Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **2** (2019) 65–94

Enumerations on bargraphs

Toufik Mansour[1,*], Armend Sh. Shabani[2]

[1]*Department of Mathematics, University of Haifa, 3498838 Haifa, Israel*
[2]*Department of Mathematics, University of Prishtina, 10000 Prishtinë, Republic of Kosovo*

(Received: 14 October 2019. Received in revised form: 17 December 2019. Accepted: 17 December 2019. Published online: 20 December 2019.)

Abstract

In this review, we present the main results related to bargraphs from the enumerative point of view. We consider several geometrically motivated statistics in combinatorial families: compositions, words, set partitions, permutations and integer partitions, when such families are presented as bargraphs. The review contains presentation of main results of each considered paper. The main discussion is preceded by short historical notes. Methods used throughout the review are discussed briefly. The review includes an up-to-date bibliography.

Keywords: bargraphs; compositions; words; set partitions; permutations; integer partitions.

2010 Mathematics Subject Classification: 05A05, 05A15, 05A16, 05A17, 05A18.

1. Introduction

The interest in bargraphs comes mainly from two different directions: enumeration of polyominoes and enumeration of compositions of a natural number. The enumeration of polyominoes has been closely related to some works in physics, percolation theory, computational science, recreational mathematics, etc. In regard to the second direction, compositions (as well as other combinatorial families: words, set partitions, permutations, integer partitions, etc.), when presented as bargraphs, create the possibility to consider them as geometric objects and to look for their geometric properties, such as area, perimeter and so on.

In this review, we will focus in the second direction, i.e. enumeration of compositions (words, set partitions, permutations, etc) presented as bargraphs, according to geometrically motivated statistics.

1.1 Polyomiones

By a *cell*, we will mean the interior and the boundary of a unit square, whose vertices belong to integer lattice. A *polyomino* is a finite connected collection of cells, such that its interior is also connected. Depending on the number of cells, polyominoes are classified as *monomino* - 1 cell, *domino* - 2 cells, *tromino* - 3 cells, *tetromino* - 4 cells, *pentomino* - 5 cells, etc. In Figure 1, we present all of the polyominoes that have three cells, i.e. all trominoes.

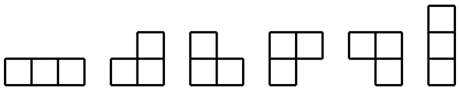

Figure 1: Trominoes.

A *column* (resp. *row*) of a polyomino is the intersection between the polyomino and any infinite vertical (resp. horizontal) strip of unit squares.

The word *polyomino* was used for the first time by Golomb in 1953 [60], although therein, it is mentioned that pentominoes date back from antiquity. The first polyomino (pentomino) problem was posed in the book *Canterbury Puzzles* [47]. Gardner [57] contributed to the popularization of the term 'polyomino' as well as to the problems related to polyominoes. For a brief historical review on polyominoes, we refer the interested reader to [62].

The enumeration of *polyominoes* did not cope with its popularization and it was not easy at all as one may have the impression in the beginning. To give some account on the difficulty of the problem, let $\mathcal{P}(n)$ denote the number of polyominoes with n cells. From Figure 1 we conclude that $\mathcal{P}(3) = 6$. It is easy to see that $\mathcal{P}(1) = 1$ and $\mathcal{P}(2) = 2$. The sequence

*Corresponding author (tmansour@univ.haifa.ac.il)

$\mathcal{P}(n)$ looks like: $1, 2, 6, 19, 63, 216, 760, 2725, 9910, 36446, 135268, 505861, \ldots$ and it stops after few more terms, and to be more precise the latest known term is $\mathcal{P}(56) = 69150714562532896936574425480218$ and it is due to Jensen [65]. The sequence $\mathcal{P}(n)$ corresponds to the sequence A001168 in OEIS (see [96]).

