \\ &= \gamma^a(\partial_\mu e_{a\nu} + \omega_{\mu ab} e^b_\nu - \Gamma^\rho_{\mu\nu} e_{a\rho})\end{aligned}$$

$\nabla_\mu \gamma_\nu = 0$ holds for any affine connection with or without torsion

The first and second order formulations of general relativity

- Fermion equation of motion

$$\gamma^\mu \nabla_\mu \Psi \;=\; 0$$

$$\gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu \Psi \;=\; \left(g^{\mu\nu} \nabla_\mu \nabla_\nu \;-\; \tfrac{1}{4} R\right) \Psi$$

The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

Euation of motion of the spin connection

Variation of the gravitational action

$$\delta S_2 = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \; e_a^\mu e_b^\nu \left(D_\mu \delta \omega_\nu{}^{ab} - D_\nu \delta \omega_\mu{}^{ab} \right)$$

We have used $\delta R_{\mu\nu ab} = D_\mu \delta\omega_{\nu ab} - D_\nu \delta\omega_{\mu ab}$

$$\delta S_2 = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \; e_a^\mu e_b^\nu \left(2\nabla_\mu \delta \omega_\nu{}^{ab} + T_{\mu\nu}{}^\rho \delta \omega_\rho{}^{ab} \right)$$

The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

Integration by parts

$$\int \mathrm{d}^D x \sqrt{-g}\, \nabla_\mu V^\mu = \int \mathrm{d}^D x\, \partial_\mu \left(\sqrt{-g}\, V^\mu\right) - \int \mathrm{d}^D x \sqrt{-g}\, K_{\nu\mu}{}^\nu V^\mu$$

$$\begin{aligned} \delta S_2 &= \frac{1}{2\kappa^2} \int \mathrm{d}^D x\, e \left(-2K_{\rho\mu}{}^\rho e_a^\mu e_b^\nu \delta\omega_\nu{}^{ab} + T_{ab}{}^\rho \delta\omega_\rho{}^{ab} \right) \\ &= \frac{1}{2\kappa^2} \int \mathrm{d}^D x\, e \left(T_{\rho a}{}^\rho e_b^\nu - T_{\rho b}{}^\rho e_a^\nu + T_{ab}{}^\nu \right) \delta\omega_\nu{}^{ab} , \end{aligned}$$

Form the fermion action

$$\begin{aligned} \delta S_{1/2} &= -\tfrac{1}{8} \int \mathrm{d}^D x\, e\, \bar{\Psi} \{\gamma^\nu , \gamma_{ab}\} \Psi\, \delta\omega_\nu^{ab} \\ &= -\tfrac{1}{4} \int \mathrm{d}^D x\, e\, \bar{\Psi}\, \gamma^\nu{}_{ab}\, \Psi\, \delta\omega_\nu^{ab} . \end{aligned}$$

The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

The equations of motion of the spin connection gives

$$T_{ab}{}^{\nu} - T_{a\rho}{}^{\rho}\, e_b^{\nu} + T_{b\rho}{}^{\rho}\, e_a^{\nu} \;= \tfrac{1}{2}\kappa^2 \bar{\Psi}\, \gamma_{ab}{}^{\nu}\, \Psi$$

the right hand side is traceless therefore also the torsion is traceless

$$T_{ab}{}^{\nu} = \tfrac{1}{2}\kappa^2 \bar{\Psi}\, \gamma_{ab}{}^{\nu}\, \Psi = -2K^{\nu}{}_{ab}$$

If we substitute $\omega = \omega(e) + K$

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\ \Big[R(g) - \kappa^2 \bar{\Psi}\gamma^{\mu}\overleftrightarrow{\nabla}_{\mu}\Psi \qquad (7$$

$$-2\nabla_{\mu}K_{\nu}{}^{\nu\mu} + K_{\mu\nu\rho}K^{\nu\mu\rho} - K_{\rho}{}^{\rho}{}_{\mu}K_{\sigma}{}^{\sigma\mu} - \tfrac{1}{2}\bar{\Psi}\gamma_{\mu\nu\rho}\Psi\, K^{\mu\nu\rho}\Big]$$

The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

The physical equivalent second order action is

$$S = \frac{1}{2}\int \mathrm{d}^D x\, e\;\left[\frac{1}{\kappa^2}R(g) - \bar{\Psi}\gamma^\mu \overleftrightarrow{\nabla}_\mu \Psi + \frac{1}{16}\kappa^2(\bar{\Psi}\gamma_{\mu\nu\rho}\Psi)(\bar{\Psi}\gamma^{\mu\nu\rho}\Psi)\right]$$

Physical effects in the fermion theories with torsion and without torsion
Differ only in the presence of quartic fermion term.
This term generates 4-point contact diagrams .

N=1 Global Supersymmetry in D=4

- Susy algebra

$$\begin{aligned}
\left\{Q_\alpha, \bar{Q}^\beta\right\} &= -\tfrac{1}{2}(\gamma_\mu)_\alpha{}^\beta P^\mu\,, \\
\left[M_{[\mu\nu]}, Q_\alpha\right] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta\,, \\
[P_\mu, Q_\alpha] &= 0\,.
\end{aligned}$$

equivalently

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\gamma_\mu C^{-1}\right)_{\alpha\beta} P^\mu \qquad C^{-1} = i\,\gamma^0$$

at quantum level

$$\left\{Q_\alpha, (Q^\dagger)^\beta\right\}_{qu} = \tfrac{1}{2}\left(\gamma_\mu \gamma^0\right)_\alpha{}^\beta P^\mu$$

N=1 Global Supersymmetry in D=4

- Susy algebra

In Weyl basis

$$\begin{aligned}\{Q_{+\alpha}, Q_{-\beta}\} &= -\frac{1}{2}\left(P_L\gamma_\mu C^{-1}\right)_{\alpha\beta} P^\mu \\ \{Q_{+\alpha}, Q_{+\beta}\} &= 0 \\ \{Q_{-\alpha}, Q_{-\beta}\} &= 0\end{aligned}$$

$$Q_+ = Q_L,\, Q_- = Q_R$$
$$P_+ = P_L, P_+ = P_R$$

In this form it is obvious the U(1) R symmetry

$$Q_\pm \to e^{\mp i\alpha} Q_\pm$$

$$[T_R, Q_\pm] = \mp i Q_\pm\,, \qquad [T_R, P_m] = 0\,, \qquad [T_R, J_{mn}] = 0$$

$$[T_R, Q_\alpha] = -\mathrm{i}(\gamma_*)_\alpha{}^\beta Q_\beta$$

N=1 Global Supersymmetry in D=4

- Basic multiplets

The states of particles with momentum $\vec{p}$ and energy $E(\vec{p}) = \sqrt{\vec{p}^2 + m^2_{B,F}}$

$$|\vec{p}, B\rangle \text{ and } |\vec{p}, F\rangle$$

$$Q_\alpha|\vec{p}, B\rangle = |\vec{p}, F\rangle \text{ and } Q_\alpha|\vec{p}, F\rangle \propto |\vec{p}, B\rangle.$$

Since $[P^\mu, Q_\alpha] = 0$, $\quad m^2_B = m^2_F$

chiral multiplet

complex spin 0 boson $Z(x) = (A(x) + \mathrm{i}B(x))/\sqrt{2}$

spin 1/2 fermion Majorana field $\chi(x)$ or Weyl spinor $P_L\chi$

N=1 Global Supersymmetry in D=4

- Susy field theories of the chiral multiplet

transformation rules

$$\begin{aligned}
\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,, \\
\delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\partial\!\!\!/ Z + F)\epsilon \\
\delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\partial\!\!\!/ P_L\chi\,.
\end{aligned}$$

