Asymptotic expansion for $\log n!$ in terms of the reciprocal of a triangular number

Gergő Nemes[1]

December 20, 2008

http://dx.doi.org/10.3247/sl2math08.004

Abstract

Ramanujan suggested an expansion for the nth partial sum of the harmonic series which employs the reciprocal of the nth triangular number. This has been proved in 2006 by Villarino, who speculated that there might also exist a similar expansion for the logarithm of the factorial. This study shows that such an asymptotic expansion really exists and provides formulas for its generic coefficient and for the bounds on its errors.

1 Introduction

1.1 Ramanujan's expansion

Ramanujan [1,6] proposed, without proof and without a formula for the general term, the following asymptotic expansion for the partial sum of the harmonic series:

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\log(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + ..., \tag{1.1}$$

where $m := \frac{n(n+1)}{2}$ is the nth triangular number and γ is the Euler-Mascheroni constant. The complete proof of this theorem was given in 2006 by M. Villarino [6] who also suggested that there might exist a series expansion for the logarithm of the factorial in terms of $\frac{1}{m}$. In this article we prove a formula which represents an affirmative answer to this conjecture.

1.2 The new asymptotic series

The following theorem gives a complete asymptotic expansion for the logarithm of the factorial in terms of the reciprocal of a triangular number. The proof of the theorem will be given in Section 2.

Theorem 1. *Let $m := \frac{n(n+1)}{2}$, where n is a positive integer. Then for every integer $r \geq 1$ there exist two numbers, ϑ_r and κ_r, $0 < \vartheta_r < 1$ and $-1 < \kappa_r < 1$, such that:*

$$\frac{2}{\sqrt{8m+1}} \log\left(\frac{n!}{\sqrt{2\pi}}\right) = \frac{1}{2}\log(2m) - 1 + \sum_{k=1}^{r} \frac{G_k}{m^k} + \vartheta_r \cdot \frac{G_{r+1}}{m^{r+1}} + \kappa_r \cdot \frac{(-1)^r}{2(r+1)\cdot 8^{r+1} m^{r+1}}, \tag{1.2}$$

where the G_k coefficients are given by

$$G_k = \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^{k} \frac{(-1)^j \left(2^{2j}-2\right) B_{2j}}{2j-1} \binom{k}{j}, \tag{1.3}$$

and B_j denotes the jth Bernoulli number (see [5]).

[1] nemesgery@gmail.com

Before we prove this, let us first briefly review the history of the Stirling series on which our proof will be based.

1.3 The history of Stirling's formula

In 1730, in his *Methodus Differentialis* [4], Stirling presented his results on the summation of logarithms. In particular, he gave a series approximation for $\log 1 + \log 2 + \ldots + \log n = \log n!$ in the form

$$\left(n+\frac{1}{2}\right)\log\left(n+\frac{1}{2}\right)-\left(n+\frac{1}{2}\right)+\frac{1}{2}\log(2\pi)-\frac{1}{24\left(n+\frac{1}{2}\right)}+\frac{7}{2880\left(n+\frac{1}{2}\right)^3}-\ldots\,. \tag{1.4}$$

Stirling gave a recurrence for the coefficients in his series, but he was not able to obtain an explicit formula for them. The general term for $k \geq 1$ is given by

$$\frac{\left(2^{1-2k}-1\right)B_{2k}}{2k\,(2k-1)\left(n+\frac{1}{2}\right)^{2k-1}}, \tag{1.5}$$

where B_k denotes the kth Bernoulli number. After seeing Stirling's results, De Moivre in his *Miscellaneis Analyticis Supplementum* discovered a much simpler approximation:

$$\left(n+\frac{1}{2}\right)\log n - n + \frac{1}{2}\log(2\pi)+\frac{1}{12n}-\frac{1}{360n^3}+\ldots\,. \tag{1.6}$$

This time the formula for the general term is

$$\frac{B_{2k}}{2k\,(2k-1)\,n^{2k-1}}. \tag{1.7}$$

De Moivre's series was associated with Stirling's name because the constant $\frac{1}{2}\log(2\pi)$ was determined by him. Both of the series are divergent for all values of n, but the partial sums can be made an arbitrarily good approximation for large enough n (see [2]).

Note that Ramanujan [7] also gave an approximation for $\log n!$ in the form

$$\log n! \approx n\log n - n + \frac{1}{2}\log\pi + \frac{1}{6}\log\left(8n^3+4n^2+n+\frac{1}{30}\right). \tag{1.8}$$

2 Proof of the new asymptotic expansion

The proof of our theorem is based on the following more precise version of the Stirling series (see [3]).

