Electronic Journal of Differential Equations, Vol. 2005(2005), No. 15, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

HOPF BIFURCATION IN THE IS-LM BUSINESS CYCLE MODEL WITH TIME DELAY

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ABSTRACT. The distinction between investment decisions and implementation leads us to formulate a new IS-LM business cycle model. It is shown that the dynamics depends crucially on the time delay parameter - the gestation time period of investment. The Hopf bifurcation theorem is used to predict the occurrence of a limit cycle bifurcation for the time delay parameter. Our analysis shows that the limit cycle behavior is independent of the assumption of nonlinearity of the investment function. An example is given to verify the theoretical results.

1. Introduction

The Hopf bifurcation theorem [8] as a tool for establishing the existence of closed orbits in dynamical systems seems to have been originally introduced to economics by Torre [10], who studied the standard IS-LM model

$$\begin{aligned} \dot{Y} &= \alpha(I(Y,r) - S(Y,r)) \\ \dot{r} &= \beta(L(Y,r) - \bar{M}) \end{aligned}$$

with Y as the gross product, I as the investment, S as the saving, L as the demand for money, and $\bar{M}$as the constant money supply. Here α and β are respectively the adjustment coefficients in the markets of goods and money. Other applications of the Hopf bifurcation theorem can be found in, Benhabib and Nishimura [2], Medio [9], Krawiec and Szydlowski [7], and Asada and Yoshida [1]. In the two-dimensional case the use of bifurcation theory actually provides no new insight into known models. The real domains of bifurcation theory are dynamical systems of dimension greater than or equal to three because the Poincare-Bendixson theorem cannot be applied anymore. Gabisch and Lorenz considered an augmented IS-LM business cycle model [4, p. 168],

$$\begin{aligned} \dot{Y} &= \alpha(I(Y,K,r) - S(Y,r)) \\ \dot{r} &= \beta(L(Y,r) - \bar{M}) \\ \dot{K} &= I(Y,K,r) - \delta K \end{aligned} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 37G15, 91B62.
Key words and phrases. Hopf bifurcation; business cycle; IS-LM model; time delay.

Submitted November 17, 2004. Published February 1, 2005.

with K as the capital stock and δ as the depreciation rate of the capital stock. The model seems to be one of the simplest complete business cycle models in the Keynesian tradition. A similar model has been studied by Boldrin [3].

In the Kalecki business cycle model [5], Kalecki assumed that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a gestation period or a time delay, after which capital equipment is available for production. Similar time lag has been introduced and discussed in [1] and [7]. In this paper, Kalecki's idea is introduced into the IS-LM model (1.1) to formulate a generalized IS-LM business cycle model as follow

$$\begin{aligned} \dot{Y} &= \alpha(I(Y,K,r) - S(Y,r)) \\ \dot{r} &= \beta(L(Y,r) - \bar{M}) \\ \dot{K} &= I(Y(t-T),K,r) - \delta K \end{aligned} \tag{1.2}$$

with T as the time delay parameter.

Investment depends on income at the time investment decisions are make and on capital stock at the time investment is finished. The latter is a consequence of the fact that at time $t-T$ there are some investments which will be finished between $t-T$ and t. We assume that capital stock produced in this period is taken into consideration when new investments are planned. The Hopf bifurcation theorem is applied to predict the occurrence of a limit cycle bifurcation for the time delay parameter. The crucial role in the creation of the limit cycle is Kalecki's time delay parameter, rather than the assumption of the S-shaped investment function. An example is given to show that a Hopf bifurcation can occur in the present IS-LM model – a linear system with time delay.

