A promenade in the garden of hook length formulas

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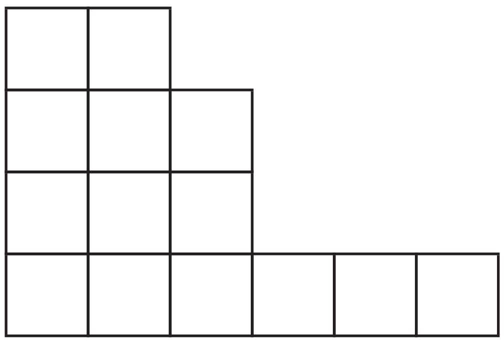
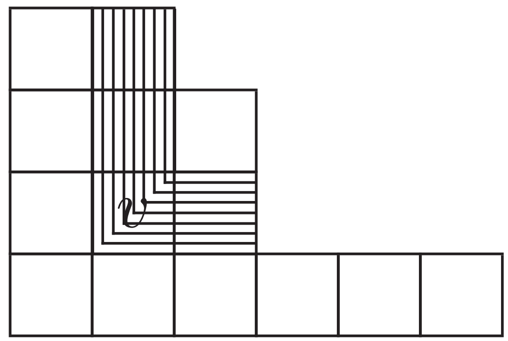
Summary:

- Some well-known examples
- How to discover new hook formulas ?
- The Main Theorem
- Specializations
- Latest news on the subject

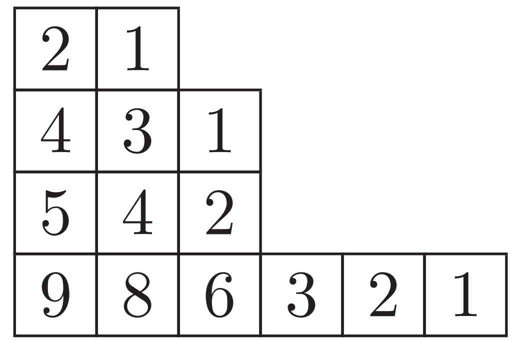
Some well-known examples: Hook length multi-set



Partition
$\lambda = (6, 3, 3, 2)$



Hook length of v
$h_v(\lambda) = 4$



Hook lengths
$\mathcal{H}(\lambda)$

The hook length multi-set of λ is

$$\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$$

Some well-known examples: permutations

f_λ : the number of standard Young tableaux of shape λ

Frame, Robinson and Thrall, 1954

$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

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Frame, Robinson and Thrall, 1954

$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

Robinson-Schensted correspondence: $\sum_{\lambda \vdash n} f_\lambda^2 = n!$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

Some well-known examples: involutions

The number of standard Young tableaux of $\{1, 2, \ldots, \}$ is equal to the number of *involutions* of order n.

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

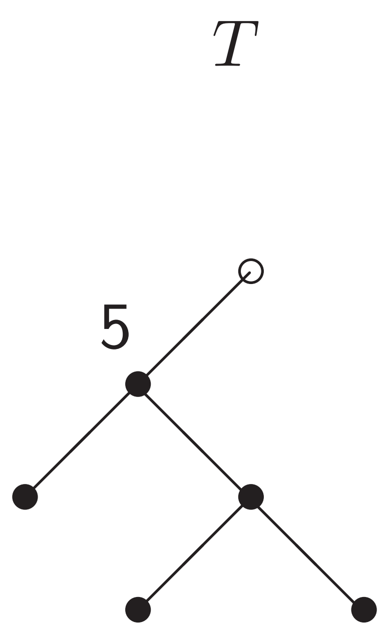
Some well-known examples: partitions

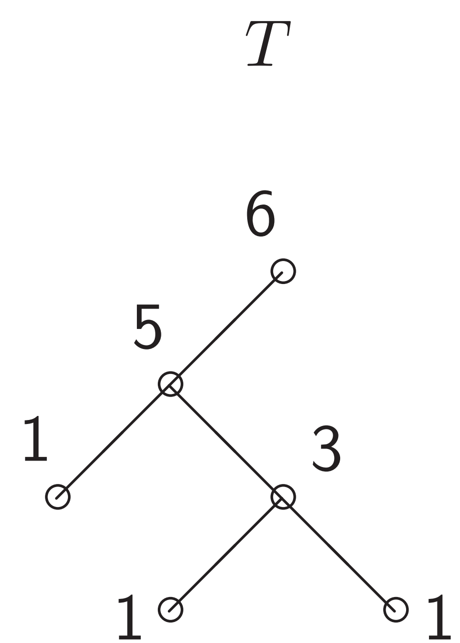
Generating function for partitions:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} 1 = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

