Mathematics. — *On* MAHLER's *partition problem.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

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1. *Introduction.*

KURT MAHLER [1]) obtained a formula for the number $p(h)$ of partitions of the natural number h into powers of a given integer $r \geq 2$, i.e. the number of solutions of

$$h = h_0 + h_1 r + h_2 r^2 + \dots \qquad (1.1)$$

in non-negative integers $h_0, h_1, h_2, \dots$ [2]). His result was

$$p(rh) = e^{O(1)} \sum_{n=0}^{\infty} r^{-\frac{1}{2}n(n-1)} h^n/n! \ ^{3)} \qquad (1.2)$$

which leads to the explicit result

$$\left.\begin{aligned} \log p(rh) = \frac{1}{2 \log r} \left(\log \frac{h}{\log h}\right)^2 + \left(\frac{1}{2} + \frac{1}{\log r} + \frac{\log \log r}{\log r}\right) \log h - \\ - \left(1 + \frac{\log \log r}{\log r}\right) \log \log h + O(1). \end{aligned}\right\} \qquad (1.3)$$

In the present paper we give a more precise analysis of the O-term in (1.3). It turns out to be of the form

$$\psi\left(\frac{\log h - \log \log h}{\log r}\right) + o(1) \qquad (1.4)$$

where ψ is a certain periodic function with period 1; the $o(1)$ term can be further investigated.

The series on the right of (1.2) has a similar asymptotic behaviour, with a different periodic function however. It is a solution of the functional equation $F'(h) = F(hr^{-1})$. We shall develop asymptotic formulae for the solutions of that equation in a separate paper.

Our results on $p(rh)$ are found in the following way. Since

$$\sum_{h=0}^{\infty} p(h)\, x^h = \prod_{k=0}^{\infty} (1 - x^{r^k})^{-1} = f(x) \qquad (|x| < 1) \qquad (1.5)$$

we have

$$p(h) = \frac{\varrho^{-h}}{2\pi} \int_0^{2\pi} f(\varrho e^{i\varphi})\, e^{-hi\varphi}\, d\varphi. \qquad (0 < \varrho < 1). \qquad (1.6)$$

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[1]) On a special functional equation, Journ. London Math. Soc. **15**, 115—123 (1940).

[2]) Of course only a finite number of the h_i are > 0.

[3]) Here $rh = h'$ must be an integer. In view of the generalisation to non-integral r we express (1.2) in terms of h instead of h'. See also (1.12).

The function $f(x)$, regular for $|x|<1$, can be calculated with great accuracy in the neighbourhood of the points $x = \exp(2\pi\nu i r^{-\mu})$ $(\nu, \mu = 0, 1, 2, \ldots)$ by a formula derived in section 2. After that, evaluation of (1.6) leads to the announced results.

It must be noted that the most important contribution to (1.6) arises from the neighbourhood of the point $x = 1$. Much smaller conributions are given by the points $\exp(2\pi\nu i r^{-1})$ $(\nu = 1, \ldots, r-1)$ and so on. These contributions are even much smaller than the errors we cannot avoid to make in the neighbourhood of $x = 1$. Therefore we restrict ourselves to a precise investigation of the neighbourhood of $x = 1$ with a relatively rough estimation of $f(x)$ on the remaining part of the circle $|x| = \varrho$.

In the sequel we shall use formula (1.16) instead of (1.6) because we want to generalise our considerations to the case that r is not an integer. First we develop the necessary formulae.

Henceforth r is an arbitrary number >1. Although not always stated explicitly, most functions in the sequel depend on r. Numbers depending on r only will be called constants.

Let $P(u)$ denote the number of solutions of

$$h_0 + h_1 r + h_2 r^2 + \ldots \leqq u \quad \ldots \ldots \quad (1.7)$$

in non-negative integers $h_0, h_1, h_2, \ldots$. The generating function is

$$F(s) = \prod_{k=0}^{\infty} (1 - e^{-sr^k})^{-1} = \int_{-\infty}^{\infty} e^{-su}\, dP(u), \quad (Re\, s > 0). \quad . \quad (1.8)$$

$F(s)$ reduces to $f(e^{-s})$ (see (1.5)) if r is an integer. The integral on the right of (1.8) is a STIELTJES integral. We notice that $P(u) = 0$ for $u < 0$, $P(0) = 1$. Furthermore we have

$$P(u) - P(u-1) = P(u/r) \qquad (-\infty < u < \infty). \quad . \quad . \quad . \quad (1.9)$$