2. Linear IS-LM model

Assume that the investment function I, the saving function S, and the demand for money L depend linearly on their arguments, that is

$$\begin{aligned} I &= \eta Y - \delta_1 K - \beta_1 r \\ S &= l_1 Y + \beta_2 r \\ L &= l_2 Y - \beta_3 r \end{aligned}$$

with $\eta, \delta_1, l_1, l_2, \beta_1, \beta_2, \beta_3$ positive constants. Now system (1.2) becomes

$$\begin{aligned} \dot{Y} &= \alpha((\eta - l_1)Y - (\beta_1+\beta_2)r - \delta_1 K) \\ \dot{r} &= \beta(l_2 Y - \beta_3 r - \bar{M}) \\ \dot{K} &= \eta Y(t-T) - \beta_1 r - (\delta+\delta_1)K \end{aligned} \tag{2.1}$$

The characteristic equation of equation (2.1) has the form

$$\det\begin{pmatrix} \alpha(\eta - l_1) - \lambda & -\alpha(\beta_1+\beta_2) & -\alpha\delta_1 \\ \beta l_2 & -\beta\beta_3 - \lambda & 0 \\ \eta e^{-\lambda T} & -\beta_1 & -(\delta+\delta_1) - \lambda \end{pmatrix} = 0$$

that is

$$\lambda^3 + A\lambda^2 + B\lambda + C + D\lambda e^{-\lambda T} + E e^{-\lambda T} = 0 \tag{2.2}$$

where

$$A = \delta + \delta_1 + \beta\beta_3 - \alpha(\eta - l_1),$$
$$B = (\delta + \delta_1)(\beta\beta_3 - \alpha(\eta - l_1)) + \alpha\beta l_2(\beta_1 + \beta_2) - \alpha\beta\beta_3(\eta - l_1),$$
$$C = -\alpha\beta\beta_1 l_2\delta_1 - (\delta + \delta_1)\alpha\beta(\beta_3(\eta - l_1) - l_2(\beta_1 + \beta_2)),$$
$$D = \alpha\eta\delta_1, \quad E = \alpha\beta\beta_3\eta\delta_1$$

Generally speaking, transcendental equation (2.2) cannot be solved analytically and has indefinite number of roots. In essence, we have two main tools besides direct numerical integration; firstly, the linear stability analysis, especially in the case of small time delay, and secondly, the Hopf bifurcation theorem. In the following sections, we discuss both approaches.

3. Linear stability analysis

For small time delay T, the method of linear stability analysis is much convenient to find the bifurcation point. To this end, let $e^{-\lambda T} \approx 1 - \lambda T$, then the eigenvalue equation (2.2) becomes

$$\lambda^3 + (A - DT)\lambda^2 + (B + D - ET)\lambda + C + E = 0 \tag{3.1}$$

By the Hopf bifurcation theorem and the Routh-Huwitz criteria, a Hopf bifurcation occurs at a value $T = T_0$ where [4, pp166],

$$A - DT_0 > 0, \quad B + D - ET_0 > 0, \quad C + E > 0 \tag{3.2}$$

and

$$(A - DT_0)(B + D - ET_0) = C + E \tag{3.3}$$

Let

$$g(\lambda, T) = \lambda^3 + (A - DT)\lambda^2 + (B + D - ET)\lambda + C + E$$

Evaluating g at $T = T_0$ yields

$$g(\lambda, T_0) = \lambda^3 + s\lambda^2 + k^2\lambda + k^2 s$$

where $s = A - DT_0$, $k^2 = B + D - ET_0$. The eigenvalues of (3.1) at T_0 are

$$\lambda_0(T_0) = -s = -(A - DT_0)$$
$$\lambda_{1,2}(T_0) = \pm ik = \pm i(B + D - ET_0)^{\frac{1}{2}}$$

where i is the imaginary unit. Differentiating implicitly $g(\lambda(T), T)$ yields

$$\frac{d\lambda}{dT} = -\frac{\partial g}{\partial T}\Big/\frac{\partial g}{\partial \lambda} = -\frac{-D\lambda^2 - E\lambda}{3\lambda^2 + 2(A - DT)\lambda + B + D - ET}$$

