# Improved Bounds on the Growth Constant of Polyiamonds*

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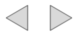
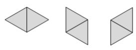
## Abstract

A polyiamond is an edge-connected set of cells on the triangular lattice. In this paper we provide improved lower and upper bounds on the asymptotic growth constant of polyiamonds, proving that it is between 2.8424 and 3.6050.

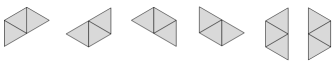

## 1 Introduction



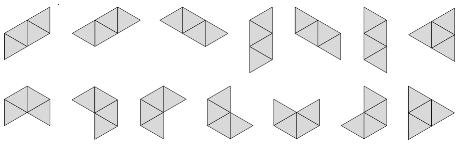
(a) $n = 1$: Two moniamonds



(b) $n = 2$: Three diamonds



(c) $n = 3$: 6 triamonds



(d) $n = 4$: 14 tetriamonds

A *polyomino* of size $n$ is an edge-connected set of $n$ cells on the square lattice $\mathbb{Z}^2$. Similarly, a *polyiamond* of size $n$ is an edge-connected set of $n$ cells on the triangular lattice. *Fixed* polyiamonds are considered distinct if they have different *shapes* or *orientations*. In this paper we consider only fixed polyiamonds, and so we refer to them simply as "polyiamonds." Figure 1 shows polyiamonds of size 1–4.

In general, a connected set of cells on a lattice is called a *lattice animal*. The fundamental combinatorial problem concerning lattice animals is "How many animals with $n$ cells are there?" The study of lattice animals began in parallel more than half a century ago in two different communities. In statistical physics, Temperley [19] investigated the mechanics of macromolecules, and Broadbent and Hammersley [5] studied percolation processes. In mathematics, Eden [6] and others analyzed cell growth problems. Since then, counting animals has attracted much attention in the literature. However, despite serious efforts over the last 50 years, counting polyominoes is still far from being solved, and is considered one of the long-standing open problems in combinatorial geometry.

The symbol $A(n)$ usually denotes the number of polyominoes of size $n$; See sequence A001168 in the On-line Encyclopedia of Integer Sequences (OEIS) [1]. Since no analytic formula for the number of animals is yet known for any nontrivial lattice, a great portion of the research has so far focused on efficient algorithms for *counting* animals on lattices, primarily on the square lattice. Elements of the sequence $A(n)$ are currently known up to $n = 56$ [11]. The growth constant of polyominoes was also treated extensively in the literature, and a few asymptotic results are known. Klarner [12] showed that the limit $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists, and the main problem so far has been to evaluate this constant. The convergence of $A(n+1)/A(n)$ to $\lambda$ (as $n\to\infty$) was proven only three decades later by Madras [15], using a novel pattern-frequency argument. The best-known lower and upper bounds on $\lambda$ are 4.0025 [4] and 4.6496 [13], respectively. It is widely believed (see, e.g., [7, 8]) that $\lambda \approx 4.06$, and the currently best estimate, $\lambda = 4.0625696 \pm 0.0000005$, is due to Jensen [11].

In the same manner, let $T(n)$ denote the number of polyiamonds of size $n$ (sequence A001420 in the OEIS). Elements of the sequence $T(n)$ were computed up to $n = 75$ [9, p. 479] using a transfer-matrix algorithn by Jensen [ibid., p. 173], adapting his original polyomino-counting algorithm [11]. Earlier counts were given by Lunnon [14] up to size 16, by Sykes and Glen [18] up to size 22, and by Aleksandrowicz and Barequet [2] (extending Redelmeier's polyomino-counting algorithm [17]) up to size 31.