N=1 Global Supersymmetry in D=4

- Transformations rules of the antichiral multiplet

$$
\begin{aligned}
\delta \bar{Z} &= \frac{1}{\sqrt{2}} \bar{\epsilon} P_R \chi \,, \\
\delta P_R \chi &= \frac{1}{\sqrt{2}} P_R (\partial\!\!\!/ \bar{Z} + \bar{F}) \epsilon \\
\delta \bar{F} &= \frac{1}{\sqrt{2}} \bar{\epsilon} \, \partial\!\!\!/ P_R \chi \,.
\end{aligned}
$$

the variations $\delta \bar{Z}$, $\delta P_R \chi$, $\delta \bar{F}$ *are precisely the adjoints of*
δZ, $\delta P_L \chi$, δF

N=1 Global Supersymmetry in D=4

- Susy algebra

$$
\begin{aligned}
[\delta_1, \delta_2] Z &= \frac{1}{\sqrt{2}} \delta_1 (\bar{\epsilon}_2 P_L \chi) - [1 \leftrightarrow 2] \\
&= \frac{1}{2} \bar{\epsilon}_2 P_L (\not{\partial} Z + F) \epsilon_1 - [1 \leftrightarrow 2] \\
&= -\frac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu Z \, .
\end{aligned}
$$

The symmetry properties of Majorana spinor bilinears has been used

N=1 Global Supersymmetry in D=4

- Action

$$S_{\rm kin} = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z}\partial_\mu Z - \bar{\chi}\not{\partial} P_L\chi + \bar{F}F\right]$$

$$S_F = \int \mathrm{d}^4x \left[FW'(Z) - \tfrac{1}{2}\bar{\chi}P_L W''(Z)\chi\right]$$

W(Z) superpotential, arbitrary holomorphic function of Z

the action S_F is not Hermitian $\qquad S_{\bar{F}} = (S_F)^\dagger$

Complete action $\qquad S = S_{\rm kin} + S_F + S_{\bar{F}}$

$F \quad \bar{F}$ Are not a dynamical field, their equations of motion are algebraic

$F = -\overline{W}'(\bar{Z}) \qquad \bar{F} = -W'(Z)$ we can eliminate them

N=1 Global Supersymmetry in D=4

- Wess-Zumino model

$$W = \tfrac{1}{2} m Z^2 + \tfrac{1}{3} g Z^3$$

Eliminating the auxiliary field F

$$\begin{aligned} S_{WZ} \;&=\; \int \mathrm{d}^4 x \left[\frac{1}{2}\left(-(\partial A)^2 - m^2 A^2 - (\partial B)^2 - m^2 B^2 - \bar{\chi}(\partial\!\!\!/ - m)\chi\right) \right. \\ &\left. \quad + \frac{g}{\sqrt{2}} \bar{\chi}(A + \mathrm{i}\gamma_* B)\chi + \frac{mg}{\sqrt{2}}(A^3 + AB^2) + \frac{g^2}{4}(A^2 + B^2)^2 \right] . \end{aligned}$$

N=1 Global Supersymmetry in D=4

- Susy algebra

Note that the anticommutator $\{Q, \bar{Q}\}$ is realized as the commutator of two variations with parameters ϵ_1, ϵ_2

$$
\begin{aligned}
[\delta_1, \delta_2]\Phi(x) &= \left[\bar{\epsilon}_1 Q, \left[\bar{Q}\epsilon_2, \Phi(x)\right]\right] - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \bar{\epsilon}_1^{\alpha}[\{Q_\alpha, \bar{Q}^\beta\}, \Phi(x)]\epsilon_{2\beta} \\
&= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\,\partial_\mu\Phi(x)\,.
\end{aligned}
$$

$\bar{\epsilon}Q = \bar{Q}\epsilon$ for Majorana spinors

If we compute the left hand side, this dones not the anticommutator of the fermionic charges because any bosonic charge that commutes with field will not contribute

N=1 Global Supersymmetry in D=4

Consider the theory after elimination of F and $\bar{F}$

$$F = -\overline{W}'(\bar{Z})$$

$$S = \int \mathrm{d}^4 x \left[-\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \not{\partial} P_L \chi - \overline{W}' W' + \tfrac{1}{2} \bar{\chi} (P_L W'' + P_R \overline{W}'') \chi \right]$$

Now the symmetry algebra only closes on-shell

$$[\delta_1, \delta_2] P_L \chi = -\tfrac{1}{4} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_L \left[\partial_\mu \chi - \gamma_\mu (\not{\partial} - \overline{W}'') \chi \right]$$

the extra factor apart from translation is a symmetric combination of the equation of the fermion field

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- Basics

supersymmetry holds *locally* in a supergravity theory.

the spinor parameters
$\epsilon(x)$ are arbitrary functions of the spacetime coordinates.

The SUSY algebra $\quad [\delta_1, \delta_2]\Phi(x) = \; -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\,\partial_\mu\Phi(x)$

will then involve local translation parameters $\bar{\epsilon}_1\gamma^\mu\epsilon_2$

Therefore we have diffeomorphism. Thus local susy requires gravity

fields $\qquad e^a_\mu(x) \qquad \Psi^i_\mu(x),\; i = 1, \ldots, \mathcal{N}$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

There are four major applications of supergravity

1) If there is some sort of broken global symmetry. N=1 D=4 supergravity coupled to chiral and gauge multiplets of global Susy could describe the physics of elementary particles

2) D=10 supergravity is the low energy limit of superstring theory. Solutions of SUGRA exhibit spacetime compactification

3) Role of D=11 supergravity for M-theory

4) AdS/CFT in the limit in which string theory is approximated by supergravity. correlations of the boundary gauge theory at strong coupling are available from weak coupling classical calculations in five and ten dimensional supergravity

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

From Supersymmetry to SUGRA

$$\{M_{ab}\,, P_a\,, Q^\alpha\} \underset{\text{gauging}}{\longrightarrow} \mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a + \bar{\psi}_{\mu\alpha}Q^\alpha$$

$$R_{\mu\nu} \equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] = \tfrac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + R_{\mu\nu}{}^a P_a + \bar{R}_{\mu\nu\alpha}Q^\alpha$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- Transformation rules for gauge theory point of view

Gauge prameters

$$\epsilon = \epsilon^a P_a + \epsilon^{ab} M_{ab} + \bar{\epsilon}^\alpha Q_\alpha = \epsilon^a P_a + \epsilon^{ab} M_{ab} + \bar{Q}^\alpha \epsilon_\alpha$$

gauge transformations

$$\delta \mathcal{A}_\mu = \partial_\mu \epsilon + [\epsilon, \mathcal{A}_\mu]$$

$$\begin{aligned} \delta e_\mu^a &= \tfrac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \,, \\ \delta \psi_\mu &= D_\mu \epsilon(x) \equiv \partial_\mu \epsilon + \tfrac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon \end{aligned}$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

First order formalism

We regard the spin connection as an independent variable from the frame field. The action is

$$
\begin{aligned}
S &= S_2 + S_{3/2} \,, \\
S_2 &= \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) \,, \\
S_{3/2} &= -\frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \, \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \,,
\end{aligned}
$$

$$D_\nu \psi_\rho \equiv \partial_\nu \psi_\rho + \tfrac{1}{4} \omega_{\nu ab} \gamma^{ab} \psi_\rho \,.$$