Despite a lot of effort this "*simply stated*" and "*elementary looking*" enumeration problem, in general case, remains unsolved. However there are algorithms to generate polyominoes inductively, and this can be easily seen from the simple fact that each polyomino of order $n+1$ can be obtained by adding a square to a polyomino of order n. The earliest method used to enumerate polyominoes is from [60]. As described there, to each polyomino in the list of polyominoes of order n, one adds squares in all possible positions in such a way that the resulting polyomino of order $n+1$ is not a duplicate of one already found. A more advanced method is due to Redelmeier [93]. His method besides the enumeration of polyominoes with $n+1$ cells also provides an upper bound on their number and it is worth mentioning that when looking for polyominoes of order $n+1$ it does not require to store the polyominoes of order n.

The most modern algorithms for enumerating fixed polyominoes are due to Conway [36] and Jensen [66], with the second one being an improvement of the first one. The common of those algorithms is that they both are exponential in n and both use the Transfer-matrix method. They count the number of polyominoes with a given certain width. Then, clearly the number for all widths is the total number of polyominoes.

Moving into asymptotic aspects of polyomino enumeration we refer to [68]. It is conjectured that

$$\mathcal{P}(n) \sim \frac{c\lambda^n}{n}$$

where $\lambda = 4.0626$ and $c = 0.3169$. However, the results of this conjectured estimate are not 'close' to the best known results. It is proved the existence of the following limit

$$\lim_{n\to\infty} \mathcal{P}(n)^{\frac{1}{n}} = \lambda.$$

The best-known lower bound for λ is 4.00253 and is due to [11]. The best known upper bound is $\lambda < 4.65$ and is due to [70].

The enumeration of polyominoes has motivated some works in physics [55, 62, 90, 92], in percolation theory [61, 97], computational science [67], recreational mathematics [87], etc.

Since the enumeration of polyominoes in general case appeared to be out of reach at present, the research was focused in studying some large subclasses of polyominoes. These subclasses have been enumerated with respect to various properties of the polyominoes. The most often considered properties that a polyomino has, include its area and its perimeter. Naturally, the area of a polyomino is the number of cells and the perimeter is the length of the border. For example, given a family of polyominoes under consideration, one might ask for the number of such polyominoes with a fixed area, or the number of polyominoes with a fixed perimeter.

In the next few lines, we present a family of polyominoes, known as *convex polyominoes* [30, 99]. A polyomino is said to be *horizontally convex* (*vertically convex*) if each of its rows (columns) is a single contiguous block of cells. In Figure 2 (leftmost and middle) it is shown an example of a horizontally convex and a vertically convex polyomino. A polyomino is said

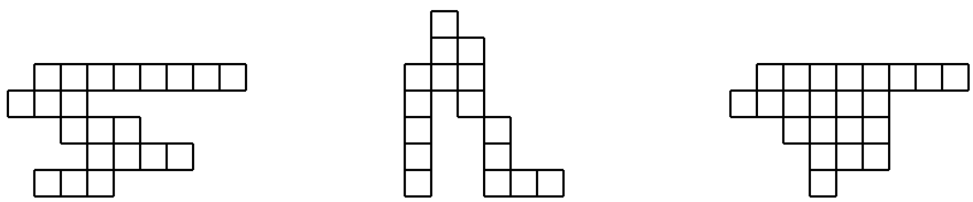

Figure 2: A horizontally convex (left), vertically convex polyomino (middle), and a convex polyomino (right).

to be *convex* if it is both vertically and horizontally convex. In Figure 2 it is shown an example of a convex polyomino. We remark that horizontally (vertically) convex polyominoes sometimes are refereed to as row (column) convex polyominoes.

A surprising simple result, proved by Delest and Viennot [41] shows that the number of convex polyominoes of perimeter $2n+8$ equals $(2n+11)4^n - 4(2n+1)\binom{2n}{n}$.

In this review, we will consider one of the very well known family of nonconvex polyominoes - *bargraphs*. We will begin with a few equivalent definitions of bargraphs (see Figure 3).