Similarly to polyominoes, the limits $\lim_{n\to\infty} \sqrt[n]{T(n)}$ and $\lim_{n\to\infty} T(n+1)/T(n)$ exist and are equal. Let, then, $\lambda_T$ denote the growth constant of polyiamonds. Klarner [12, p. 857] showed that $\lambda_T \geq 2.13$ by taking the square root of 4.54, a lower bound he computed for the growth constant of animals on the rhomboidal lattice, using the fact that a rhombus is made of two neighboring equilateral triangles. This bound is also mentioned by Lunnon [14, p. 98]. Rands and Welsh [16] used

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renewal sequences in order to show that

$$\lambda_T \geq (T(n)/(2(1+\lambda_T)))^{1/n} \quad (1)$$

for any $n \in \mathbb{N}$. Substituting the easy upper bound $\lambda_T \leq 4$ (see below) in the right-hand side of this relation, and knowing at that time elements of the sequence $T(n)$ for $1 \leq n \leq 20$ only (data provided by Sykes and Glen [18]), they used $T(20) = 173,338,962$ to show that $\lambda_T \geq (T(20)/10)^{1/20} \approx 2.3011$. Nowadays, since we know $T(n)$ up to $n = 75$,[1] we can obtain, using the same method, that $\lambda_T \geq (T(75)/10)^{1/75} \approx 2.7714$. We can even do slightly better than that. Substituting in Equation (1) the upper bound we obtain in Section 3 ($\lambda_T \leq 3.6050$), we see that $\lambda_T \geq (T(75)/(2(1+3.6050)))^{1/75} \approx 2.7744$. However, we can still improve on this.

An easy upper bound, based on an idea of Eden [6] (originally applied to the square lattice for setting an upper bound on $\lambda$), was described by Lunnon [14, p. 98]. Every polyiamond $P$ can be built according to a set of $n-1$ "instructions" taken from a superset of size $2(n-1)$. Each instruction tells us how to choose a lattice cell $c$, neighboring a cell already in $P$, and add $c$ to $P$. (Some of these instruction sets are illegal, and other sets produce the same polyiamonds, but this only helps.) Hence, $\lambda_T \leq \lim_{n\to\infty} \binom{2(n-1)}{n-1}^{1/n} = 4$.

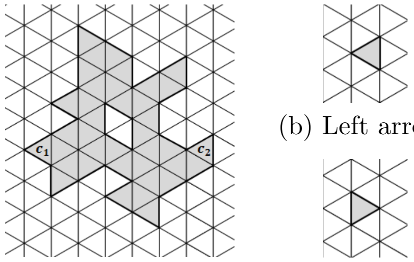
As can be seen, there is a large gap between the lower and upper bounds on $\lambda_T$. Based on existing data, it is believed [18] (but has never been proven) that $\lambda_T = 3.04 \pm 0.02$. In this paper we improve both lower and upper bounds on $\lambda_T$, showing that $2.8424 \leq \lambda_T \leq 3.6050$. The new lower bound is obtained by using a concatenation argument tailored to the triangular lattice, and the new upper bound is obtained by investigating the growth constant of a sequence dominating the enumerating sequence of polyiamonds.

## 2 Lower Bound

A concatenation of two polyiamonds $P_1, P_2$ is the translation of $P_1$ relative to $P_2$, so that $P_1, P_2$ do not overlap but their union is a valid (connected) polyiamond, and all the translated versions of the cells of $P_1$ are smaller than the cells of $P_2$ under a proper definition of a lexicographic order on the cells of the lattice. We use a concatenation argument in order to improve the lower bound on $\lambda_T$.