Namely, $P(u) - P(u-1)$ denotes the number of solutions of

$$u - 1 < h_0 + h_1 r + h_2 r^2 + \ldots \leqq u, \quad . \quad . \quad . \quad (1.10)$$

which equals the number of solutions of

$$h_1 r + h_2 r^2 + \ldots \qquad \leqq u \quad \ldots \ldots \quad (1.11)$$

since for any solution $h_1, h_2, \ldots$ of (1.11) just one non-negative integer h_0 satifying (1.10) can be found. Since (1.11) has $P(u/r)$ solutions we obtain (1.9).

If r and $u = h$ are integers (1.10) and (1.1) are equivalent, and so

$$P(h) = p(rh). \quad \ldots \ldots \ldots \quad (1.12)$$

Moreover $P(y) = P([y])$ in that case, and

$$p(rh) = p(rh+1) = \ldots = p(rh+r-1). \quad . \quad . \quad . \quad . \quad (1.13)$$

Again considering the general case we first notice that by a well-known inversion formula we have

$$\frac{P(u-0)+P(u+0)}{2}=\frac{1}{2\pi i}\lim_{A\to+\infty}\int_{a-iA}^{a+iA} F(s)\,e^{us}\frac{ds}{s} \qquad (a>0). \quad (1.14)$$

Here the path of integration is the straight line.

It will be convenient to operate with $P_1(u)$ instead of $P(u)$, where

$$P_1(u)=\int_{u-1}^{u} P(v)\,dv=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\,e^{us}(1-e^{-s})\frac{ds}{s^2} \qquad (a>0). \quad (1.15)$$

The latter integral is absolutely convergent.

By (1.8) we have $(1-e^{-s})\,F(s)=F(rs)$. Consequently $P_1(u)$ can be written in the form

$$P_1(u)=\frac{r}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\,e^{us/r}\frac{ds}{s^2} \qquad (a>0). \quad . \quad . \quad . \quad (1.16)$$

From this formula we shall deduce the asymptotic behaviour of $P_1(u)$. The way back from $P_1(u)$ to $P(u)$ will be an easy one.

2. *An exact formula for $F(s)$.*

We shall derive a useful expression (formula (2.15) [4])) for the function (1.8), i.e.

$$F(s)=\prod_{k=0}^{\infty}(1-e^{-sr^k})^{-1} \qquad (Re\, s>0). \quad . \quad . \quad . \quad . \quad (2.1)$$

in the neighbourhood of $s=0$. For a moment we restrict ourselves to real and positive values of s. We have

$$\log F(s)=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} m^{-1}e^{-sr^k m}.$$

From MELLIN's formula

$$e^{-w}=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(z)\,w^{-z}\,dz \qquad (a>0,\ w>0)$$

we now infer that

$$\log F(s)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\Gamma(z)\,m^{-1}\,(sr^k m)^{-z}\,dz, \quad . \quad . \quad (2.2)$$

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[4]) The present author who found this formula in 1944 was informed after the war by Mr. MAHLER that a similar formula was communicated to him about 1923 by C. L. SIEGEL, who never published it. Therefore we present a full proof here.

the operation carried out being allowed in virtue of the convergence of

$$\int_{a-i\infty}^{a+i\infty} \sum_{0}^{\infty}\sum_{1}^{\infty} |\Gamma(z)\, m^{-1} (s\, r^k m)^{-z}|\, dz.$$

Finally (2.2) leads to

$$\log F(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^{-z} \frac{\Gamma(z)\,\zeta(1+z)}{1-r^{-z}}\, dz, \qquad (a>0,\, s>0) \quad (2.3)$$

where ζ denotes the RIEMANN ζ-function. Its functional equation gives

$$s^{-z}\,\Gamma(z)\,\zeta(1+z) = \frac{-\pi}{z \sin \frac{1}{2}\pi z} \cdot \left(\frac{2\pi}{s}\right)^z \zeta(-z), \quad . \; . \; . \; . \quad (2.4)$$

and it is easily derived that for $0 < s < 2\pi r$ we have

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{-n+\frac{1}{2}-i\infty}^{-n+\frac{1}{2}+i\infty} s^{-z} \frac{\Gamma(z)\,\zeta(1+z)}{1-r^{-z}}\, dz = 0$$

if n runs through the positive integers. It now follows from (2.3), (2.4) and from the estimation $\zeta(\sigma + it) = O(|t|)$ $(0 \leq \sigma \leq 1)$ that, for $0 < s < 2\pi r$, $\log F(s)$ equals the sum of the residues of

$$s^{-z}\,\Gamma(z)\zeta(1+z)(1-r^{-z})^{-1}. \quad . \; . \; . \; . \; . \; . \; . \quad (2.5)$$