Theorem 2. *Let n be a positive integer. Then for every integer $r \geq 1$ there exists a number ρ_r, $0 < \rho_r < 1$, for which the following expression holds:*

$$\frac{1}{\left(n+\frac{1}{2}\right)}\log\left(\frac{n!}{\sqrt{2\pi}}\right)=\log\left(n+\frac{1}{2}\right)-1+\sum_{k=1}^{r}\frac{S_k}{\left(n+\frac{1}{2}\right)^{2k}}+\rho_r\cdot\frac{S_{r+1}}{\left(n+\frac{1}{2}\right)^{2r+2}}, \tag{2.1}$$

where the S_k coefficients are given by

$$S_k=\frac{\left(2^{1-2k}-1\right)B_{2k}}{2k\,(2k-1)}.\ \blacksquare \tag{2.2}$$

Proof. First, let us modify the term $\log\left(n+\frac{1}{2}\right)$:

$$\begin{aligned}\log\left(n+\frac{1}{2}\right) &= \frac{1}{2}\log\left(n+\frac{1}{2}\right)^2 = \frac{1}{2}\log\left(2m+\frac{1}{4}\right) = \frac{1}{2}\log\left(2m\left(1+\frac{1}{8m}\right)\right)\\ &= \frac{1}{2}\log(2m)+\frac{1}{2}\log\left(1+\frac{1}{8m}\right) = \frac{1}{2}\log(2m)+\frac{1}{2}\sum_{i\geq 1}\frac{(-1)^{i-1}}{i\cdot 8^i m^i}\\ &= \frac{1}{2}\log(2m)+\sum_{i=1}^{r}\frac{(-1)^{i-1}}{2i\cdot 8^i m^i}+\delta_r,\end{aligned}$$

where

$$\delta_r := \sum_{i\geq r+1}\frac{(-1)^{i-1}}{2i\cdot 8^i m^i}.$$

To obtain the series expansion in terms of $\frac{1}{m}$, we apply the binomial theorem:

$$\begin{aligned}\sum_{k=1}^{r}\frac{S_k}{\left(n+\frac{1}{2}\right)^{2k}} &= \sum_{k=1}^{r}\frac{S_k}{\left(2m+\frac{1}{4}\right)^{k}} = \sum_{k=1}^{r}\frac{S_k}{(2m)^k}\left(1+\frac{1}{8m}\right)^{-k}\\ &= \sum_{k=1}^{r}\frac{S_k}{(2m)^k}\sum_{l\geq 0}\binom{-k}{l}\frac{1}{8^l m^l}\\ &= \sum_{k=1}^{r}\frac{S_k}{2^k}\sum_{l\geq 0}\binom{k+l-1}{l}\frac{(-1)^l}{8^l m^{k+l}}.\end{aligned}$$

Replacing every k by $k-l$, and every l by $k-j$ we obtain

$$\sum_{k=1}^{r}\frac{S_k}{\left(n+\frac{1}{2}\right)^{2k}} = \sum_{k=1}^{r}\left\{\sum_{j=1}^{k}\frac{S_j}{2^j}\binom{k-1}{k-j}\frac{(-1)^{k-j}}{8^{k-j}}\right\}\frac{1}{m^k}+\varepsilon_r,$$

where

$$\varepsilon_r := \sum_{k=1}^{r}\frac{S_k}{2^k}\sum_{l\geq r-k+1}\binom{k+l-1}{l}\frac{(-1)^l}{8^l m^{k+l}}.$$

Now, collecting all the partial results

$$\begin{aligned}\frac{1}{\left(n+\frac{1}{2}\right)}\log\left(\frac{n!}{\sqrt{2\pi}}\right) =& \frac{1}{2}\log(2m)+\sum_{i=1}^{r}\frac{(-1)^{i-1}}{2i\cdot 8^i m^i}-1+\sum_{k=1}^{r}\left\{\sum_{j=1}^{k}\frac{S_j}{2^j}\binom{k-1}{k-j}\frac{(-1)^{k-j}}{8^{k-j}}\right\}\frac{1}{m^k}\\ &+\delta_r+\varepsilon_r+\rho_r\cdot\frac{S_{r+1}}{\left(n+\frac{1}{2}\right)^{2r+2}}\\ =& \frac{1}{2}\log(2m)-1+\sum_{k=1}^{r}\left\{\frac{(-1)^{k-1}}{2k\cdot 8^k}+\sum_{j=1}^{k}\frac{S_j}{2^j}\binom{k-1}{k-j}\frac{(-1)^{k-j}}{8^{k-j}}\right\}\frac{1}{m^k}\\ &+\delta_r+\varepsilon_r+\rho_r\cdot\frac{S_{r+1}}{\left(n+\frac{1}{2}\right)^{2r+2}}.\end{aligned}$$