$\kappa^2 = 8\pi G_N$ is the gravitational coupling constant

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- First order formalism

  Let us compute the equations of the spin connection

$$\delta S_{3/2} = -\frac{1}{8\kappa^2} \int \mathrm{d}^D x \, e \, (\bar{\psi}_\mu \gamma^{\mu\nu\rho} \gamma_{ab} \psi_\rho) \delta\omega_\nu{}^{ab}$$

valid for D=2,3,4,10, 11 where Majorana spinors exist

spinor bilinears of rank 3 are symmetric ,therefore we have

$$\bar{\psi}_\mu \gamma^{\mu\nu\rho} \gamma_{ab} \psi_\rho = \bar{\psi}_\mu \left(\gamma^{\mu\nu\rho}{}_{ab} + 6\gamma^{[\mu} e^\nu{}_{[b} e^{\rho]}{}_{a]} \right) \psi_\rho$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

$$\begin{aligned}
\delta S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \left(-2K_{\rho\mu}{}^{\rho} e_a^{\mu} e_b^{\nu}\delta\omega_{\nu}{}^{ab} + T_{ab}{}^{\rho}\delta\omega_{\rho}{}^{ab}\right) \\
&= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \left(T_{\rho a}{}^{\rho} e_b^{\nu} - T_{\rho b}{}^{\rho} e_a^{\nu} + T_{ab}{}^{\nu}\right)\delta\omega_{\nu}{}^{ab}\,,
\end{aligned}$$

The spin connection equation of motion is

$$\delta S_2 + \delta S_{3/2} = 0$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- First order formalism

$$T_{ab}{}^{\nu} = \frac{1}{2}\bar{\psi}_a\gamma^{\nu}\psi_b + \frac{1}{4}\bar{\psi}_{\mu}\gamma^{\mu\nu\rho}{}_{ab}\psi_{\rho}$$

The fifth rank tensor vanishes for D=4. For dimensions D>4 this term is not Vanishing and is one the complications of supergravity

The equivalent second order action of gravity is

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\, \left[R(e) - \bar{\psi}_{\mu}\gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho} + \mathcal{L}_{\mathrm{SG,torsion}}\right]$$

$$\mathcal{L}_{\mathrm{SG,torsion}} = -\frac{1}{16}\left[(\bar{\psi}^{\rho}\gamma^{\mu}\psi^{\nu})(\bar{\psi}_{\rho}\gamma_{\mu}\psi_{\nu} + 2\bar{\psi}_{\rho}\gamma_{\nu}\psi_{\mu}) - 4(\bar{\psi}_{\mu}\gamma\cdot\psi)(\bar{\psi}^{\mu}\gamma\cdot\psi)\right]$$

With

$$D_{\nu}\psi_{\rho} \equiv \partial_{\nu}\psi_{\rho} + \frac{1}{4}\omega_{\nu ab}(e)\gamma^{ab}\psi_{\rho}$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- Local supersymmetry transformations

The second order action for N=1 D=4 is supergravity is complete and it is local supersymmetry

$$\delta e_\mu^a = \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \,,$$

$$\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon \equiv \partial_\mu\epsilon + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon \,, \\
\omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab} \,, \\
K_{\mu\nu\rho} &= -\tfrac{1}{4}(\bar{\psi}_\mu\gamma_\rho\psi_\nu - \bar{\psi}_\nu\gamma_\mu\psi_\rho + \bar{\psi}_\rho\gamma_\nu\psi_\mu)
\end{aligned}$$

which includes the gravitino torsion

The variation of the action contains terms which are first, third and fifth order in the gravitino field. The terms are independent and must cancel separately

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- The universal part of supergravity. Second order formalism

$$
\begin{aligned}
S &= S_2 + S_{3/2}\,, \\
S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega)\,, \\
S_{3/2} &= -\frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho\,,
\end{aligned}
$$

$D_\nu\psi_\rho \equiv \partial_\nu\psi_\rho + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\psi_\rho\,.$ We not need to include the connection $\Gamma^\sigma_{\nu\rho}(g)\psi_\sigma$ due to symmetry properties

$\omega_{\nu ab}(e)$; Is the torsion-free spin connection

$\kappa^2 = 8\pi G_N$ is the gravitational coupling constant

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- Transformation rules

The variation of the action consists of terms linear in ψ_μ From the frame field variation and the gravitino variation and cubic terms from the field variation of the gravitino action

Variation of the gravitational action

$$\delta S_2 = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) (-\bar{\epsilon}\gamma^\mu \psi^\nu)$$

$\mathcal{N}=1$ pure supergravity in 4 dimensions

gravitino variation

In the second order formalism, partial integration is valid, so we compute $\delta\bar{\psi}_\mu$ by two

$$\begin{aligned}\delta S_{3/2} &= -\frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\epsilon}\overleftarrow{D}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \\ &= \frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\epsilon}\gamma^{\mu\nu\rho} D_\mu D_\nu \psi_\rho = \frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\epsilon}\gamma^{\mu\nu\rho} R_{\mu\nu ab}\gamma^{ab}\psi_\rho\end{aligned}$$

We now need some Dirac algebra to evaluate the product $\gamma^{\mu\nu\rho}\gamma^{ab}$

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

$$\begin{aligned}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab} &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 6R_{\mu\nu}{}^{[\rho}{}_b\gamma^{\mu\nu]b} + 6\gamma^{[\mu}R_{\mu\nu}{}^{\rho\nu]} \\ &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 2R_{\mu\nu}{}^{\rho}{}_b\gamma^{\mu\nu b} + 4R_{\mu\nu}{}^{\mu}{}_b\gamma^{\nu\rho b} \\ &\quad +4\gamma^{\mu}R_{\mu\nu}{}^{\rho\nu} + 2\gamma^{\rho}R_{\mu\nu}{}^{\nu\mu}\,,\end{aligned}$$

First Bianchi identity without torsin

$$R_{\mu\nu\rho}{}^{a} + R_{\nu\rho\mu}{}^{a} + R_{\rho\mu\nu}{}^{a} = 0\;, \quad R_{\mu\nu\rho}{}^{a} = R_{\mu\nu b}{}^{a}e^{b}_{\rho} \qquad R_{\nu b} = R_{\mu\nu}{}^{\mu}{}_{b}$$

Finaly we have

$$\delta S_{3/2} = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\,(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(\bar{\epsilon}\gamma^{\mu}\psi^{\nu})$$

Therefore the linear terms cancel

Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

The algebra of local supersymmetry

$$
\begin{aligned}
[\delta_1,\delta_2]\, e_\mu^a &= \tfrac{1}{2}\delta_1\bar{\epsilon}_2\gamma^a\psi_\mu - (1\leftrightarrow 2) = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\delta_1\psi_\mu - (1\leftrightarrow 2) \\
&= \tfrac{1}{2}\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 - (1\leftrightarrow 2) \\
&= \tfrac{1}{2}(\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 + D_\mu\bar{\epsilon}_2\gamma^a\epsilon_1) \\
&= D_\mu\xi^a\,, \qquad \xi^a = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1 = -\tfrac{1}{2}\bar{\epsilon}_1\gamma^a\epsilon_2\,.
\end{aligned}
$$

Infinitesimal transformation of the frame field

$$\delta_\xi e_\mu^a = \xi^\rho\partial_\rho e_\mu^a + \partial_\mu\xi^\rho e_\rho^a$$

covariant form

$$
\begin{aligned}
\delta_\xi e_\mu^a &= \xi^\rho\nabla_\rho e_\mu^a - \xi^\rho\omega_\rho{}^a{}_b e_\mu^b + \xi^\rho\Gamma^\sigma_{\rho\mu}e_\sigma^a + \nabla_\mu\xi^\rho\, e_\rho^a - \Gamma^\rho_{\mu\sigma}\xi^\sigma e_\rho^a \\
&= \nabla_\mu\xi^\rho e_\rho^a - \xi^\rho\omega_\rho{}^a{}_b e_\mu^b + \xi^\rho T_{\rho\mu}{}^a\,.
\end{aligned}
$$

Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

$$\nabla_\rho e^a_\mu \;=\; 0 \qquad e^a_\rho \nabla_\mu \xi^\rho = D_\mu \xi^a \qquad \xi^\rho T_{\rho\mu}{}^a = \tfrac{1}{2}(\xi^\rho \bar{\psi}_\rho)\gamma^a \psi_\mu$$

$$[\delta_1, \delta_2]\, e^a_\mu \;=\; \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) e^a_\mu$$

the susy parameter $\hat{\epsilon} = \xi^\rho \psi_\rho$

$$[\delta_1, \delta_2]\, \psi_\mu = \xi^\rho (D_\rho \psi_\mu - D_\mu \psi_\rho) + \ldots \, .$$

$$[\delta_1, \delta_2]\, \psi_\mu \;=\; \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) \psi_\mu + \ldots \, .$$

the dots means a symmetric combination of the equations of motion

Local supersymmetry of $\mathcal{N} = 1$, $D = 4$ supergravity

- Generalizations

one can couple the gravity multiplet (e_μ^a, ψ_μ)

to gauge (A_μ^A, λ^A) and chiral $(z^\alpha, P_L\chi^\alpha)$ multiplets

Supergravity in dimensions different from four

D=10 supergravities Type IIA and IIB are the low energy limits of superstring theories of the same name

Type II A and gauged supergravities appear in ADS/CFT correspondence

D=11 low energy limit of M theory that it is not perturbative

$D = 11$ supergravity

Field content

128 bosons degrees of freedom
graviton $g_{\mu\nu}$, **44** of SO(9): $\square\square$

3-form $C_{\mu\nu\rho}$, $C_{(3)}$, **84** of SO(9): $\begin{array}{c}\square\\\square\\\square\end{array}$

128 fermions degrees of freedom

$D = 11$ supergravity

Construction of the action and transformation rules

Gauge transformation of 3-form

$$
\begin{aligned}
\delta A_{\mu\nu\rho} &= 3\partial_{[\mu}\theta_{\nu\rho]} \equiv \partial_\mu\theta_{\nu\rho} + \partial_\nu\theta_{\rho\mu} + \partial_\rho\theta_{\mu\nu}\,, \\
F_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}A_{\nu\rho\sigma]} \equiv \partial_\mu A_{\nu\rho\sigma} - \partial_\nu A_{\rho\sigma\mu} + \partial_\rho A_{\sigma\mu\nu} - \partial_\sigma A_{\mu\nu\rho}\,, \\
\partial_{[\tau}F_{\mu\nu\rho\sigma]} &\equiv 0\,.
\end{aligned}
$$

Bianchi dentity

Ansatz action

$$
S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \left[e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} + \dots\right]
$$

Initially we use second order formalism with torsion-free spin connection $\omega_{\mu ab}(e)$

$D = 11$ supergravity

- Ansatz transformations

$$
\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,,\\
\delta\psi_\mu &= D_\mu\epsilon + \left(a\,\gamma^{\alpha\beta\gamma\delta}{}_\mu + \,b\,\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)F_{\alpha\beta\gamma\delta}\epsilon\,,\\
\delta A_{\mu\nu\rho} &= -c\,\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} = -\tfrac{1}{3}c\,\bar{\epsilon}(\gamma_{\mu\nu}\psi_\rho + \gamma_{\nu\rho}\psi_\mu + \gamma_{\rho\mu}\psi_\nu)
\end{aligned}
$$

Useful relations

$$
\bar{\chi}\gamma^{\mu_1\mu_2\ldots\mu_r}\lambda = t_r\bar{\lambda}\gamma^{\mu_1\mu_2\ldots\mu_r}\chi\,,\qquad t_0 = t_3 = 1\,,\quad t_1 = t_2 = -1\,,\quad t_{r+4} = t_r
$$

To determine the constants we consider the free action (global susy)

$$
S_0 = \frac{1}{2}\int \mathrm{d}^{11}x\left[-\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\right]
$$

$D = 11$ supergravity

- transformations

and Bianchi identity we get $a = c/216, \; b = -8a$

$$
\begin{aligned}
\delta\psi_\mu &= \partial_\mu \epsilon + \frac{c}{216}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right) F_{\alpha\beta\gamma\delta}\epsilon \\
\delta A_{\mu\nu\rho} &= -c\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} \, .
\end{aligned}
$$

To determine c we compute the commutator of two susy transformations

$$[\delta_1, \delta_2] A_{\mu\nu\rho} = -\tfrac{4}{9} c^2 \bar{\epsilon}_1 \gamma^\sigma \epsilon_2 F_{\sigma\mu\nu\rho}$$

If $c^2 = 9/8$

With gauge transformation given the parameter $\theta_{\mu\nu} = -\bar{\epsilon}_1 \gamma^\sigma \epsilon_2 A_{\sigma\mu\nu}$

$$[\delta_1, \delta_2] = \delta_{susy} + \delta_{gauge}$$

$D = 11$ supergravity

- transformations

  the conserved Noether current is (coefficient of $D_\nu \epsilon$

$$\mathcal{J}^\nu = \frac{\sqrt{2}}{96}\left(\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12\gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho}\right)\psi_\rho$$

Ansatz for the action and transformations in the interacting case. We introduce the frame field $e_\mu^a(x)$ and a gauge susy parameter $\epsilon(x)$

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\
\delta\psi_\mu &= D_\mu\epsilon + \frac{\sqrt{2}}{288}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right) F_{\alpha\beta\gamma\delta}\epsilon \\
\delta A_{\mu\nu\rho} &= -\frac{3\sqrt{2}}{4}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}\,,
\end{aligned}$$

$D = 11$ supergravity

- Action

$$
\begin{aligned}
S &= \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \Big[e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \\
&\qquad - \frac{\sqrt{2}}{96} \bar{\psi}_\nu \left(\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12\gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho} \right) \psi_\rho + \ldots \Big] \\
&= \frac{1}{\kappa^2}\int \mathrm{d}^{11}x\, e\; L
\end{aligned}
$$

We need to find the dots

$D = 11$ supergravity

- Action

$$
\begin{aligned}
S_{\mathrm{C-S}} &= -\frac{\sqrt{2}}{(144\kappa)^2}\int \mathrm{d}^{11}x\,\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\,,\\
&= -\frac{\sqrt{2}}{6\kappa^2}\int F^{(4)}\wedge F^{(4)}\wedge A^{(3)}\,,
\end{aligned}
$$

Full action

$$
\begin{aligned}
S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\,\Bigg[& e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\tfrac{1}{2}(\omega+\hat{\omega}))\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\\
&- \frac{\sqrt{2}}{192}\bar{\psi}_\nu\left(\gamma^{\alpha\beta\gamma\delta\nu\rho}+12\gamma^{\alpha\beta}g^{\gamma\nu}g^{\delta\rho}\right)\psi_\rho(F_{\alpha\beta\gamma\delta}+\hat{F}_{\alpha\beta\gamma\delta}) \qquad (10.26)\\
&- \frac{2\sqrt{2}}{(144)^2}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\Bigg]\,. \qquad (10.27)
\end{aligned}
$$

$D = 11$ supergravity

$$\begin{aligned}
\omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab} \,, \\
\hat{\omega}_{\mu ab} &= \omega_{\mu ab}(e) - \tfrac{1}{4}(\bar{\psi}_\mu \gamma_b \psi_a - \bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_b \gamma_a \psi_\mu) \,, \\
K_{\mu ab} &= -\tfrac{1}{4}(\bar{\psi}_\mu \gamma_b \psi_a - \bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_b \gamma_a \psi_\mu) + \tfrac{1}{8}\bar{\psi}_\nu \gamma^{\nu\rho}{}_{\mu ab} \psi_\rho \,, \\
\hat{F}_{\mu\nu\rho\sigma} &= 4\,\partial_{[\mu} A_{\nu\rho\sigma]} + \tfrac{3}{2}\sqrt{2}\,\bar{\psi}_{[\mu} \gamma_{\nu\rho} \psi_{\sigma]} \,.
\end{aligned}$$