Some well-known examples: binary trees

hook length for unlabeled binary trees





$\mathcal{H}_v(T) = 5$

$\mathcal{H}(T) = \{1, 1, 1, 3, 5, 6\}$

Some well-known examples: binary trees

f_T : the number of *increasing labeled binary trees*

$$f_T = \frac{n!}{\prod_{h \in \mathcal{H}(T)} h}$$

Some well-known examples: binary trees

Each labeled binary tree with n vertices is in bijection with a permutation of order n

$$\sum_{T \in \mathcal{B}(n)} n! \prod_{v \in T} \frac{1}{h_v} = n!$$

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Generating function form:

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \frac{1}{1-x}$$

Some well-known examples: binary trees, Catalan

The number of binary trees with n vertices is equal to the n-th Catalan number

$$\sum_{T \in \mathcal{B}(n)} 1 = \frac{1}{n+1}\binom{2n}{n}$$

Generating function form:

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} 1 = \frac{1-\sqrt{1-4x}}{2x}$$

Some well-known examples: tangent numbers

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T), h \geq 2} \frac{1}{2h} = \tan(x) + \sec(x)$$

Question: What is its combinatorial interpretation ?

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Answer : B !

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But what is A ?

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$$\sum_{T \in \mathcal{C}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \tan(x) + \sec(x)$$

$\mathcal{C}$: complete binary trees

Discover new hook formulas

	$Partitions$	$Trees$
Discovering		
Proving		

Discover new hook formulas

	$Partitions$	$Trees$
Discovering	Hard	
Proving		

Discover new hook formulas

	Partitions	*Trees*
Discovering	Hard	Hard
Proving		

Discover new hook formulas

	$Partitions$	$Trees$
Discovering	Hard	Hard
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Discover new hook formulas

	Partitions	*Trees*
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Proving	Hard	Easy

Discover new hook formulas

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hook length expansion

Discover new hook formulas: expansion

$\rho(h)$: weight function
$f(x)$: formal power series
They are connected by the relation:

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Discover new hook formulas: expansion

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- $hook\ length\ expansion: \ \rho \longleftarrow f$

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They are connected by the relation:

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- *generating function* : $\rho \longrightarrow f$
- *hook length expansion* : $\rho \longleftarrow f$
- *hook length formula* : when both ρ and f have "nice" forms

Discover new hook formulas: algorithm

- Does the *hook length expansion* exist ? Yes.

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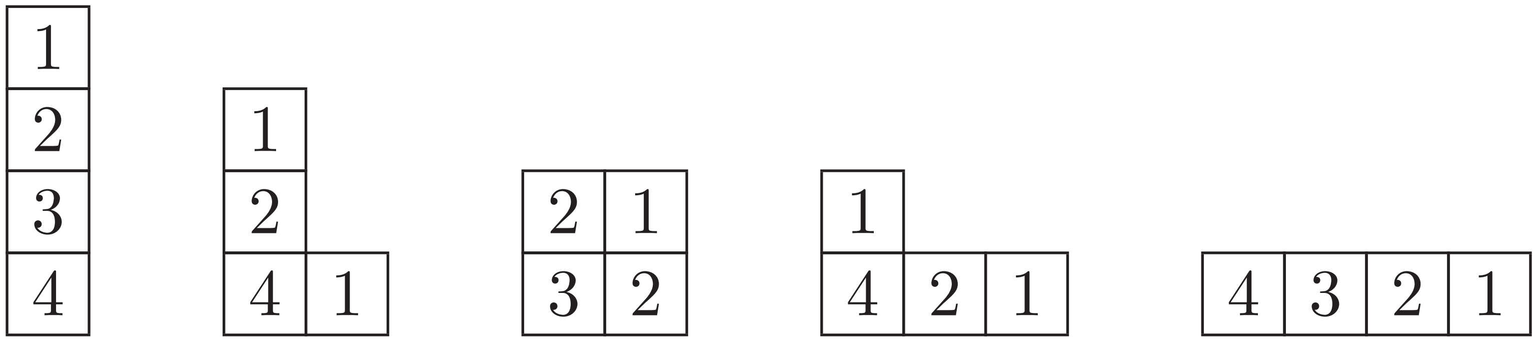