**Theorem 1** $\lambda_T \geq 2.8424$.

**Proof.** We orient the triangular lattice as is shown in Figure 1(a), and define a lexicographic order on the cells of the lattice as follows: A cell $c_1$ is *smaller* than cell $c_2 \neq c_1$ (denoted as $c_1 < c_2$) if the lattice column of $c_1$ is to the left of the column of $c_2$, or if $c_1, c_2$ are in the same column and $c_1$ is below $c_2$. Denote triangles which look like a "left arrow" (Figure 1(b)) as triangles of Type 1, and triangles which look like a "right arrow" (Figure 1(c)) as triangles of Type 2. Let $T_1(n)$ be the number of polyiamonds of size $n$ whose *largest* (top-right) triangle is of Type 1, and let $T_2(n)$ be the number of polyiamonds of size $n$ whose *largest* triangle is of Type 2. Obviously, we have $T(n) = T_1(n) + T_2(n)$.[2] An interesting observation is that by rotational symmetry, the number of polyiamonds of size $n$, whose *smallest* (bottom-left) triangle is of Type 2, is also $T_1(n)$, and so the number of polyiamonds, whose *smallest* triangle is of Type 1, is $T_2(n)$.




(a) A polyiamond (b) Left arrow (c) Right arrow

Figure 1: Polyiamonds on the triangular lattice

We now proceed with a standard concatenation argument, tailored to the specific case of the triangular lattice. Interestingly, not all pairs of polyiamonds of size $n$ can be concatenated. In addition, there exist many polyiamonds of size $2n$ which cannot be represented as the concatenation of two polyiamonds of size $n$. Let us count carefully the amount of pairs of polyiamonds that can be concatenated.

- Polyiamonds, whose largest triangle is of Type 1, can be concatenated only to polyiamonds whose smallest triangle is of Type 2, and this can be done in two different ways (see Figure 2(a)). There are $2(T_1(n))^2$ concatenations of this kind.

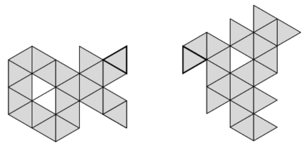
- Polyiamonds, whose largest triangle is of Type 2, can be concatenated, in a single way, only to polyiamonds whose smallest triangle is of Type 1 (see Figure 2(b)). There are $(T_2(n))^2$ concatenations of this kind.

Altogether, we have $2(T_1(n))^2+(T_2(n))^2$ possible concatenations, and, as argued above,

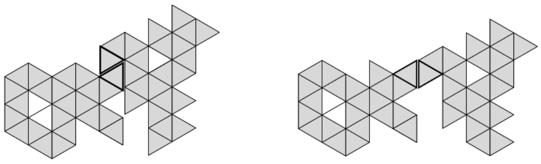
$$2(T_1(n))^2 + (T_2(n))^2 \leq T(2n). \quad (2)$$

[1] $T(75) = 15,936,363,137,225,733,301,433,441,827,683,823$.

[2] Observe that $T_1(n) = T_2(n-1)$ and, hence, $T(n) = T_2(n) + T_2(n-1)$. Indeed, when the largest cell of a polyiamond $P$ is of Type 1, its only possible neighboring cell within $P$ is the cell immediately below it. Therefore, the number of polyiamonds of size $n$ whose largest cell is of Type 1 is equal to the number of polyiamonds of size $n-1$ whose largest cell is of Type 2.

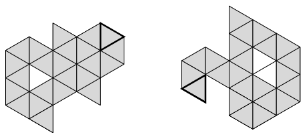


Two polyiamonds

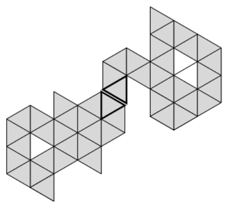


Vertical concatenation Horizontal concatenation

(a) Two concatenations of Type-1 and Type-2 triangles



Two polyiamonds



Vertical concatenation

(b) A single concatenation of Type-2 and Type-1 triangles

Figure 2: Possible concatenations of polyiamonds

Let us now find a lower bound on the number of concatenations. Let $x = x(n)$ be the fraction of polyiamonds of Type 1 out of all polyiamonds of size $n$, i.e., $T_1(n) = xT(n)$ and $T_2(n) = (1-x)T(n)$. Eq. (2) can then be rewritten as $T(2n) \geq 2(xT(n))^2 + ((1-x)T(n))^2 = (3x^2 - 2x + 1)T^2(n)$. Elementary calculus shows that the function $f(x) = 3x^2 - 2x + 1$ assumes its minimum at $x = 1/3$ and that $f(1/3) = 2/3$. Hence,

$$\frac{2}{3}T^2(n) \leq T(2n).$$

By manipulating this relation, we obtain that

$$\left(\frac{2}{3}T(n)\right)^{1/n} \leq \left(\frac{2}{3}T(2n)\right)^{1/(2n)}$$