We have to consider the roots $z = 2\pi i k/\log r$ of $1-r^{-z}$ $(k = 0, \pm 1, \pm 2, \ldots)$ and the poles $z = 0, -1, -2, \ldots$ of $\Gamma(z)$. The point $z = 0$ is the only pole of $\zeta(1+z)$ and thus it is a triple pole of (2.5).

First we evaluate the residue at $z = 0$. We have

$$z\,\zeta(z+1) = 1 + \gamma z + \gamma_2 z^2 + \ldots \quad . \; . \; . \; . \quad (2.6)$$

where $\gamma = 0{,}5772157\ldots$ is EULER's constant, and $\gamma_2 = 0{,}0728158\ldots$. Furthermore

$$z\,\Gamma(z) = 1 - \gamma z + \tfrac{1}{2}\Gamma''(1)\, z^2 + \ldots, \quad . \; . \; . \; . \quad (2.7)$$

where $\Gamma''(1) = \gamma^2 + \frac{1}{6}\pi^2$,

$$s^{-z} = 1 - z\log s + \tfrac{1}{2}(\log s)^2 z^2 - \ldots \quad . \; . \; . \; . \quad (2.8)$$

$$\frac{z \log r}{1-r^{-z}} = 1 + \tfrac{1}{2} z \log r + \tfrac{1}{12} z^2 (\log r)^2 + \ldots \quad . \; . \; . \quad (2.9)$$

It follows from (2.6), (2.7), (2.8) and (2.9) that the residue of (2.5) at $z = 0$ equals

$$(\log s)^2/2\log r - \tfrac{1}{2}\log s + \alpha_0 \quad . \; . \; . \; . \; . \; . \quad (2.10)$$

where

$$a_0 = \{\gamma_2 - \tfrac{1}{2}\gamma^2 + \tfrac{1}{12}\pi^2 + \tfrac{1}{12}(\log r)^2\}/\log r. \quad . \quad . \quad . \quad (2.\ 11)$$

The residues at $z = 2\pi ik/\log r$ $(k = \pm 1, \pm 2, \ldots)$ are easily found to be

$$a_k\, s^{-2\pi ik/\log r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.\ 12)$$

where

$$a_k = \Gamma\left(\frac{2\pi ik}{\log r}\right)\zeta\left(1 + \frac{2\pi ik}{\log r}\right)\Big/\log r. \quad . \quad . \quad . \quad . \quad (2.\ 13)$$

Finally we consider the poles $n = -1, -2, \ldots$ of $\Gamma(z)$. The residue of $\Gamma(z)$ at $z = -n$ is $(-1)^n/n!$; furthermore $\zeta(1-n) = (-1)^{n-1}\, n^{-1} B_n$, where the B_n are the BERNOULLI numbers defined by

$$y/(e^y - 1) = \sum_0^\infty y^n B_n/n! \quad ; \quad B_0 = 1, B_1 = -\tfrac{1}{2}, B_2 = \tfrac{1}{6}, B_3 = 0, \ldots$$

The residue of (2.5) at $z = -n$ thus amounts to

$$\beta_n s^n = \frac{B_n}{n \cdot n!} \cdot \frac{s^n}{r^n - 1} \qquad (n = 1, 2, 3, \ldots). \quad . \quad . \quad . \quad (2.\ 14)$$

By taking the sum of the residues (2.10), (2.12), (2.14) we find

$$\log F(s) = \frac{(\log s)^2}{2\log r} - \tfrac{1}{2}\log s + \sum_{k=-\infty}^{\infty}{}' a_k\, s^{-2\pi ik/\log r} + \sum_{n=1}^{\infty} \beta_n s^n. \quad (2.\ 15)$$