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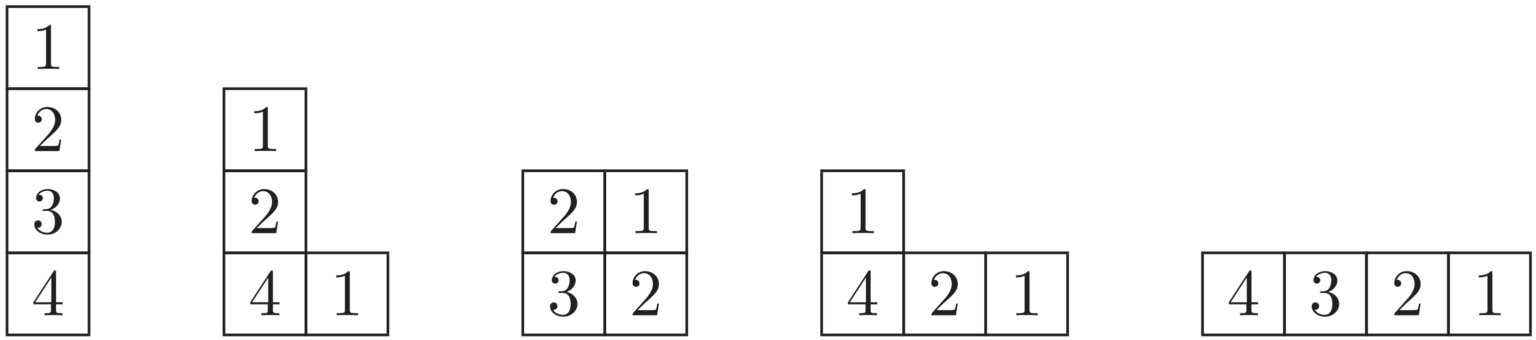

$$\rho_4\rho_3\rho_2\rho_1 + \rho_4\rho_2\rho_1\rho_1 + \rho_3\rho_2\rho_2\rho_1 + \rho_4\rho_2\rho_1\rho_1 + \rho_4\rho_3\rho_2\rho_1 = f_4$$

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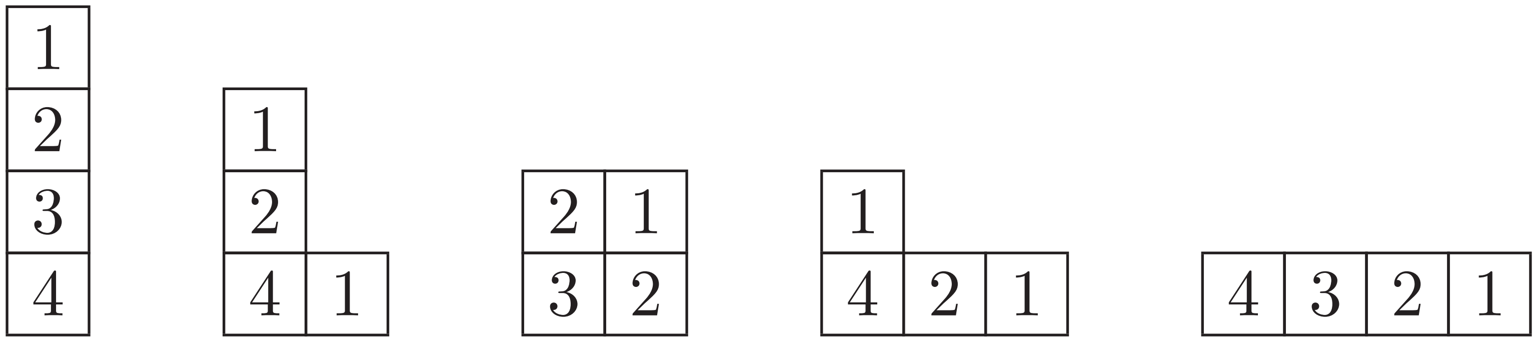