This action is invariant under the transformation rules

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \,, \\
\delta \psi_\mu &= D_\mu(\hat{\omega})\epsilon + \frac{\sqrt{2}}{288}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)\hat{F}_{\alpha\beta\gamma\delta}\epsilon \\
\delta A_{\mu\nu\rho} &= -\frac{3\sqrt{2}}{4} \bar{\epsilon} \gamma_{[\mu\nu} \psi_{\rho]} \,.
\end{aligned}$$

$D = 11$ supergravity

The algebra of $D = 11$ supergravity

$$
\begin{aligned}
[\delta_Q(\epsilon_1),\, \delta_Q(\epsilon_2)] \;&=\; \delta_{\rm gct}(\xi^\mu) + \delta_L(\lambda^{ab}) + \delta_Q(\epsilon_3) + \delta_A(\theta_{\mu\nu})\,, \\
\xi^\mu &= \tfrac{1}{2}\epsilon_2\gamma^\mu\epsilon_1\,, \\
\lambda^{ab} &= -\xi^\mu\hat\omega_\mu{}^{ab} + \frac{1}{288}\sqrt{2}\bar\epsilon_1\left(\gamma^{ab\mu\nu\rho\sigma}\hat F_{\mu\nu\rho\sigma} + 24\gamma_{\mu\nu}\hat F^{ab\mu\nu}\right)\epsilon_2\,, \\
\epsilon_3 &= -\xi^\mu\psi_\mu\,, \\
\theta_{\mu\nu} &= -\xi^\rho A_{\rho\mu\nu} + \tfrac{1}{4}\sqrt{2}\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2\,.
\end{aligned}
$$

The spinor bilinears $\bar\epsilon_1\Gamma^{(2)}\epsilon_2$ and $\bar\epsilon_1\Gamma^{(6)}\epsilon_2$ have a special role. They are non-vanishing fot the classical BPS M2 and M5 solutions .

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

Toroidal reduction of D=11 Supergravity

$$
\begin{aligned}
ds_{11}^2 &= e^{2\alpha\phi} ds_{10}^2 + e^{2\beta\phi}(dy + A_\mu dx^\mu)^2 \\
C_{(p)}^{11} &= C_{(p)}^{10} + C_{(p-1)}^{10} \wedge dy
\end{aligned}
$$

IIA SUGRA bosonic fields

$\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \boldsymbol{7}, \boldsymbol{8}, \boldsymbol{9}.$ 5,6

Fermionic fields, non-chiral gravitino, non-chiral dilatino

Bogomol'ny bound

- Consider an scalar field theory in 4d flat space time

$$L = -\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \quad V(\phi) = \phi^2(m + g\phi)^2$$

There are two vacua at $\phi = 0, \quad \phi = -\frac{m}{g}$

We expect a domain wall separating the region of two vacua

We look for an static configuration connecting the two vacua

$$\phi(x) \longrightarrow -\frac{m}{g}, \quad x \to \infty$$

$$\phi(x) \longrightarrow 0, \quad x \to -\infty$$

Bogomol’ny bound

- BPS procedure

The potential V can be wriiten in terms of superpotential W

$$W = \frac{1}{2}\phi^2 + \frac{1}{3}g\phi^3 \qquad\qquad V = W'^2$$

Energy density in terms energy momentum tensor

$$\begin{aligned} T_{00} &= |\dot{\phi}|^2 + \nabla\phi\cdot\nabla\phi + |W'|^2 \\ &= |\partial_x\phi|^2 + |W'|^2 \qquad \text{for domain wall} \end{aligned}$$

Total energy

$$\begin{aligned} \mathcal{E} &= \int_{-\infty}^{\infty} dx\, \left[|\partial_x\phi|^2 + W'^2\right] \\ &= \int_{-\infty}^{\infty} dx\, (\partial_x\phi - W')^2 + 2\Delta W \end{aligned}$$

Bogomol'ny bound

where $$\Delta W = [W(\phi(x))]_{x=-\infty}^{x=\infty}$$

We have an energy bound

$$\mathcal{E} \geq 2|\Delta W|$$

which is saturated if the first order equation, BPS equation is verified

$$\partial_x \phi = W'$$

In this case the energy is $$\mathcal{E} = \frac{m^3}{3g^2}.$$

Domain wall as a BPS solution

$$\phi(x) = -\frac{\mu}{2g}\left[\tanh\left(\mu\left(x - x_0\right)/2\right) + 1\right]$$

One can prove that this BPS solution is also a solution of the second order equations of motion

Notice that the domain wall is non-perturbative solution of the equations of motion

If the theory can be embbed in a supersymmetric theory, the solutions of the BPS equations will preserve some supersymmetry

Effective Dynamics of the domain wall

The width of the domanin wall is $L \sim \dfrac{1}{m}$ If we consider fluctuations of the scalar filed with wave length >>L the dynamics of the will be Independent of of the details of the wall.

$$\phi(t,x,y,z) = \phi_{cl}(x) + \delta\phi(t,x,y,z)$$

The lagrangian up to quadratic fluctuations is

$$L = L(\phi_{cl}) - \frac{1}{2}\partial_i(\delta\phi)\partial^i(\delta\phi) - \frac{1}{2}(-\partial_x^2 - m^2 + 3m^2 tan^2 h(\frac{(x-x_0)m}{2})(\delta\phi)^2$$

Let us do the separation of variables

$$\delta\phi(t,x,y,z) = X(t,y,z)Z(x)$$

Effective Dynamics of the domain wall

To study the small perturbations we we should study the eigenvalue problem

$$(-\partial_x^2 - m^2 + 3m^2 tan^2 h(\frac{(x-x_0)m}{2}))Z_n(x) = w_n Z_n(x)$$

Exits a zero mode $Z_0(x) = \phi'_{cl}(x)$

This zero mode corresponds to a massless excitation and it is associated with the broken translation invariance

The action for these fluctuations given by

$$S = -T \int dt dy dz (1 + \frac{1}{2}\partial_i X \partial^i X) \quad T = -\int dz L(\phi_{cl})$$

It describe the accion of a membrane, 2-brane, at low energies

Effective Dynamics of the domain wall

The membrane action to all orders is given by

$$S = -T \int d^3\xi \sqrt{-\det g}$$

where is teh determinat of the induced metric

$$g_{\mu\nu}(\xi) = \partial_\mu X^m \partial_\nu X^n \eta_{mn}$$

Supersymmetric domain wall

- WZ Action

$$S_{\mathrm{kin}} = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \not{\partial} P_L \chi + \bar{F} F \right]$$

$$S_F = \int \mathrm{d}^4x \left[F W'(Z) - \tfrac{1}{2} \bar{\chi} P_L W''(Z) \chi \right]$$

W(Z) superpotential, arbitrary holomorphic function of Z $\quad S_{\bar{F}} = (S_F)^\dagger$

Complete action $\quad S = S_{\mathrm{kin}} + S_F + S_{\bar{F}}$

F $\bar{F}$ Are not a dynamical field, their equations of motion are algebraic

$F = -\overline{W}'(\bar{Z}) \qquad \bar{F} = -W'(Z)$ we can eliminate them

Domain wall ½ BPS

The susy transformations for the WZ model are

$$\begin{aligned} \delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,, \\ \delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\not{\partial}Z + F)\epsilon\,, \\ \delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\not{\partial}P_L\chi\,. \end{aligned}$$

$$\begin{aligned} \delta \bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi\,, \\ \delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\not{\partial}\bar{Z} + \bar{F})\epsilon \\ \delta \bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\not{\partial}P_R\chi\,. \end{aligned}$$