The range of validity of (2.15), for which we took thus far $0 < s < 2\pi r$, can be continued over the semi-circle $Re\, s > 0$, $|s| < 2\pi r$. For, the last series on the right of (2.15) converges for $|s| < 2\pi r$, and the first one for $Re\, s > 0$. We have namely, for $k \to \pm\infty$,

$$a_k = O\left(e^{-\pi^2 |k|/\log r}\, |k|^{-\frac{1}{2}} \log |k|\right) \quad . \quad . \quad . \quad . \quad . \quad (2.\ 16)$$

since $|\Gamma(it)| \backsim \sqrt{2\pi}\, e^{-\frac{1}{2}\pi|t|}\, |t|^{-\frac{1}{2}}$ and $\zeta(1 + it) = O(\log |t|)$ for $t \to \pm\infty$.

3. *Preliminary estimations.*

In orther to deal with (1.16) we first derive some rough estimations for $F(\sigma + it)/F(\sigma)$. Here σ and t are real, $\sigma > 0$, and F is defined by (1.8) = = (2.1).

Lemma. We have [5])

$$|F(\sigma + it)/F(\sigma)| \leqslant 1 \qquad (\sigma > 0, -\infty < t < \infty) \quad . \quad . \quad . \quad . \quad (3.\ 1)$$

$$|F(\sigma + it)/F(\sigma)| \leqslant \sigma^{c_1} \qquad (0 < \sigma \leqslant |t| \leqslant e^{-1}) \quad . \quad . \quad . \quad . \quad . \quad (3.\ 2)$$

$$|F(\sigma + it)/F(\sigma)| \leqslant \sigma^{c_2 y^2} \qquad (0 \leqslant |t| \leqslant \sigma \leqslant 1, \sigma > 0, y = t/\sigma). \quad (3.\ 3)$$

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[5]) The numbers $c_1, c_2, \ldots$ denote positive constants. If such a c occurs for the first time in a formula or statement we mean that it can be given a positive value such that the formula or statement is correct. At a second occurrence it keeps the value given to it the first time. The same applies to positive functions $c(\lambda)$ etc.

Proof. (3.1) immediately follows from

$$|1-e^{-\sigma r^k}| \leqslant |1-e^{-(\sigma+it)r^k}|.$$

More generally we have ($-\infty < t < \infty$, $\sigma > 0$, $K = 0, 1, 2, \ldots$)

$$\left|\frac{F(\sigma+it)}{F(\sigma)}\right| \leqslant \prod_{k=0}^{K} \left|\frac{1-e^{-(\sigma+it)r^k}}{1-e^{-\sigma r^k}}\right|^{-1}. \quad . \quad . \quad . \quad . \quad (3.4)$$

It is easily seen that

$$\left|\frac{1-e^{-(\sigma+it)r^k}}{1-e^{-\sigma r^k}}\right|^{-1} = \left\{1+\frac{4e^{-\sigma r^k}\sin^2(\frac{1}{2}tr^k)}{(1-e^{-\sigma r^k})^2}\right\}^{-\frac{1}{2}}.$$

Assuming that

$$|t|\, r^K \leqslant 1 \quad , \quad \sigma r^K \leqslant 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

we have for $0 \leq k \leq K$

$$|\sin(\tfrac{1}{2}tr^k)| \geqslant \pi^{-1}|t|\, r^k,$$

$$\frac{(\sigma r^k)^2 e^{-\sigma r^k}}{(1-e^{-\sigma r^k})^2} \geqslant \operatorname*{Min}_{e^{-1}\leq x\leq 1} \left(\frac{\log x^{-1}}{1-x}\right)^2 \cdot x > e^{-1}.$$

So if (3.5) holds, the right-hand-side of (3.4) is less than

$$\left\{1+4\left(\frac{tr^k}{\pi}\right)^2 \cdot (\sigma r^k)^{-2} e^{-1}\right\}^{-\frac{1}{2}(K+1)} \leqslant \left\{1+\left(\frac{t}{4\sigma}\right)^2\right\}^{-\frac{1}{2}(K+1)}.$$