Evaluating the required derivatives of g at T_0, we have

$$\frac{d\lambda_1(T_0)}{dT} = -\frac{(Dk^2 - Eki)(-3k^2 + B + D - ET_0 - 2k(A - DT_0)i)}{P^2 + R^2} \tag{3.4}$$

where $P = -3k^2 + B + D - ET_0$, $R = 2(A - DT_0)k$. The real part of (3.4) is

$$\operatorname{Re}\big(\frac{d\lambda_1(T_0)}{dT}\big) = -\frac{Dk^2(-3k^2 + B + D - ET_0) - 2Ek^2(A - DT_0)}{P^2 + R^2}$$

and $\operatorname{Re}(\frac{d\lambda_1(T_0)}{dT}) > 0$ is equivalent to

$$-D(B + D - ET_0) < Ek^2(A - DT_0) \tag{3.5}$$

Noting that D and E are positive, inequality (3.5) holds if the following conditions are fulfilled

$$A - DT_0 > 0 \quad \text{and} \quad B + D - ET_0 > 0$$

So inequality (3.2) is sufficient to have positive slope of the real part of the eigenvalue $\lambda_1(T)$. This fact guarantees the bifurcation to a limit cycle for $T = T_0$ according to the Hopf bifurcation theorem.

4. Hopf bifurcation analysis

For larger time delay T, the linear stability analysis of above section is no longer effective and another approach is needed. Let $\lambda = \sigma + i\omega$ and rewrite (2.2) in terms of its real and imaginary parts as

$$\sigma^3 - 3\sigma\omega + A\sigma^2 - A\omega^2 + B\sigma + C + e^{-\sigma T}(D\sigma\cos\omega T + D\omega\sin\omega T + E\cos\omega T) = 0$$

$$3\sigma^2\omega - \omega^3 + 2A\sigma\omega + B\omega + e^{-\sigma T}(D\omega\cos\omega T - D\sigma\sin\omega T - E\sin\omega T) = 0$$

To find the first bifurcation point, we set $\sigma = 0$. Then the above two equations reduce to

$$-A\omega^2 + C + D\omega\sin\omega T + E\cos\omega T = 0 \tag{4.1}$$

$$-\omega^3 + B\omega + D\omega\cos\omega T - E\sin\omega T = 0 \tag{4.2}$$

These two equations can be solved easily numerically. If the first bifurcation point is $(\omega_{\rm bif}, T_{\rm bif})$, then the other bifurcation points (ω, T) satisfy

$$\omega T = \omega_{\rm bif} T_{\rm bif} + 2n\pi, \quad n = 1, 2, \dots$$

By squaring (4.1) and (4.2), and then adding them, it follows that

$$\omega^6 + (A^2 - 2B)\omega^4 + (B^2 - 2AC - D^2)\omega^2 + C^2 - E^2 = 0 \tag{4.3}$$

This is a cubic equation in ω^2 and the left side is positive for large values of ω^2 and negative for $\omega = 0$ if $C^2 < E^2$. Hence, if the above condition is met, then (4.3) has at least one positive real root. Moreover, we have the following lemma [6].

Lemma 4.1. *Define*

$$\Delta = \frac{4}{27}a_2^3 - \frac{1}{27}a_1^2a_2^2 + \frac{4}{27}a_1^3a_3 - \frac{2}{3}a_1a_2a_3 + a_3^3$$

and suppose that $a_3 > 0$. Then necessary and sufficient conditions for the cubic equation

$$z^3 + a_1z^2 + a_2z + a_3 = 0$$

to have at least one single positive root for z are
(1) either $(a) a_1 < 0, a_2 \geq 0$, and $a_1^2 > 3a_2$, or $(b) a_2 < 0$; and
(2) $\Delta < 0$.

Denote

$$G(\lambda, T) = \lambda^3 + A\lambda^2 + B\lambda + C + D\lambda e^{-\lambda T} + Ee^{-\lambda T}$$

then

$$\frac{d\lambda}{dT} = -\frac{\partial G}{\partial T}\Big/\frac{\partial G}{\partial \lambda} = \frac{(D\lambda^2 + E\lambda)e^{-\lambda T}}{3\lambda^2 + 2A\lambda + B + (D - DT\lambda - ET)e^{-\lambda T}} \tag{4.4}$$