Definition 1.1. *A bargraph is a vertically convex polyomino, whose lower edge lies on the horizontal axis.*

An alternative definition of the bargraph is the following one:

Definition 1.2. *A* bargraph *(or a skyline polyomino) is a lattice path in $\mathbb{N}_0^2$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, that starts with $(0, 0)$ and ends upon their first return to the x-axis (sometimes, we require that after the first return to the x-axis the progression* *ends at* $(0,0)$*). Each step is an up step* $(0,1)$*, a down step* $(0,-1)$*, a right horizontal step* $(1,0)$*, or a left horizontal step* $(-1,0)$*. The first step has to be an up step, the right horizontal steps must all lie above the* x*-axis, and the left horizontal steps must all lie on the* x*-axis. An up step cannot directly follow a down step and vice versa. Clearly, the number of down steps must equal the number of up steps.*

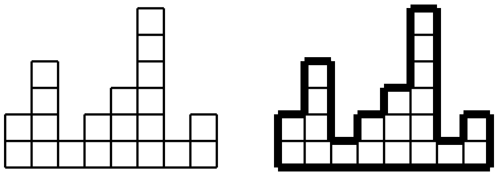

Figure 3: A bargraph (left) and its lattice path (right).

In Figure 3 it is presented an example of a bargraph as a lattice path. A *semi-perimeter* of a bargraph π is half of the perimeter of π, while the *perimeter* of π is the number of steps in the border of π. For instance, the semi-perimeter and the perimeter of the bargraph in Figure 3 are 18 and 36 (including the steps on the x-axis), respectively.

We remark that a bargraph can be described as a sequence (called composition, see below) of columns $c = c_1c_2\dots c_m$ such that the jth column (from the left) contains c_j *cells* or *unit squares*, where m denotes the number of horizontal steps of the bargraph. That is, a bargraph with m columns (horizontal steps) can be described as a composition with m parts. For example, the sequence 24123612 corresponds to the bargraph in Figure 3.

1.2 Compositions

A *composition* of a positive integer n is a sequence $\sigma = \sigma_1\sigma_2\dots\sigma_m$ of positive integers such that $\sigma_1 + \cdots + \sigma_m = n$. The $\sigma_1,\dots,\sigma_m$ are called the *parts*, and m denotes the number of parts of σ. We refer to n as the *order* of the composition. For example, the compositions of 3 are 111, 12, 21 and 3.

In the following few lines, we provide very brief information on compositions and refer the reader for more details to a recent book [63]. The study of compositions can be traced back to Percy Alexander MacMahon, who in 1893, published his work, entitled *Memoir on the Theory of Compositions of a Number* [72], which later on would become influential. Therein compositions are described as partitions in which the order of occurrence of the parts is essential. Several results related to compositions were derived. We mention here, for example, the total number of compositions, the number of compositions with a given number of parts, presentation of a composition as a graph and so on. He was also able to provide answers to some of the questions he tackled, by using simple combinatorial arguments. After MacMahon's work [72] there was a long break in research in enumerative combinatorics, with sporadic works, mainly on partitions and permutations, partially influenced by [72]. The break was even longer in relation to compositions since till the late 1960s, there were only individual studies on various aspects of them, but this constituted neither focused nor systematic research.

The situation changed positively in the 1970s when several groups of authors developed new research directions. The study of compositions and words, besides enumerating the total number of those objects, also involved certain of their characteristics (statistics). Several generalizations of previous results and the introduction of new concepts have appeared in the last decade. In particular, the research on pattern avoidance in words and compositions has followed earlier very active research on pattern avoidance in permutations.

In this review, we consider all the known studies on the enumeration of bargraphs according to some statistics. Due to the geometric nature of bargraphs, we will mainly focus on those statistics having a geometric flavour.

In the following example, we show some interactions of the same objects: compositions - bargraphs, by giving situations where statistics on bargraphs are easy to be expressed compared to analogue statistics on compositions and the other way around.

Example 1.1. *See below the statistics "perimeter", "site-perimeter", "water cells", "number of pushes", etc.*

Figure 4 presents the all bargraphs with one, two and three cells. Clearly, to the these bargraphs correspond the following compositions $1, 11, 2, 111, 21, 12, 3$.

It is very common to encode the steps in a lattice path with letters as following: "up step" – "u", "down step"–"d", "horizontal step"–"h". Using this notation the bargraphs in Figure 4 will be encoded as: uhd, $uhhd$, $uuhdd$, $uhhhd$, $uuhdhd$, $uhuhdd$ and $uuuhddd$, respectively.