For the domain wall ansatz the transformation of the should be

$$0 = \delta\chi = \frac{1}{2}(\gamma^1\partial_x Z - W')\epsilon_R + \frac{1}{2}(\gamma^1\partial_x \bar{Z} - \bar{W}')\epsilon_l$$

Domain wall ½ BPS

This condition implies

$$\partial_x Z = W'$$

and

$$(\gamma_x - 1)\epsilon_R = (\gamma_x - 1)\epsilon_L = 0$$

As $\gamma_x^2 = 1$ and tr $\gamma_x = 0$ the space of solutions for ϵ is 2-dimensional

Note that this supersymmetric calculation recovers the result of the bosonic BPS Calculation. Therefore the domain wall is ½ BPS

This result can be deduced from the anticommutator of spinorial charges

Classical Solutions of Supergravity

- The solutions of supergravity give the metric, vector fields and scalar fields.
- The preserved supersymmetry means some rigid supersymmetry

$$\delta(\epsilon)\,\mathrm{boson} = \epsilon\,\mathrm{fermion}\,, \qquad \delta(\epsilon)\,\mathrm{fermion} = \epsilon\,\mathrm{boson}$$

Killing Spinors and BPS Solutions

- N=1 D=4 supergravity

Flat metric with fermions equal to zero is a solution of supergravity with

$$g_{\mu\nu} = \eta_{\mu\nu}$$ Vacuum solution

The residual global transformations are determined by the conditions

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu = 0 , \qquad \delta\psi_\mu = D_\mu\epsilon = 0$$

The Killing spinors of the Minkowski background are the set of 4 independent constant Majorana spinors. We have D=4 Poincare Susy algebra

Killing vectors and Killing spinors

$$k_A = k_A^\mu \frac{\partial}{\partial x^\mu}, \quad \mathcal{L}_{k_A} g_{\mu\nu} = \nabla_\mu k_{\nu A} + \nabla_\nu k_{\mu A} = 0$$

$$[k_A\, ,\, k_B] = f_{AB}{}^C k_C$$

Killing Spinors and BPS Solutions

- The integrability condition for Killing spinors

$\text{Killing spinor condition } D_\mu \epsilon = 0$

Integrability condition

$$[D_\mu, D_\nu]\epsilon = \frac{1}{4} R_{\mu\nu ab}\gamma^{ab}\epsilon = 0$$

$$\textit{Suppose that } \epsilon \textit{ and } \epsilon' \textit{ are both Killing spinors}$$

$$-\frac{1}{2}\bar{\epsilon}'\gamma^\rho\gamma^\nu R_{\mu\nu ab}\gamma^{ab}\epsilon = R_{\mu\nu}\bar{\epsilon}'\gamma^\rho\gamma^\nu\epsilon = 0$$

$$R_{\mu\nu}\bar{\epsilon}'\{\gamma^\rho, \gamma^\nu\}\epsilon = 2R^\rho_\mu\,\bar{\epsilon}'\epsilon = 0$$

A spacetime with Killing spinors satisfies $R_{\mu\nu} = 0$ only if $\bar{\epsilon}'\epsilon \neq 0$

Killing spinors for pp-waves

Ansatz for the metric

$$\mathrm{d}s^2 = 2H(u,x,y)\mathrm{d}u^2 + 2dudv + \mathrm{d}x^2 + \mathrm{d}y^2$$

For H=0 reduces to Minkowski spacetime in light-cone coordinates

$$u = (x-t)/\sqrt{2},\ v = (x+t)/\sqrt{2}.$$

Flat metric in these coordinates $\hat{\eta}_{ab}$, where $a,b = +,-,1,2$

$$\hat{\eta}_{+-} = \hat{\eta}_{-+} = \hat{\eta}_{11} = \hat{\eta}_{22} = 1$$

Note that $K = \partial/\partial v$ is a covariant constant null vector

$$k^M \frac{\partial}{\partial x^M} = \frac{\partial}{\partial v},\ k_{M;N} = 0,\ g_{MN}k^M k^N = 0$$

Killing spinors for pp-waves

The frame 1-forms are

$$e^- = du\,, \qquad e^+ = dv + H du\,, \qquad e^1 = dx\,, \qquad e^2 = dy$$

From the first Cartan structure equation we get the torsion free spin connection one forms

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a$$

$$\omega^{+1} = H_x e^-\,, \qquad \omega^{+2} = H_y e^-$$

and from the second one $\quad \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab}$

$$\rho^{+1} = H_{xx} e^1 \wedge e^- + H_{xy} e^2 \wedge e^-, \quad \rho^{+2} = H_{yy} e^2 \wedge e^- + H_{xy} e^1 \wedge e^-$$

Killing spinors for pp-waves

The Killing spinor conditions are

$$D_\mu \epsilon = (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\epsilon = 0$$

explicitely

$$\begin{aligned} D_u\epsilon &= (\partial_u - \tfrac{1}{2}H_x\gamma^1\gamma^- - \tfrac{1}{2}H_x\gamma^2\gamma^-)\epsilon = 0\,, \\ D_v\epsilon &= \partial_v\epsilon = 0\,, \qquad D_x\epsilon = \partial_x\epsilon = 0\,, \qquad D_y\epsilon = \partial_y\epsilon = 0\,. \end{aligned}$$

All conditions are verified if we take constant spinors with constraint

$$\gamma^-\epsilon = 0, \quad \gamma_0\gamma^1\epsilon = \epsilon$$

Since $(\gamma_0\gamma^1)^2, \mathrm{tr}\,\gamma_0\gamma^1 = 0$ there are two Killing spinors.

Killing spinors for pp-waves

Notice

$$2\bar{\epsilon}'\epsilon \ = \ \bar{\epsilon}'(\gamma^+\gamma^- + \gamma^-\gamma^+)\epsilon = 0$$

To complete the analysis we need the Ricci tensor. The non-trvial component is

$$R_{--} = R_{-1-}{}^{1} + R_{-2-}{}^{2} = -(H_{xx} + H_{yy}).$$

Therefore the pp-wave is Ricic flat if and only if H is harmonic in the variables x,y

pp-waves in D=11 supergravity

Eleven dimensional supergravity with bosonic fileds the metric and the four-form field strength F_4 has pp-wave solutions

$$ds^2 = 2dx^+dx^- + H(x^i, x^-)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

$$F_4 = dx^- \wedge \varphi$$

where $H(x^i, x^-)$ obeys $\quad \triangle H = \frac{1}{12}|\varphi|^2$

$\triangle$ is the laplacian in the transverse euclidean space $\mathbb{E}^9$

3-form

pp-waves in D=11 supergravity

$\partial/\partial x^+$ is a covariantly constant null vector

If we choose

$$H(x^i, x^-) = \sum A_{ij} x^i x^j$$

where $A_{ij} = A_{ji}$ is a constant symmetric matrix

They have at least 16 Killing spinors. If one choose

$$A_{ij} = \begin{cases} -\frac{1}{9}\mu^2 \delta_{ij} & i,j = 1,2,3 \\ -\frac{1}{36}\mu^2 \delta_{ij} & i,j = 4,5,\ldots,9 \end{cases}$$

$$\varphi = \mu dx^1 \wedge dx^2 \wedge dx^3 ,$$

The number of Killing spinors is 32!, like $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$