We have thus derived a series expansion in terms of $\frac{1}{m}$, but the general term looks different than in (1.3). However, inserting (2.2) into our general coefficient, we get

$$
\begin{aligned}
G_k &= \frac{(-1)^{k-1}}{2k \cdot 8^k} + \sum_{j=1}^{k} \frac{S_j}{2^j} \binom{k-1}{k-j} \frac{(-1)^{k-j}}{8^{k-j}} \\
&= (-1)^{k-1} \left\{ \frac{1}{2k \cdot 8^k} + \sum_{j=1}^{k} \frac{\left(2^{1-2j}-1\right) B_{2j}}{2^j \cdot 2j\,(2j-1)} \binom{k-1}{k-j} \frac{(-1)^{1-j}}{8^{k-j}} \right\} \\
&= (-1)^{k-1} \left\{ \frac{1}{2k \cdot 8^k} + \sum_{j=1}^{k} \frac{\left(2^{1-2j}-1\right) B_{2j}}{2^j\,(2j-1)} \frac{1}{2k} \binom{k}{j} \frac{(-1)^{1-j}}{8^{k-j}} \right\} \\
&= \frac{(-1)^{k-1}}{2k \cdot 8^k} \left\{ 1 + \sum_{j=1}^{k} \frac{\left(2^{1-2j}-1\right) B_{2j}}{2^j\,(2j-1)} \binom{k}{j} \frac{(-1)^{1-j}}{8^{-j}} \right\} \\
&= \frac{(-1)^{k-1}}{2k \cdot 8^k} \left\{ 1 + \sum_{j=1}^{k} \frac{(-1)^j \left(2^{2j}-2\right) B_{2j}}{2j-1} \binom{k}{j} \right\} \\
&= \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^{k} \frac{(-1)^j \left(2^{2j}-2\right) B_{2j}}{2j-1} \binom{k}{j},
\end{aligned}
$$

which is the desired form of G_k.

Since $\left(n+\frac{1}{2}\right)^{-1} = \frac{2}{\sqrt{8m+1}}$, we now have

$$
\begin{aligned}
\frac{2}{\sqrt{8m+1}} \log\left(\frac{n!}{\sqrt{2\pi}}\right) =& \frac{1}{2}\log(2m) - 1 + \sum_{k=1}^{r} \left\{ \frac{(-1)^{k-1}}{2k \cdot 8^k} \sum_{j=0}^{k} \frac{(-1)^j \left(2^{2j}-2\right) B_{2j}}{2j-1} \binom{k}{j} \right\} \frac{1}{m^k} \\
&+ \delta_r + \varepsilon_r + \rho_r \cdot \frac{S_{r+1}}{\left(n+\frac{1}{2}\right)^{2r+2}}.
\end{aligned}
$$

It remains to show that for $r \geq 1$ the error term has the properties specified in the theorem. To estimate the error, we use the fact that a convergent alternating series whose terms decrease in absolute value monotonically to zero evaluates to any of its partial sums and a remainder comprised between zero and the first neglected term. It follows that

$$
\begin{aligned}
\upsilon_r &:= \rho_r \cdot \frac{S_{r+1}}{\left(n+\frac{1}{2}\right)^{2r+2}} = \rho_r \cdot \frac{S_{r+1}}{(2m)^{r+1}} \left(1+\frac{1}{8m}\right)^{-(r+1)} = \sigma_r \cdot \frac{S_{r+1}}{(2m)^{r+1}}, \\
\delta_r &= \sum_{i \geq r+1} \frac{(-1)^{i-1}}{2i \cdot 8^i m^i} = \tau_r \cdot \frac{(-1)^r}{2(r+1) \cdot 8^{r+1} m^{r+1}}, \\
\varepsilon_r &= \sum_{k=1}^{r} \frac{S_k}{2^k} \sum_{l \geq r-k+1} \binom{k+l-1}{l} \frac{(-1)^l}{8^l m^{k+l}} = \sum_{k=1}^{r} \frac{S_k}{2^k} \left\{ \omega_k \cdot \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}} \\
&= \Omega_r \cdot \sum_{k=1}^{r} \frac{S_k}{2^k} \left\{ \binom{r}{r-k+1} \frac{(-1)^{r-k+1}}{8^{r-k+1}} \right\} \frac{1}{m^{r+1}},
\end{aligned}
$$