$$\rho_4\rho_3\rho_2\rho_1 + \rho_4\rho_2\rho_1\rho_1 + \rho_3\rho_2\rho_2\rho_1 + \rho_4\rho_2\rho_1\rho_1 + \rho_4\rho_3\rho_2\rho_1 = f_4$$

We can solve ρ_4 when knowing $\rho_1, \rho_2, \rho_3, f_4$, because there is at most one "4" in each partition (linear equation with one variable)

Discover new hook formulas: maple package

Maple package for the hook length expansion

`HookExp`

Two procedures

`hookgen`: $\rho \longrightarrow f$

`hookexp`: $\rho \longleftarrow f$

Discover new hook formulas: permutation

Example : permutations

```
> read("HookExp.mpl"):
> hookexp(exp(x), 8);
```

$$[1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}]$$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

Discover new hook formulas: involution

Example: involutions

```
> hookexp(exp(x+x^2/2), 8);
```

$$[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}]$$

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

Discover new hook formulas: interpolation

permutations : $$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

involutions : $$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

♡♡♡ What about the interpolation

$$e^{x+zx^2/2} \quad ?$$

Discover new hook formulas: interpolation

Try

```
> hookexp(exp(x+z*x^2/2), 8);
```

$$\left[1,\ \frac{1+z}{4},\ \frac{3z+1}{9+3z},\ \frac{z^2+6z+1}{16+16z},\ \frac{5z^2+10z+1}{5z^2+50z+25},\ \frac{z^3+15z^2+15z+1}{120z+36z^2+36},\ \frac{7z^3+35z^2+21z+1}{7z^3+147z^2+245z+49}\right]$$

Many binomial coefficients, so that ...

Discover new hook formulas: interpolation

Interpolation between permutations and involutions:

First Conjecture (H., 2008)

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \rho(z;h) = e^{x+zx^2/2}$$

where

$$\rho(z;n) = \frac{\displaystyle\sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} z^k}{\displaystyle n \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{2k+1} z^k}$$

Discover new hook formulas: partition

Another example: generating function for partitions

```
> hookexp(product(1/(1-x^k), k=1..9), 9);
```

$$[1, 1, 1, 1, 1, 1, 1, 1, 1]$$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} 1 = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

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♡♡♡ What about $\prod_k (1-x^k)$?

♡♡♡ or more generally $\prod_k (1-x^k)^{z}$?

Discover new hook formulas: partition

Try it by using HookExp

Discover new hook formulas: partition

Try it by using HookExp

```
> hookexp(product((1-x^k)^z, k=1..7), 7);
```

$$\left[-z, \frac{3-z}{4}, \frac{8-z}{9}, \frac{15-z}{16}, \frac{24-z}{25}, \frac{35-z}{36}, \frac{48-z}{49}\right]$$

Discover new hook formulas: partition

Try it by using HookExp

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We see that the ρ has a very simple expression:

$$\rho(h) = \frac{h^2 - 1 - z}{h^2} = 1 - \frac{z+1}{h^2}$$

Discover new hook formulas: Nekrasov-Okounkov

The previous hook length expansion suggests:

Theorem (Nekrasov-Okounkov, 2003; H., 2008)

$$\sum_{\lambda \in \mathcal{P}} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z+1}{h^2}\right) x = \prod_{k \geq 1} \left(1 - x^k\right)^z$$

Discover new hook formulas: proofs

How to prove ?

Discover new hook formulas: proofs

How to prove ?

The Russian-Physics Proof

Nekrasov, Okounkov (2003): arXiv: hep-th/0306238, 90 pages

(The last formula is deeply hidden in N-O's paper. See formula (6.12) on page 55)

Discover new hook formulas: proofs

How to prove ?