For K we may take

$$K = [\operatorname{Min}(\log\sigma^{-1}, \log|t|^{-1})/\log r]$$

so that for $0 < \sigma \leqslant 1, -1 \leqslant t \leqslant 1$ we obtain

$$\left|\frac{F(\sigma+it)}{F(\sigma)}\right| \leqslant \left\{1+\left(\frac{t}{4\sigma}\right)^2\right\}^{\frac{\log \operatorname{Max}(\sigma, |t|)}{2\log r}} \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

We can now prove (3.2) and (3.3). First suppose $0 < \sigma \leq |t| \leq e^{-1}$. Putting $|t|\,\sigma^{-1} = y \geq 1$ we have, as $1+2^{-4}y^2 > (ey)^{c_3}$,

$$\log|F(\sigma+it)/F(\sigma)| \leqslant \frac{\log|t|}{2\log r} \cdot \log(1+2^{-4}y^2) \leqslant$$

$$\leqslant c_4 \log|t| \cdot \log ey = c_4(\log\sigma + \log y)(1+\log y).$$

The latter expression is a concave function of $\log y$, whose maximum in the interval $1 \leq y \leq e^{-1}\sigma^{-1}$ is attained either at $y = 1$ or at $y = e^{-1}\sigma^{-1}$. Both points give the value $c_4 \log \sigma$. This proves (3.2).

Second suppose $0 \leq |t| \leq \sigma \leq 1$, $\sigma > 0$. Since

$$1+2^{-4}y^2 \geqslant e^{c_5 y^2} \qquad (0 \leqslant y \leqslant 1)$$

we infer from (3.6)

$$|F(\sigma+it)/F(\sigma)| \leqslant e^{c_6 y^2 \log \sigma}$$

which proves (3.3).

4. *The asymptotic behaviour of $P_1(u)$.*

According to (2.15), a first approximation to $F(s)$ for s in the neighbourhood of the origin is $\exp\{(\log s)^2/2\log r\}$. If we had to evaluate the integral (cf. (1.16))

$$\int\limits_{a-i\infty}^{a+i\infty} e^{(\log s)^2/2\log r}\, e^{us/r}\, ds \qquad (a>0)$$

for large positive values of u we would choose an integration path passing through the saddle-point near the origin, i.e. the point $s=\sigma$, where σ satisfies

$$\frac{\log \sigma}{\sigma \log r} + u r^{-1} = 0, \qquad \sigma > 0. \quad \ldots\ldots \quad (4.1)$$

For $u>0$ the number σ is uniquely determined by (4.1). *Henceforth σ denotes this special function of u.* We now take $a=\sigma$ in (1.16) also:

$$P_1(u) = \frac{r}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} F(s)\, e^{us/r} \frac{ds}{s^2}. \quad \ldots\ldots \quad (4.2)$$

With the abbreviations

$$s=\sigma+it, \qquad t=y\sigma$$

$$\varkappa(u,y) = F(s)/F(\sigma)\cdot e^{iuy\sigma/r}(1+iy)^{-2} \quad \ldots\ldots \quad (4.3)$$

$$K(u) = \int\limits_{-\infty}^{\infty} \varkappa(u,y)\, dy. \quad \ldots\ldots \quad (4.4)$$

we have

$$P_1(u) = \frac{r}{2\pi\sigma} F(\sigma)\, e^{u\sigma/r} K(u) \quad \ldots\ldots \quad (4.5)$$

First we prove that for any number $\lambda>0$ we have

$$\left| K(u) - \int\limits_{-\lambda}^{\lambda} \varkappa(u,y)\, dy \right| \leqslant c_7\, \sigma^{c_8(\lambda)} \qquad (u>c_9) \quad \ldots \quad (4.6)$$

For $u\to\infty$ we have $\sigma\to 0$, and so c_9 can be chosen such that $u>c_9$ implies $\sigma<e^{-1}$. Now by (3.2), (3.3) and (4.3) we have for $u>c_9$

$$|\varkappa(u,y)| \leqslant \sigma^{c_{10}(\lambda)}(1+y^2)^{-1} \qquad (\lambda \leqslant |y| \leqslant e^{-1}\sigma^{-1})$$

with $c_{10}(\lambda) = \text{Min}\,(c_1, c_2\,\lambda^2)$ if $\lambda < 1$, $c_{10}(\lambda) = c_1$ if $\lambda \geq 1$. It follows, that, if $\lambda < e^{-1}\,\sigma^{-1}$,

$$\left| \int_{\lambda}^{e^{-1}\sigma^{-1}} \varkappa(u, y)\, dy \right| \leqslant \tfrac{1}{2}\,\pi\,\sigma^{c_{10}(\lambda)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