where $0 < \sigma_r,\ \tau_r,\ \omega_k,\ \Omega_r < 1$ for $1 \leq k \leq r$ and $r \geq 1$. Moreover, $0 < \Omega_r < 1$ because for a fixed r, all the terms with the ω_k weights have the same sign. Since υ_r and ε_r have always the same sign (positive for odd r and negative for even r), we have

$$\begin{aligned}\upsilon_r+\varepsilon_r &= \sigma_r\cdot\frac{S_{r+1}}{(2m)^{r+1}}+\Omega_r\cdot\sum_{k=1}^{r}\frac{S_k}{2^k}\left\{\binom{r}{r-k+1}\frac{(-1)^{r-k+1}}{8^{r-k+1}}\right\}\frac{1}{m^{r+1}}\\ &=\vartheta_r\cdot\sum_{k=1}^{r+1}\frac{S_k}{2^k}\left\{\binom{r}{r-k+1}\frac{(-1)^{r-k+1}}{8^{r-k+1}}\right\}\frac{1}{m^{r+1}},\end{aligned}$$

where $0<\vartheta_r<1$. But δ_r has always the opposite sign so that we can set $\tau_r=\vartheta_r+\kappa_r$, where $-1<\kappa_r<1$. Finally

$$\begin{aligned}&\vartheta_r\cdot\left(\frac{(-1)^r}{2\,(r+1)\cdot 8^{r+1}m^{r+1}}+\sum_{k=1}^{r+1}\frac{S_k}{2^k}\left\{\binom{r}{r-k+1}\frac{(-1)^{r-k+1}}{8^{r-k+1}}\right\}\frac{1}{m^{r+1}}\right)\\ &=\vartheta_r\cdot\frac{(-1)^r}{2\,(r+1)\cdot 8^{r+1}m^{r+1}}\left(1+\sum_{k=1}^{r+1}\frac{\left(2^{1-2k}-1\right)B_{2k}}{2^k\cdot(2k-1)}\left\{\binom{r+1}{k}\frac{(-1)^{-k+1}}{8^{-k}}\right\}\right)\\ &=\vartheta_r\cdot\frac{(-1)^r}{2\,(r+1)\cdot 8^{r+1}m^{r+1}}\left(1+\sum_{k=1}^{r+1}\frac{(-1)^k\left(2^{2k}-2\right)B_{2k}}{(2k-1)}\binom{r+1}{k}\right)\\ &=\vartheta_r\cdot\frac{(-1)^r}{2\,(r+1)\cdot 8^{r+1}m^{r+1}}\sum_{k=0}^{r+1}\frac{(-1)^k\left(2^{2k}-2\right)B_{2k}}{(2k-1)}\binom{r+1}{k}\\ &=\vartheta_r\cdot\frac{G_{r+1}}{m^{r+1}}.\end{aligned}$$

This completes the proof of the theorem. ■

Conjecture. Numerical computations imply that for $r\geq 2$ we can use $\kappa_r=0$ in (1.2).

Appendix A: The following table gives the first ten coefficients of the asymptotic series.

G_1	$\frac{1}{24}$	G_6	$\frac{7619}{138378240}$
G_2	$-\frac{1}{1440}$	G_7	$-\frac{7439}{92252160}$
G_3	$-\frac{1}{4032}$	G_8	$\frac{2302207}{13442457600}$
G_4	$\frac{1}{8960}$	G_9	$-\frac{39982913}{81265766400}$
G_5	$-\frac{23}{380160}$	G_{10}	$\frac{242180131}{132433100800}$

Table with numerical values of G_k.

References

[1] B. Berndt. *Ramanujan's Notebooks.* Volume 5, Springer, New York, 1998.

[2] E. T. Copson. *Asymptotic Expansions.* Cambridge University Press, 1965.

[3] G. Allasia, C. Giordano and J. Pečarić. *Inequalities for the Gamma function relating to asymptotic expansions.* Volume 5, Mathematical Inequalities and Applications, 2002.

[4] J. Stirling. *Methodus differentialis, sive tractatus de summation et interpolation serierum infinitarium.* London, 1730.

[5] M. Abramowitz and I. A. Stegun (eds.). *Handbook of Mathematical Functions.* Dover, 1965.

[6] M. B. Villarino. *Ramanujan's Harmonic Number Expansion into Negative Powers of a Triangular Number.* arXiv:0707.3950v2 [math.CA] 28 Jul 2007.

[7] S. Ragahavan and S. S. Rangachari (eds.). *S. Ramanujan: The lost notebook and other unpublished papers.* Springer, 1988.