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The Lotharingian-Combinatorics Proof

H. (2008): arXiv:0805.1398 [math.CO], 28 pages

Discover new hook formulas: Combinatorial proof

Basic tools in the Lotharingian-Combinatorics Proof

- Macdonald's identities (1972): *Affine root systems and Dedekind's η-function*
- Garvan, Kim, Stanton's bijection (1990): *Cranks and t-cores*
- Lagrange interpolation formula

Discover new hook formulas

"This is great !

But can we do more ?"

Main Theorem: $1 + x^k$

We have (when $z = 1$):

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 - x^k).$$

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♡♡♡ What about

$$\prod (1 + x^k) \ ?$$

Main Theorem: $1+x^k$

Try

```
> hookexp(product(1+x^k, k=1..14),14);
```

$$[1, \frac{1}{2}, 1, \frac{7}{8}, 1, \frac{17}{18}, 1, \frac{31}{32}, 1, \frac{49}{50}, 1, \frac{71}{72}, 1, \frac{97}{98}]$$

We have

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda), h \text{ even}} (1-\frac{2}{h^2}) = \prod_{k\geq 1}(1+x^k).$$

Main Theorem: $1 + x^k$

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- No, with the right-hand side by `hookexp`($\longleftarrow$), because no "nice" expansion for

$$\prod \frac{1}{1+x^k} \quad \text{or} \quad \prod (1+x^k)^z.$$

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- Yes, with the left-hand side by hookgen($\longrightarrow$).

Main Theorem: $1+x^k$ variation

We have just seen:

$$\rho = [1, \frac{1}{2}, 1, \frac{7}{8}, 1, \frac{17}{18}, 1, \frac{31}{32}, 1, \frac{49}{50}, 1, \frac{71}{72}, 1, \frac{97}{98}] \longrightarrow \prod_{k\geq 1}(1+x^k).$$

Try the following variations of ρ with `hookgen`:

$$[1,\ 1-\frac{z}{2},\ 1,\ 1-\frac{z}{8},\ 1,\ 1-\frac{z}{18},\ 1,\ 1-\frac{z}{32},\ 1,\ 1-\frac{z}{50},\ 1]$$

$$[1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1]$$

$$[1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z]$$

Main Theorem: $1+x^k$ variation

```
> ...
```

$$\left[1,\ 1-\frac{z}{2},\ 1,\ 1-\frac{z}{8},\ 1,\ 1-\frac{z}{18},\ 1,\ 1-\frac{z}{32},\ 1,\ 1-\frac{z}{50},\ 1\right]$$

```
> hookgen(%):  etamake(%, x, 10):  simplify(%);
```

$$\prod_{k\geq 1}\frac{(1-x^{2k})^z}{1-x^k}$$

When $z=1$

$$\prod_{k\geq 1}\frac{(1-x^{2k})^z}{1-x^k}=\prod_{k\geq 1}\frac{1-x^{2k}}{1-x^k}=\prod_{k\geq 1}(1+x^k)$$

Main Theorem: $1 + x^k$ variation

```
> r:=n-> if n mod 3=0 then -1 else 1 fi:
> [seq(r(i), i=1..17)];
```

$$[1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1]$$

```
> hookgen(%):  etamake(%, x, 17):  simplify(%);
```

$$\prod_{k\geq 1} \frac{(1-x^{12k})^3(1-x^{3k})^6}{(1-x^{6k})^9(1-x^k)}$$

Main Theorem: $1+x^k$ variation

```
> f := k -> (1-x^(3*k))^3/(1-(z*x^3)^k)^3/(1-x^k):
> hookexp(product(f(k),k=1..15), 15);
```

$$[1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z]$$

```
> ...
```

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} z^{\mathrm{hmul}_t(\lambda)} = \prod_{k \geq 1} \frac{(1-x^{tk})^t}{(1-(zx^t)^k)^t (1-x^k)}$$

where $\mathrm{hmul}_t(\lambda)$ is the number of boxes v such that $h_v(\lambda)$ is a multiple of t.