Furthermore we have, by (3.1), $|\varkappa(u, y)| \leq (1 + y^2)^{-1}$ for any u and y, and it follows that

$$\left| \int_{e^{-1}\sigma^{-1}}^{\infty} \varkappa(u, y)\, dy \right| \leqslant \int_{e^{-1}\sigma^{-1}}^{\infty} \frac{dy}{1+y^2} \leqslant e\,\sigma. \quad . \quad . \quad . \quad . \quad (4.8)$$

and also

$$\left| \int_{\lambda}^{\infty} \varkappa(u, y)\, dy \right| \leqslant e\,\sigma \text{ if } \lambda \geqslant e^{-1}\,\sigma^{-1}. \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

From (4.7), (4.8), (4.9) and the analogous inequalities for $y < 0$, (4.6) follows immediately ($c_7 = 2e$, $c_8(\lambda) = \text{Min}\,\{1, c_{10}(\lambda)\}$).

For a closer investigation of $\varkappa(u, y)$ for small values of $|y|$ we use (2.15). Introducing the abbreviations ($s = \sigma + i\,\sigma\,y$)

$$\sum_{-\infty}^{\infty} \alpha_k\, s^{-2\pi i k/\log r} = g(s), \qquad \sum_{1}^{\infty} \beta_n\, s^n = \omega(s), \quad . \quad . \quad . \quad (4.10)$$

$$\left.\begin{aligned} \varrho(u, y) = \exp\,[\,\{\log^2(1 + iy) + 2\log\sigma \cdot \log(1 + iy)\}/2\log r - \\ - \tfrac{5}{2}\log(1 + iy) + i\,u\,y\,\sigma/r + g(s)] \end{aligned}\right\} (4.11)$$

we have

$$\varkappa(u, y) = \varrho(u, y)\, e^{-g(\sigma) + \omega(s) - \omega(\sigma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.12)$$

If $\sigma < e^{-1}$, $|y| \leq 1$ we have

$$|\omega(s)| < c_{11}\,\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.13)$$

and by (2.16) and (4.10)

$$|g(\sigma + i\sigma y)| < c_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.14)$$

In virtue of (3.1) and (4.3) we have

$$|\varkappa(u, y)| \leqslant |\varkappa(u, 0)| = 1 \qquad (-\infty < y < \infty) \quad . \quad . \quad . \quad (4.15)$$

and it follows from (4.12), (4.13) and (4.14) that

$$\left| \int_{-\lambda}^{\lambda} \varkappa(u, y)\, dy - e^{-g(\sigma)} \int_{-\lambda}^{\lambda} \varrho(u, y)\, dy \right| < c_{13}\,\sigma \text{ for } 0 < \lambda < 1,\ u > c_9. \quad (4.16)$$

We transform $\int_{-\lambda}^{\lambda} \varrho(u, y)\, dy$ by introducing a new variable z by

$$\tfrac{1}{2} z^2 = \log(1 + iy) - iy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.17)$$

$$y = z + \mu_2 z^2 + \mu_3 z^3 + \ldots \qquad (|z| < c_{14}) \quad . \quad . \quad . \quad (4.18)$$

On expressing u in terms of σ by (4.1) we obtain from (4.11)

$$\varrho(u, y) = \exp\left\{\frac{\log^2(1 + iy) + z^2 \log\sigma}{2 \log r} - \tfrac{5}{2}\log(1 + iy) + g(\sigma + i\sigma y)\right\} (4.19)$$

If we put, for a moment,

$$\log\sigma / \log r = v$$

the function $e^{g(s)}$ becomes an analytical function of the variables v and y in the range $|Im\, v| < \frac{1}{2}\pi/\log r$, $|y| < \cos(Im\, v \log r)$, since $g(s)$ is analytical for $Re\, s > 0$ (cf. (2.16)). Moreover it is a periodic function of v with period 1. It follows that $e^{g(s)}$ can be written in the form

$$e^{g(\sigma + i\sigma y)} = \sum_{n=0}^{\infty} \chi_n(v)\, y^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.20)$$

where the functions $\chi_n(v)$ are analytical in the strip $|Im\, v| < \frac{1}{2}\pi/\log r$, periodical mod 1, and satisfy

$$|\chi_n(v)| < c_{15}^{n+1} \qquad (-\infty < v < \infty,\ n = 0, 1, 2, \ldots).$$

If v is real (4.20) converges for $|y| < 1$.