Spheres

the unit sphere S^2 is the surface $(x^1)^2+(x^2)^2+(x^3)^2 = 1$ embedded in flat Euclidean space $\mathbb{R}^3$

$$x^1 = \sin\theta^2 \sin\theta^1, \ x^2 = \sin\theta^2 \cos\theta^1, \ x^3 = \cos\theta^2$$

$$0 \leq \theta^1 \leq 2\pi, \quad 0 \leq \theta^2 \leq \pi$$

The metric of the sphere is obtained as induced metric of the flat $\mathbb{R}^3$

$$\begin{aligned} d\Omega_2^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= (d\theta^2)^2 + \sin^2\theta^2 (d\theta^1)^2 . \end{aligned}$$

Spheres

Frame one forms

$$\bar{e}^2 = \mathrm{d}\theta^2 \,, \qquad \bar{e}^1 = \sin\theta^2 \mathrm{d}\theta^1$$

Spin connection. First structure equation

$$\bar{\omega}^{12} = \cos\theta^2 \mathrm{d}\theta^1$$

Curvature. Second curvature equation

$$\bar{\rho}^{ab} = \bar{e}^a \wedge \bar{e}^b$$

Constant positive curvature

Spheres

Recursive proocedure for higher dimensional sphres

$$x^{n+1}_{(n)} = \cos\theta^n , \qquad x^a_{(n)} = \sin\theta^n x^a_{(n-1)} , \qquad a \leq n$$

Frame and connection forms are

$$\begin{aligned}
\bar{e}^n_{(n)} &= \mathrm{d}\theta^n , \qquad \bar{e}^a_{(n)} = \sin\theta^n \bar{e}^a_{(n-1)} , \qquad a \leq n-1 \\
\bar{\omega}^{ab}_{(n)} &= \bar{\omega}^{ab}_{(n-1)} , \qquad \bar{\omega}^{an}_{(n)} = \cos\theta^n \bar{e}^a_{(n-1)} .
\end{aligned}$$

$$\begin{aligned}
\bar{e}^a &= \Big(\prod_{j=a+1}^{n} \sin\theta^j\Big)\mathrm{d}\theta^a , \qquad a \leq n , \\
\bar{\omega}^{ab} &= \cos\theta^b \Big(\prod_{j=a+1}^{b-1} \sin\theta^j\Big)\mathrm{d}\theta^a , \qquad 1 \leq a < b \leq n
\end{aligned}$$

Anti-de Sitter space

AdS_D for the D-dimensional case. simple solutions of supergravity

with negative constant solution

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g}\,(R - \Lambda)$$

$$\Lambda = -(D-1)(D-2)/L^2$$

$$R_{\mu\nu} = -\frac{D-1}{L^2} g_{\mu\nu}$$

AdS is an example of a maximally symmetric spacetime

$$R_{\mu\nu\rho\sigma} = k\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

k is a constant of dimension $1/(\mathrm{length})^2$ $\quad k = -1/L^2$

Anti-de Sitter space

Einstein metric

$$\begin{aligned} R_{\mu\nu} &= k(D-1)g_{\mu\nu} \\ R &= k\,D(D-1)\,. \end{aligned}$$

Ads as a coset space

$$[P_0, K] = iP_1$$

$$[P_1, K] = iP_0 \qquad\qquad K = J_{01}.$$

$$[P_0, P_1] = -i\frac{1}{R^2}K$$

$$g = e^{iP_1 x^1} e^{iP_0 x^0}$$

Anti-de Sitter space

- MC 1-form

$$\begin{aligned}\Omega &= P_0\ dx^0\ +\ P_1\ (dx^1 \cos\frac{x^0}{R})\ -\ K\ (\frac{dx^1}{R}\sin\frac{x^0}{R}) \\ &\equiv P_0\ e^0\ +\ P_1\ e^1\ +\ K\ \omega^{01}\ .\end{aligned}$$

Ads metric

$$ds^2 = -e^0 e^0\ +\ e^1 e^1\ =\ -{dx^0}^2\ +\ {dx^1}^2\ \cos^2\frac{x^0}{R}$$

Ads can be embedded in pseuo-Euclidean space

$$\eta_{ab}u^a u^b = -(u^0)^2 + (u^1)^2 - (u^2)^2 = -R^2$$

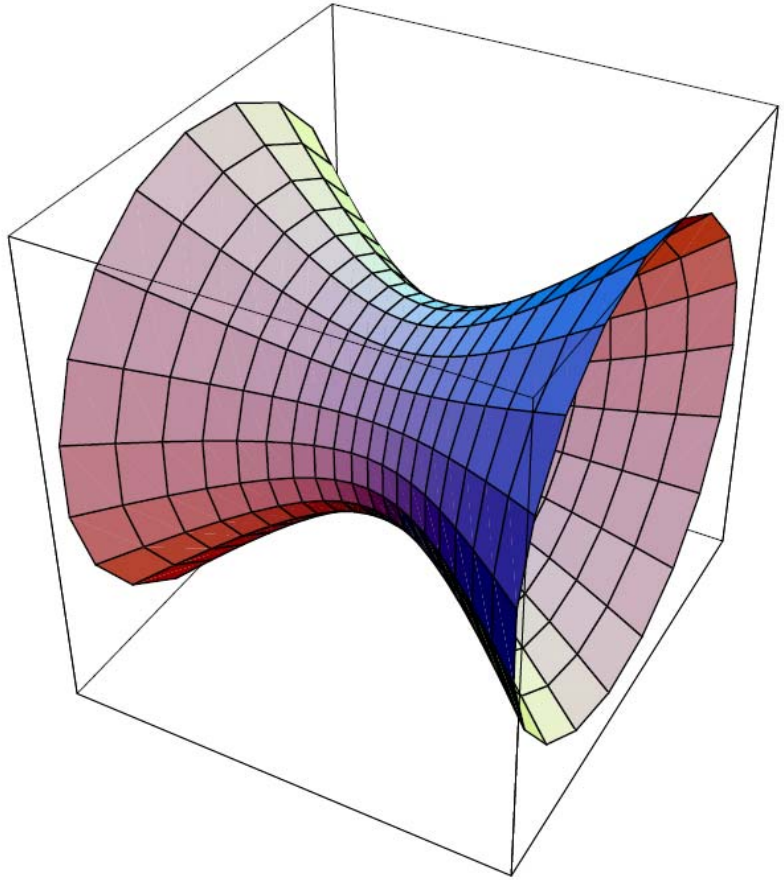
Anti-de Sitter space

$$u^0 = R \ \sin \frac{x^0}{R}$$
$$u^1 = R \ \cos \frac{x^0}{R} \ \ \sinh \frac{x^1}{R}$$
$$u^2 = R \ \cos \frac{x^0}{R} \ \ \cosh \frac{x^1}{R}$$

metric

$$ds^2 = -e^0 e^0 \ + \ e^1 e^1 \ = \ -dx^{0\,2} \ + \ dx^{1\,2} \ \cos^2 \frac{x^0}{R}$$