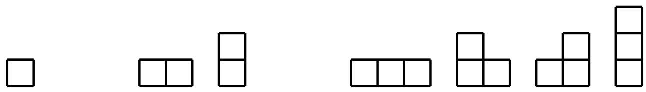

Figure 4: All the bargraphs with one, two or three cells.

1.3 The structure of the paper

The structure of the paper is as follows. We begin the next section by introducing the notations for the further sections, listing all the statistics that we will cover through the review and providing some preliminary results. At the end of the section, we will emphasize, without many details, the methods and techniques that will be used throughout the review. From Section 3 and afterward, in the beginning, we will introduce some definitions and notations.

We consider Bargraphs in Section 3. We begin this section with the table presenting the time line of research for bargraphs. We continue the work by presenting the main results for most important statistics on bargraphs, in most of the cases each of them occupying a subsection. We will cover the following statistics: number of horizontal steps, number of up steps, perimeter, site-perimeter, lattice paths (with focus on Dyck paths and Motzkin paths), inner-site perimeter, height, width, walls, descents, up steps, levels, peaks, water cells, protected cells, corners, interior vertices and edges, depth, durfee squares and counting bargraphs (bargraphs in bargraphs). Other statistics appearing throughout the paper, although not as individual sections include the number of cells in the rightmost column, horizontal half-perimeter, vertical half-perimeter, the position of the first level, height of the first level, position of the first peak, the distance between leftmost and rightmost peaks, the height of the first column, double rises, peaks/valleys of width ℓ, corners of type dh and uh, horizontal segments, horizontal segments of length ℓ, length of the first descent, number of columns of height h, least column height, the width of the leftmost horizontal segment, occurrences of UHU, length of the initial staircase, area, initial u steps, final d steps, sum of heights of valleys, grounded squares, bargraph c of one column, bargraph $11\ldots1$ of one row, bargraph $c1$, bargraph cd, border cells, and tangent cells.

We consider Words in Section 4. After the table presenting the time line of research for words, we present the main results for the most important statistics on words. We will cover the following statistics: perimeter, site-perimeter, water cells and shedding light.

In Section 5 we consider Set partitions. As in other previous sections, we begin with the time line of research for set partitions. Then we present the main results for the most important statistics on set partitions. We will cover the following statistics: area and up steps, perimeter, site-perimeter, interior vertices, water cells, corners, 1×2 rectangles and 2×2 squares.

Statistics on Permutations presented as bargraphs are covered in Section 6. After the time line of research for permutations, we present the following statistics: descents and up steps, site-perimeter, water cells and pushes.

In Section 7, we cover statistics: perimeter and corners in integer partitions presented as bargraphs.

We end the review with Section 8, in which we present several extensions and generalizations for bargraphs. More precisely, we consider x-bargraphs, cylindrical bargraphs, superdiagonal bargraphs, symmetric bargraphs, weakly alternating bargraphs and t-bargraphs, bargraphs and a topological index.

We end the paper by presenting an updated bibliography of research covered in the review and related either directly or indirectly to bargraphs.

2. Notations, preliminaries, statistics and methods

Let $\mathbb{N}$ denote, as usual, the set of all nonnegative integers. For $n\in\mathbb{N}$ denote by $[n]$ the set of integers $\{1,2,\ldots,n\}$. We denote the set of all bargraphs (including the empty bargraph) by $\mathcal{B}$, the set of all bargraphs with n cells by $\mathcal{B}_n$ and the set of all bargraphs with n cells and m columns by $\mathcal{B}_{n,m}$. Clearly, $\mathcal{B}=\cup_{n\geq0}\mathcal{B}_n$ and $\mathcal{B}_n=\cup_{m=0}^{n}\mathcal{B}_{n,m}$.