This implies that the sequence $\left(\frac{2}{3}T(k)\right)^{1/k}$, $\left(\frac{2}{3}T(2k)\right)^{1/(2k)}$, $\left(\frac{2}{3}T(4k)\right)^{1/(4k)}, \ldots$ is monotone increasing for any value of $k$, and, as a subsequence of $\left(\left(\frac{2}{3}T(n)\right)^{1/n}\right)$, it converges to $\lambda_T$ too. Therefore, any term of the form $\left(\frac{2}{3}T(n)\right)^{1/n}$ is a lower bound on $\lambda_T$. In particular, $\lambda_T \geq \left(\frac{2}{3}T(75)\right)^{1/75} \approx 2.8424$. $\square$

## 3 Upper Bound

We follow the method used recently [3] for polyominoes (animals on the *square* lattice).[3]

### 3.1 Number of Compositions

**Definition 2** *A polyiamond $P$ can be* decomposed *into two polyiamonds $P_1, P_2$ if the cell set of $P$ can be split into two complementing non-empty subsets, such that each subset is a valid (connected) polyiamond. We also say that the polyiamonds $P_1, P_2$ can be* composed *so as to yield the polyiamond $P$.*

A *composition* of two polyiamonds is a natural generalization of the widely-used notion of the *concatenation* of polyiamonds. In fact, concatenation is a composition in lexicographic order.

**Theorem 3** *(Composition) Let $P_1, P_2$ be two polyiamonds of sizes $n_1$ and $n_2$, respectively. Then, $P_1$ and $P_2$ can be composed and yield at most $(n_1+2)(n_2+2)/2$ different polyiamonds.*[4]

**Proof.** Refer again to Figure 1(a). A boundary edge of a polyiamond can be either vertical, ascending, or descending. The inside of the polyiamond can be either to the left or to the right of a boundary edge,[5] where the latter case is marked below by overlining. Denote, then, the number of boundary edges of the various types by $x$ and $\bar{x}$, where $x \in \{v, a, d\}$. Thus, if the perimeter of a polyiamond is $p$, we can classify its boundary by the vector $(v, a, d, \bar{v}, \bar{a}, \bar{d})$, where $v + a + d + \bar{v} + \bar{a} + \bar{d} = p$. Suppose we are given two polyiamonds $P_1, P_2$ with respective perimeters $p_1, p_2$ and associated perimeter vectors $(v_i, a_i, d_i, \bar{v_i}, \bar{a_i}, \bar{d_i})$ (for $i = 1, 2$). Then, a trivial upper bound on the number of compositions of $P_1, P_2$ is $\sum_{t \in \{v,a,d,\bar{v},\bar{a},\bar{d}\}} t_1 \bar{t_2}$, using the convention $\bar{\bar{t_i}} = t_i$. Note that the number of boundary edges of any type of a polyiamond of perimeter $p$ cannot exceed $p/2$. Therefore, by convexity, the number of compositions of $P_1, P_2$ is bounded from above by $2(p_1/2 \cdot p_2/2) = p_1p_2/2$. (A slightly sharper upper bound which takes into account odd perimeters is $\lceil p_1/2 \rceil \lceil p_2/2 \rceil + \lfloor p_1/2 \rfloor \lfloor p_2/2 \rfloor$.) The perimeter of a polyiamond of size $n$ is maximized when the cell-adjacency graph of the polyiamond is a tree, in which case the perimeter is $n+2$. (Indeed, the perimeter of a single triangle is 3, and each of the additional $n-1$ triangles adds at most 1 to the perimeter.) The claim follows. $\square$

[3]There is a gap in Theorem 3 in this reference. Therefore, we take a different approach in Theorem 3 below.