Main Theorem

The previous and many other experimentations suggest:

Main Theorem (H. 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z}(1 - x^k)}$$

$$\mathcal{H}_t(\lambda) = \{h \mid h \in \mathcal{H}(\lambda), h \equiv 0 \pmod t\}.$$

Main Theorem: fields of interest

This work has some links with the following fields:

• General Mathematical Community: Euler, Jacobi, Gauss

• High Energy Physics Theory: Nekrasov, Okounkov

• Lie Algebra and Representation Theory: Macdonald, Dyson, Kostant, Milne, Adin, Schlosser

• Modular Forms and Number Theory: Ramanujan, Lehmer, Ono, Stanton

• q-Series, Combinatorics: (Here we are !)

• Algorithm, Computer Algebra: RSK, Krattenthaler (`rate`), Garvan (`qseries`), Sloane

• Plane Trees: Viennot, Foata, Schützenberger, Strehl, Gessel, Postnikov

Main Theorem: Specializations

The Main Theorem has so many specializations:

- the Jacobi triple product identity $\longrightarrow$
- the Gauss identity $\longrightarrow$
- the Nekrasov-Okounkov formula
- the generating function for partitions
- the Macdonald identity for $A_\ell^{(a)}$
- the classical hook length formula
- the marked hook formula $\longrightarrow$
- the generating function for t-cores
- the t-core analogues of the hook formula
- the t-core analogues of the marked hook formula
- ...

Specializations, Jacobi + Gauss

The Main Theorem unifies Jacobi and Gauss identities.

$t = 1, y = 1, z = 4$:

Jacobi

$$\prod_{m\geq 1}(1-x^m)^3 = \sum_{m\geq 0}(-1)^m(2m+1)x^{m(m+1)/2}$$

$t = 2, y = 1, z = 2$:

Gauss

$$\prod_{m\geq 1}\frac{(1-x^{2m})^2}{1-x^m} = \sum_{m\geq 0} x^{m(m+1)/2}$$

Specializations, t-cores

Let $\{z = t \text{ or } \ y = 0\}$, we get the well known formula:

$$\sum_{\lambda:\ t\text{-cores}} x^{|\lambda|} = \prod_{k\geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

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♡♡♡ What about

$$\prod_{k\geq 1} \frac{(1 + x^{tk})^t}{1 - x^k} \ ?$$

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♡♡♡ What about

$$\prod_{k\geq 1} \frac{(1 + x^{tk})^t}{1 - x^k} \ ?$$

♡♡♡ How to generalize it ?

Specializations, t-cores

First, try `hookexp` ($\longleftarrow$):

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} 2^{\#\{h\in\mathcal{H}(\lambda), h=t\}} = \prod_{k\geq 1} \frac{(1+x^{tk})^t}{1-x^k}.$$

Specializations, t-cores

First, try `hookexp` ($\longleftarrow$):

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} 2^{\#\{h\in\mathcal{H}(\lambda),h=t\}} = \prod_{k\geq 1} \frac{(1+x^{tk})^t}{1-x^k}.$$

Then, try `hookgen` ($\longrightarrow$):

Theorem (H. 2008)

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} y^{\#\{h\in\mathcal{H}(\lambda),h=t\}} = \prod_{k\geq 1} \frac{(1+(y-1)x^{tk})^t}{1-x^k}$$

Specializations, marked hook formula

- $\{z = -b/y, y \to 0\}$ in Main Theorem:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{tb}{h^2} = e^{bx^t} \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

- Compare the coefficients of $b^n x^{tn}$:

$$\sum_{\lambda \vdash tn, \# \mathcal{H}_t(\lambda) = n} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t^n n!}$$

- $t = 1$:

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Specializations, marked hook formula

- Compare the coefficients of $(-z)^{n-1}x^{nt}y^n$

$$\sum_{\lambda\vdash nt,\#\mathcal{H}_t(\lambda)=n} \prod_{h\in\mathcal{H}_t(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}_t(\lambda)} h^2 = \frac{3n-3+2t}{2(n-1)!}$$

- $t=1$:

Marked hook formula (H. 2008)

$$\sum_{\lambda\vdash n} f_\lambda^2 \sum_{h\in\mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2} n!$$