Evaluating the real part of this equation at $T = T_{\rm bif}$ and setting $\lambda = i\omega_{\rm bif}$ yield

$$\frac{d\sigma}{dT}\Big|_{T=T_{\rm bif}} = Re\Big(\frac{d\lambda}{dT}\Big)\Big|_{T=T_{\rm bif}} = \frac{\omega_{\rm bif}^2(3\omega_{\rm bif}^4 + 2\omega_{\rm bif}^2(A^2 - 2B) + B^2 - 2AC - D^2)}{P_1^2 + Q_1^2}$$

where

$$P_1 = -3\omega_{\rm bif}^2 + B + T_{\rm bif}(-A\omega_{\rm bif}^2 + C) + D\cos\omega_{\rm bif}T_{\rm bif}$$
$$Q_1 = 2A\omega_{\rm bif} + T_{\rm bif}(-\omega_{\rm bif}^3 + B\omega_{\rm bif}) - D\sin\omega_{\rm bif}T_{\rm bif}$$

Let $x = \omega_{\rm bif}^2$, then (4.3) reduces to

$$f(x) = x^3 + (A^2 - 2B)x^2 + (B^2 - 2AC - D^2)x + C^2 - E^2$$

then

$$f'(x) = 3x^2 + 2(A^2 - 2B)x + B^2 - 2AC - D^2$$

If $\omega_{\rm bif}$ is the least positive simple root of equation (4.3), unless this is a double root when we must take $\omega_{\rm bif}$ as the next root, then

$$f'(x)\big|_{T=T_{\rm bif}} > 0$$

Hence,

$$\frac{d\sigma}{dT}\Big|_{T=T_{\rm bif}} = \frac{\omega_{\rm bif}^2 f'(\omega_{\rm bif}^2)}{P_1^2 + Q_1^2} > 0$$

According to the Hopf bifurcation theorem, we come to the main result of this paper.

Theorem 4.2. *Assume that the conditions of Lemma 4.1 are satisfied and $\omega_{\rm bif}$ is the least positive root of equation (4.3) unless this is a double root when we must take $\omega_{\rm bif}$ as the next root which is simple, then a Hopf bifurcation occurs as T passes through $T_{\rm bif}$.*

A similar phenomenum appeared in the model of multiparty political system studied in [6].

Example. When $\alpha = 3, \beta = 2$, $\delta = 0.1, \delta_1 = 0.5$, $\eta = 0.3, l_1 = 0.2, l_2 = 0.1$, $\bar{M} = 0.05$, $\beta_1 = \beta_2 = \beta_3 = 0.2$, system (2.1) becomes

$$\dot{Y} = 0.3Y - 0.4r - 0.5K$$
$$\dot{r} = 0.2Y - 0.4r - 0.1$$
$$\dot{K} = 0.3Y(t-T) - 0.2r - 0.6K$$

The characteristic equation (2.2) becomes

$$\lambda^3 + 0.7\lambda^2 + 0.18\lambda + 0.012 + 0.45\lambda e^{-\lambda T} + 0.18e^{-\lambda T} = 0$$

It is easy to verify that the conditions of Theorem 4.2 are fulfilled, so a limit cycle bifurcation occurs when the time delay parameter T passes through $T_{\rm bif} = 0.740471$ where the relative eigenvalues are $\lambda_0 = -0.382583$, $\lambda_{1,2} = \pm 0.6993i$. Moreover, we can determine the approximate period of the closed orbit by

$$\tilde{T} = \frac{2\pi}{|\lambda(T_{\rm bif})|} = \frac{2\pi}{0.6993} = 8.98496$$

which implies the period of the economical system is about 9.

Conclusion. When we take into account the distinction between investment decision and expenditure, we come to the problem of gestation lags in investment. This leads us to formulate the generalized IS-LM business cycle model with time delay. The Hopf bifurcation theorem is used to predict the occurrence of a bifurcation to a limit cycle for some values of the time delay parameter. Our model admits the limit cycle behavior even for a linear investment function instead of a S-shaped one. It is also shown that Kalecki's time delay parameter plays the crucial role of existence of limit cycle behavior. The example in section 5 verifies the analytical results.

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