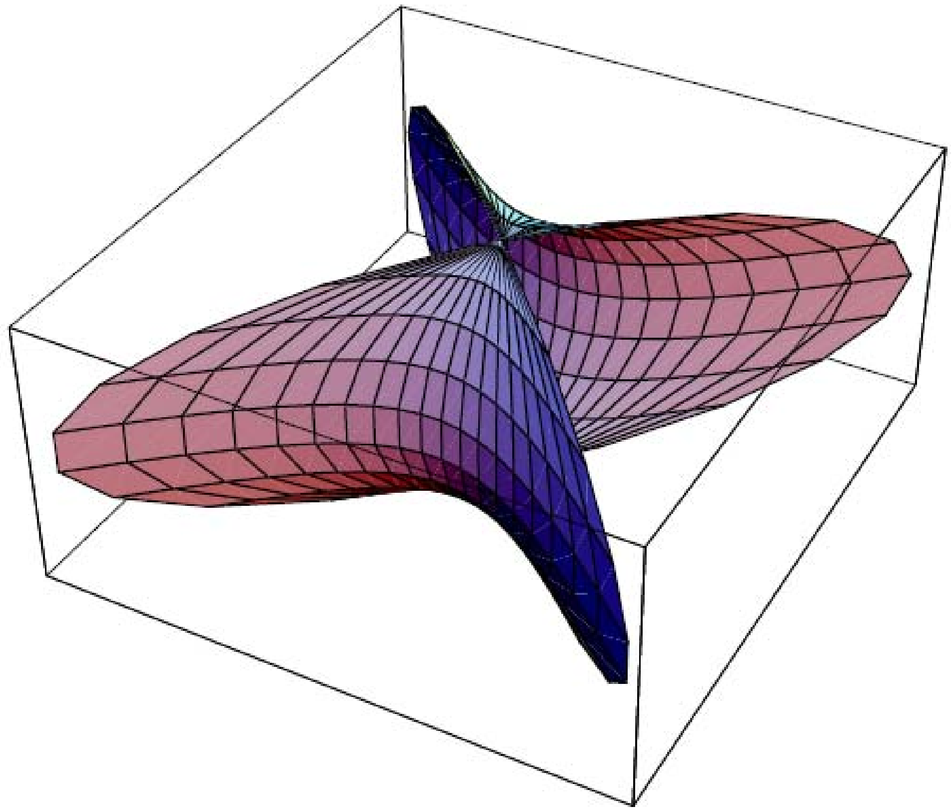
Anti-de Sitter space



Anti-de Sitter space

Note that u^0 varies in (-R.R) $x^0 \in (0, 2\pi R)$ i $x^1 \in (-\infty, \infty)$

Local parametrization



Global parametrization

$$g = e^{iP_0 x^0} e^{iP_1 x^1}$$

$$\begin{aligned} \Omega : \quad &= P_1\ dx^1 \ + \ P_0\ (dx^0 \cosh\frac{x^1}{R}) \ + \ K\ (\frac{dx^0}{R}\sinh\frac{x^1}{R}). \\ &\equiv P_0\ e^0 \ + \ P_1\ e^1 \ + \ K\ \omega^{01}\ . \end{aligned}$$

$$\begin{aligned} u^0 =& R\ \cosh\frac{x^1}{R}\sin\frac{x^0}{R} \\ u^1 =& R\ \sinh\frac{x^1}{R} \\ u^2 =& R\ \cosh\frac{x^1}{R}\cos\frac{x^0}{R}\ . \end{aligned}$$

$$ds^2 = -\ \cosh^2\frac{x^1}{R}dx^{0\,2} \ + \ dx^{1\,2}$$

Anti-de Sitter space

$$Y^A \eta_{AB} Y^B = -(Y^0)^2 + \sum_{i=1}^{D-1} (Y^i)^2 - (Y^D)^2 = -L^2$$

the coordinates $Y'^A = \lambda^A{}_B Y^B$ provided that $\lambda^A{}_B$ is a matrix

$\mathrm{SO}(D-1,2)$

Different embeddings

$$\begin{aligned} Y^i &= r\bar{x}^i \qquad \text{with} \qquad \sum_{i=1}^{D-1} (\bar{x}^i)^2 = 1\,, \\ Y^0 &= \sqrt{L^2+r^2}\sin(t/L) \qquad Y^D = \sqrt{L^2+r^2}\cos(t/L) \end{aligned}$$

$\bar{x}^i$ parameterizes the unit sphere S^{D-2}

Anti-de Sitter space

This coordinate system is global , covers the whole hyperbolid for

$0 \leq r < \infty, \quad 0 \leq t < 2\pi L$ the algunlar variables the whole S^{D-2}

New radial coordinate $\cosh(y/L) = \sqrt{1 + r^2/L^2}$

$$ds^2 = -\cosh^2(y/L)dt^2 + dy^2 + L^2 \sinh^2(y/L)\, d\Omega^2_{D-2}$$

Another possibility $\cosh(y/L) = 1/\cos\rho \quad t = L\tau$

$$ds^2 = \frac{L^2}{\cos^2\rho}\left[-d\tau^2 + \left(d\rho^2 + \sin^2\rho\, d\Omega^2_{D-2}\right)\right]$$

It is conformalto the direct product of the real line, time coordinate, times the Sphere in D-1 dimensions

Anti-de Sitter space

Poincaré patch

$$
\begin{aligned}
Y^0 &= Lux^0\,, \\
Y^i &= Lux^i\,, \qquad i = 1, \ldots, D-2 \\
Y^{D-1} &= \frac{1}{2u}\left(-1 + u^2(L^2 - x^2)\right)\,, \\
Y^D &= \frac{1}{2u}\left(1 + u^2(L^2 + x^2)\right)\,, \\
x^2 &= -(x^0)^2 + \sum (x^i)^2\,.
\end{aligned}
$$

$$
\mathrm{d}s^2 = L^2\left[\frac{\mathrm{d}u^2}{u^2} + u^2\left(-(\mathrm{d}x^0)^2 + \sum_i (\mathrm{d}x^i)^2\right)\right]
$$

Anti-de Sitter space

$$z = 1/u$$

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left[\mathrm{d}z^2 - (\mathrm{d}x^0)^2 + \sum_i (\mathrm{d}x^i)^2\right]$$

The metric is conformal to the positive region of D dimensional Minlowski space with coordinates (x^0, x^i, z)

Killing spinors for anti-de Sitter space

The bosonic action that leads to AdS space is

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g}\,(R - \Lambda)$$

Killing spinor are solutions of

$$\hat{D}_\mu \epsilon \equiv (D_\mu - \frac{1}{2L}\gamma_\mu)\epsilon = 0$$

Integrability condition

$$[\hat{D}_\mu\,,\,\hat{D}_\nu]\epsilon \;\; = \;\; \left(\frac{1}{4} R_{\mu\nu ab}\gamma^{ab} + \frac{1}{2L^2}\gamma_{\mu\nu}\right)\epsilon$$

If we insert $R_{\mu\nu ab} = -(e_{a\mu}e_{b\nu} - e_{a\nu}e_{b\mu})/L^2$ vanishes identically
It is a hint that AdS is a maximally supersymmetric space
I

Killing spinors for anti-de Sitter space

We will study Killing spinors in the Poincaré patch of AdS_D

$$z = L\mathrm{e}^{-r/L}$$

$$\mathrm{d}s^2 = \mathrm{e}^{2r/L}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}r^2$$

Frame fields $$e^{\hat{\mu}} = \mathrm{e}^{r/L}\mathrm{d}x^\mu\,, \qquad e^r = \mathrm{d}r$$

Spin connection $$\omega^{\hat{\mu}r} = \frac{1}{L}e^{\hat{\mu}}\,, \qquad \omega^{\mu\nu} = 0$$

$$\hat{D}_r = (\partial_r - \frac{1}{2L}\gamma_r)\epsilon = 0\,,$$

$$\hat{D}_\mu = (\partial_\mu + \frac{1}{2L}(\gamma_r - 1))\epsilon = 0$$

Killing spinors for anti-de Sitter space

we introduce constant spinors $\eta_\pm$ which satisfy $\gamma_r \eta_\pm = \pm\eta_\pm$

$$\begin{aligned} \epsilon_+ &= \mathrm{e}^{r/2L}\eta_+ \,, \\ \epsilon_- &= (\mathrm{e}^{-r/2L} + \frac{1}{L}\mathrm{e}^{r/2L} x^\mu \gamma_{\hat{\mu}})\eta_- \end{aligned}$$

The last term includes transverse indexes.