We now easily deduce from (4.19) and (4.20) that $\varrho(u, y)\dfrac{dy}{dz}$ can be written in the form (σ real)

$$\varrho(u, y)\frac{dy}{dz} = e^{z^2 \log\sigma / 2\log r} \sum_{n=0}^{\infty} \psi_n\left(\frac{\log\sigma}{\log r}\right) z^n \qquad (|z| < c_{16}). \quad . \quad (4.21)$$

Again, $\psi_n(v)$ is analytical for $|Im\, v| < \frac{1}{2}\pi/\log r$ and

$$|\psi_n(v)| < c_{17}^{n+1} \qquad (-\infty < v < \infty,\ n = 0, 1, 2, \ldots).$$

It is easily verified that

$$\psi_0(\log\sigma/\log r) = e^{g(\sigma)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.22)$$

Now take $c_{18} < 1$ such that $-c_{18} \leq y \leq c_{18}$ implies $|z| < \frac{1}{2}c_{14}$, $c_{17}|z| < \frac{1}{2}$ and either $|\arg z| < \frac{1}{8}\pi$ or $|\arg -z| < \frac{1}{8}\pi$. Let ζ_1 and ζ_2 denote the values of z for $y = -c_{18}$ and $y = +c_{18}$, respectively, and put $Re\, \zeta_1^2 = Re\, \zeta_2^2 = c_{19}$. Now the integral

$$\int_{-c_{18}}^{c_{18}} \varrho(u, y)\, dy$$

can be expanded in a familiar way by means of (4.21): for any positive integer N we have $(u > c_9,\ \sigma < e^{-1})$

$$\left.\begin{aligned}&\left|\int_{-c_{18}}^{c_{18}} \varrho(u, y)\, dy - \sum_{n=0}^{2N+1} \psi_n\left(\frac{\log \sigma}{\log r}\right) \int_{-\infty}^{\infty} e^{-z^2 \frac{\log \sigma^{-1}}{2\log r}} z^n\, dz\right| < \\ &< c_{20}(N)\, e^{-c_{19}\log \sigma^{-1}/2\log r} + \sum_{2N+2}^{\infty} c_{17}^{n+1} \left|\int_{\zeta_1}^{\zeta_2} e^{-z^2 \frac{\log \sigma^{-1}}{2\log r}} z^n\, dz\right|.\end{aligned}\right\} \quad (4.23)$$

On taking for our integration path the broken line consisting of the segments $(\zeta_1, 0)$ and $(0, \zeta_2)$, we find because of

$$|c_{17}\,\zeta_{1,2}| < \tfrac{1}{2},\ |\arg -\zeta_1| < \tfrac{1}{8}\pi,\ |\arg \zeta_2| < \tfrac{1}{8}\pi$$

(put $|z| = t$ on both parts of the path) if $n \geq 2N + 2$:

$$\left|\int_{\zeta_1}^{\zeta_2} e^{-z^2\log \sigma^{-1}/2\log r}\, z^n\, dz\right| <$$

$$< \int_{-|\zeta_1|}^{|\zeta_2|} (2\,c_{17})^{-n+2N+2}\, e^{-t^2 \log \sigma^{-1}/4\log r}\, |t|^{2N+2}\, dt < \int_{-\infty}^{\infty}.$$

It now follows from (4.23) that $(u > c_9,\ \sigma < e^{-1})$

$$\left.\begin{aligned}&\left|\int_{-c_{18}}^{c_{18}} \varrho(u, y)\, dy - \right. \\ &\left. - \left(\frac{2\pi \log r}{\log \sigma^{-1}}\right)^{\frac{1}{2}} \sum_{m=0}^{N} \left(\frac{\log r}{\log \sigma^{-1}}\right)^m \frac{(2m)!}{2^m m!} \psi_{2m}\left(\frac{\log \sigma}{\log r}\right)\right| < \frac{c_{21}(N)}{(\log \sigma^{-1})^{N+\frac{3}{2}}}.\end{aligned}\right\} \quad (4.24)$$