[4]A slightly sharper upper bound is $(\lceil n_1/2 \rceil + 1)(\lceil n_2/2 \rceil + 1) + (\lfloor n_1/2 \rfloor + 1)(\lfloor n_2/2 \rfloor + 1)$.

[5]The "left" (resp., "right") side of an ascending edge means *above* (resp., *below*) the edge, while the "left" (resp., "right") side of a descending edge means *below* (resp., *above*) the edge.

### 3.2 Balanced Decompositions

**Definition 4** *A decomposition of a polyiamond of size $n$ into two polyiamonds $P_1, P_2$ is $k$-*balanced *if $k \le |P_i| \le n - k$ (for $i = 1, 2$).*

**Theorem 5** *Every polyiamond of size $n$ has at least one $\lceil (n-1)/3 \rceil$-balanced decomposition.*

**Proof.** Let us rephrase the claim in graph terminology. In fact, we prove a stronger claim which states that every connected graph $G$, for which $\Delta(G) \le 3$, can be partitioned into two vertex-disjoint subgraphs $G_1, G_2$, such that (1) $G_1, G_2$ are connected; and (2) $\lceil (n-1)/3 \rceil \le |G_i| \le \lfloor (2n+1)/3 \rfloor$ (for $i = 1, 2$). This can be done constructively by considering a spanning tree of $G$, marking an arbitrary vertex as its root, and traversing the tree downwards from the root while keeping the invariant that either the already-traversed subgraph meets the size requirement or the untraversed part contains a subgraph with this property. When the process stops, which must be the case, the desired decomposition is found. □

### 3.3 The Bound

**Theorem 6** $\lambda_T \le 3.6050$.

**Proof.** First, the combination of Theorems 3 and 5 implies that

$$T(n) \le \sum_{k=\lceil \frac{n-1}{3} \rceil}^{\lfloor n/2 \rfloor} (1 - \frac{\delta_{k,n/2}}{2}) \frac{(k+2)(n-k+2)}{2} T(k)T(n-k).$$

Indeed, every polyiamond of size $n$ can be decomposed in at least one $\lceil (n-1)/3 \rceil$-balanced way into a pair of polyiamonds $P_1, P_2$ of sizes $n_1$ and $n_2$, respectively (where $n_1 + n_2 = n$), and a code with up to $(n_1+2)(n_2+2)/2$ options will tell us uniquely how to compose $P_1, P_2$ in order to reconstruct $P$. (The factor $(1 - \delta_{k,n/2}/2)$ compensates for double counting which occurs when $P_1, P_2$ are of the same size.) Naturally, $P$ can be decomposed in more than one way, and the number of compositions of $P_1, P_2$ can be smaller than $(n_1 + 2)(n_2 + 2)/2$, but this only helps.

Second, define the sequence $T'(n)$ as follows.

$$T'(n) = \begin{cases} T(n) & 1 \le n \le 75; \\ \sum_{k=\lceil \frac{n-1}{3} \rceil}^{\lfloor n/2 \rfloor} (1 - \frac{\delta_{k,n/2}}{2}) \frac{(k+2)(n-k+2)}{2} T'(k)T'(n-k) & n > 75. \end{cases}$$

(Recall that the sequence $T(n)$ is known for $1 \le n \le 75$.) Since $T'(n) \ge T(n)$ for any value of $n \in \mathbb{N}$ (this can be proven by a simple induction on $n$), the growth constant of $T'(n)$, if it exists, is an upper bound on $\lambda_T$. Numerical calculations show that $T'(n)$ does have an asymptotic growth constant which is about 3.6050, implying the claim. □

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