Specializations, marked hook formula

- Direct combinatorial proof ? Not yet
- Generalizations ? Yes

Specializations, marked hook formula

- Direct combinatorial proof ? Not yet
- Generalizations ? Yes

Specializations, marked hook formula

$$\sum_{\lambda\vdash n}\prod_{h\in\mathcal{H}(\lambda)}\frac{1}{h^2}\sum_{h\in\mathcal{H}(\lambda)}h^{2}=\frac{3n-1}{2(n-1)!}$$

$$\sum_{\lambda\vdash n}\prod_{h\in\mathcal{H}(\lambda)}\frac{1}{h^2}\sum_{h\in\mathcal{H}(\lambda)}h^{4}=\frac{40n^2-75n+41}{6(n-1)!}$$

$$\sum_{\lambda\vdash n}\prod_{h\in\mathcal{H}(\lambda)}\frac{1}{h^2}\sum_{h\in\mathcal{H}(\lambda)}h^{6}=\frac{1050n^3-4060n^2+5586n-2552}{24(n-1)!}$$

Second Conjecture (H. 2008)

$$P_k(n)=(n-1)!\sum_{\lambda\vdash n}\Big(\prod_{v\in\lambda}\frac{1}{h_v^2}\Big)\Big(\sum_{u\in\lambda}h_u^{2k}\Big)$$

is a polynomial in n of degree k.

Specializations, Bessenrodt

- $\{y = 1;$ compare the coefficients of $z\}$ in Main Theorem

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t} \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^{tk}}{k(1 - x^{tk})}.$$

- $t = 1$:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^{k}}{k(1 - x^{k})}.$$

Specializations, Bessenrodt

Direct proof.

By using an elegant result on multi-sets of hook lengths and multi-sets of partition parts.

It is amusing to see that this result is rediscovered periodically:

- Stanley (1972, partial)
- Kirdar, Skyrme (1982, partial)
- Elder (1984, partial)
- Hoare (1986, partial)
- Bessenrodt (1998)
- Bacher, Manivel (2002)
- H. (2008)

New hook length formulas for

plane trees

SLC 62 (2009)

Latest news on the subject

The First Conjecture has been proved by:

Kevin Carde, Joe Loubert, Aaron Potechin, Adrian Sanborn

under the guidance of Dennis Stanton and Vic Reiner

(the Minnesota school)

Latest news on the subject

Ameya Velingker, Emily Clader, Yvonne Kemper, Matt Wage

was working on these new hook length formulas and found interesting applications on Modular Forms and Number Theory

under the guidance of Ken Ono

(the Wisconsin school)

Latest news on the subject

The Second Conjecture has been proved by Richard Stanley

Tewodros Amdeberhan slightly simplified Stanley's proof

(the MIT school)

Latest news on the subject

Laura Yang, Bruce Sagan

have found generalizations and other proofs of certain hook length formulas for plane trees

Papers

- Discovering hook length formulas by expansion technique
- New hook length formulas for binary trees
- Yet another generalization of Postnikov's hook length formula for binary trees
- Some conjectures and open problems on partition hook lengths
- The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications
- An explicit expansion formula for the powers of the Euler product in terms of partition hook lengths
- (with Ken Ono) Hook lengths and 3-cores
- Hook lengths and shifted parts of partitions

All papers are available on:

http://www-irma.u-strasbg.fr/~guoniu/hook

References

- Laura Yang, Generalizations of Han's hook length identities
- Bruce Sagan, Probabilistic proofs of hook length formulas involving trees
- Richard Stanley, Some combinatorial properties of hook lengths, contents, and parts of partitions
- Tewodros Amdeberhan, Differential operators, shifted parts, and hook lengths
- Gil Kalai, Powers of Euler products and Han's marked hook formula (blog)
- Emily Clader, Yvonne Kemper, Matt Wage, Lacunarity of certain partition theoretic generating functions arising from Han's generalization of the Nekrasov-Okounkov formula
- Ameya Velingker, An exact formula for the coefficients of Han's generating function
- Kevin Carde, Joe Loubert, Aaron Potechin, Adrian Sanborn, Proof of Han's hook expansion conjecture