On taking $\lambda = c_{18}$ in (4.6) and (4.16) we find (cf. (4.5), (4.1), (2.15), (4.13) and (4.14))

$$\left.\begin{aligned}P_1(u) = r \sqrt{\frac{\log r}{2\pi}}\, e^{\frac{(\log \sigma)^2}{2\log r} - \left(\frac{3}{2} + \frac{1}{\log r}\right)\log \sigma - \frac{1}{2}\log\log \sigma^{-1}} \times \\ \times \left\{\sum_{m=0}^{N} \frac{\varphi_m\left(\frac{\log \sigma}{\log r}\right)}{(\log \sigma^{-1})^m} + O\left(\frac{1}{(\log \sigma^{-1})^{N+1}}\right)\right\},\end{aligned}\right\} \quad . \quad . \quad (4.25)$$

where the functions

$$\varphi_m(v) = (\log r)^m \frac{(2\,m)!}{2^m\, m!}\, \psi_{2m}(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.26)$$

are periodic functions of v with period 1, analytical in $|Im\, v| < \frac{1}{2}\pi/\log r$. Especially

$$\varphi_0\left(\frac{\log \sigma}{\log r}\right) = e^{g(\sigma)} = \exp\left\{\sum_{-\infty}^{\infty} a_k\, e^{-2\pi i k \log \sigma/\log r}\right\}. \quad . \quad . \quad (4.27)$$

Formula (4.25) is our final result for the asymptotic behaviour of $P_1(u)$; σ is related to u by (4.1) and cannot be expressed explicitly in terms of u in a simple way.

5. *Final results concerning* $P(u)$.

The difference $P(u)-P_1(u)$ is relatively small. We have, by (1.15)

$$P(u-1) \leqslant P_1(u) \leqslant P(u), \quad \ldots \ldots \quad (5.1)$$

and on the other hand, by (1.9),

$$P(u)-P(u-1)=P(u/r) \leqslant P_1(ur^{-1}+1). \quad \ldots \quad (5.2)$$

It follows that

$$0 \leqslant P(u)-P_1(u) \leqslant P_1(ur^{-1}+1) \quad \ldots \ldots \quad (5.3)$$

In order to show that $P_1(ur^{-1}+1)/P_1(u)$ is small we first give a first-order asymptotic expression for $P_1(u)$ explicitly in terms of u. It is readily derived from (4.1),

$$\sigma^{-1}=ur^{-1}\log r \cdot \log \sigma^{-1},$$

that, if $u \to \infty$,

$$\log \sigma^{-1}=\log u-\log\log u+\log\log r-\log r+O(\log\log u/\log u),$$

$$\log\log \sigma^{-1}=\log\log u+O(\log\log u/\log u),$$

$$\log^2 \sigma=(\log u-\log\log u)^2-2\log(r/\log r)\cdot(\log u-\log\log u)+$$
$$+2\log\log u+\log^2(r/\log r)+2\log(r/\log r)+O\{(\log\log u)^2/\log u\},$$

and we obtain from (4.25)

$$\left.\begin{aligned}\log P_1(u)=&\frac{(\log u-\log\log u+\log\log r)^2}{2\log r}+\\&+\left\{\frac{1}{2}+\frac{1}{\log r}\right\}\log u-\log\log u+\log\log r-\tfrac{1}{2}\log 2\pi+\\&+\sum_{-\infty}^{\infty}a_k \exp\left\{2\pi i k\left(\frac{\log u-\log\log u+\log\log r}{\log r}\right)\right\}+\\&+O\{(\log\log u)^2/\log u\}.\end{aligned}\right\} \quad (5.4)$$

The a_n are defined by (2.11) and (2.13).

From (5.4) it is easily deduced that

$$P_1(ur^{-1}+1)/P_1(u)=O\{\exp(-\log u+\log\log u)\}=O(u^{-\frac{1}{2}}).$$

Now (5.3) shows that (4.25) and (5.4) remain valid if $P_1(u)$ is replaced by $P(u)$.

If r and rh are integers we have $P(h)=p(rh)$; thus (5.4) proves (1.4). The more precise expansion (4.25) however cannot easily be expressed explicitly in terms of u (or rh).

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Mathematisch Instituut der Technische Hogeschool, Delft.