A *combinatorial class* $\mathcal{F}$ is a set such that each element in it has a size in $\mathbb{N}$. In this context the size of element $f\in\mathcal{F}$ is denoted $size(f)$. A statistic st is a function from $\mathcal{F}$ to $\mathbb{N}$. For given statistics $st_1,st_2,\ldots,st_d$ on $\mathcal{F}$, we define the generating function $B_{st_1,st_2,\ldots,st_d}(t,x_1,x_2,\ldots,x_d)$ where t marks the size and x_i marks st_i for all $i=1,2,\ldots,d$. That is,

$$F_{st_1,st_2,\ldots,st_d}(t,x_1,x_2,\ldots,x_d)=\sum_{\pi\in\mathcal{F}}t^{size(\pi)}\prod_{i=1}^{d}x_i^{st_i(\pi)}.$$

We are not only interested in finding an explicit formula for this generating function, but would also like to find the expected value of a given fixed statistic. The expected value of the statistic st_1 (for example) on all members of $\mathcal{F}$ having size n is given by

$$\frac{[t^n]\frac{\partial}{\partial q}F_{st_1,st_2,\ldots,st_d}(t,q,1,1,\ldots,)\mid_{q=1}}{[t^n]F_{st_1,st_2,\ldots,st_d}(t,1,1,1,\ldots,)}.$$

We will use this observation throughout the paper, where $\mathcal{F}$ can be either set of bargraphs, set partitions, k-ary words, permutations, or set of integer partitions (see below).

The aim of this paper is to review all the known results up to today in the area of statistics on bargraphs. Below we list most of the statistics (in alphabet order) that this paper covers. For $B \in \mathcal{B}$, we define the following statistics:

- $\#L$ denotes the number of occurrences of the word L. For example, $\#u$ is the number of up steps, $\#h$ is the number of horizontal steps, etc.
- *Border cell* of B is a cell inside of B that has at least one edge in common with an outside cell of B.
- *Corner* of B is a point of intersection of an u step or a d step with a h step.
- *Corner of type* dh in B is a point proceeded by d step and followed by h step.
- *Corner of type* uh in B is a point proceeded by u step and followed by h step.
- *Depth* of a cell c in B, denoted by $dep(c)$, is the minimum number of horizontal or up steps to exit B starting from c. The depth of B is defined as $dep(B) = \max_c dep(c)$, where the maximum is over all cells c of B.
- *Descent* in B is two consecutive columns such that the size of the left column is greater than the size of the right column.
- *Edge visiting* is a horizontal step of the bargraph which lies on the x-axis.
- *Height of a peak/valley* of B is the y-coordinate of its h steps.
- *Height* of B is the maximum of y-coordinate that it reaches, that is, the size of the highest column in it.
- *Height of a column* in B is the size of the column.
- *Horizontal half-perimeter* of B is the half of $\#h$ in the perimeter of B.
- *Horizontal segment* in B is an occurrence h^j in B with j maximal, where $j \geq 1$.
- *Horizontal segment of length* ℓ in B is an occurrence h^ℓ in B with ℓ maximal.
- *Inner site-perimeter* of B is the number of cells inside B that have at least one edge in common with an outside cell.
- *Interior vertex* of B is a vertex that is adjacent to exactly four different cells of B.
- *Interior edge* of B is an edge that is formed from interior vertices.
- *Length of first descent* in B is the number of first consecutive d steps.
- *Length of the longest initial staircase* in B is the length of the longest initial sequence of the form $uhuh\cdots$ in B.
- *Level* in B is a pair of consecutive columns with the same size.
- *Number of cells in the rightmost column* in B is the size of the rightmost column of B.
- *Number of double rises* in B is the number of triples of consecutive columns such that the size of the columns is increasing.
- *Number of double falls* in B is the number of triples of consecutive columns such that the size of the columns is decreasing.
- *Peak* in B is an occurrence of uh^jd for some $j \geq 1$.
- *Peak of width* ℓ in B is an occurrence of $uh^\ell d$.
- *Perimeter* of B is the length of its border.
- *Rise* in B is formed by two consecutive columns such that the size of the left column is smaller than the size of the right column.
- *Shedding light cell* is a cell whose edge facing north or the edge facing west or both is hit by the ray of light under the assumption that a light source at infinity in the North-West direction sheds parallel light rays onto the bargraph. It is also called *a lit cell*.
- *Site-perimeter* of B is the number of nearest-neighboring cells outside the boundary of B.
- *Tangent cell* of B is a cell inside of B which is not a border cell of B and that has at least one vertex in common with an outside cell of B.
- *Valley* of B is an occurrence of dh^ju for some $j \geq 1$.
- *Valley of width* ℓ of B is an occurrence of $dh^\ell u$.
- *Vertical half-perimeter* of B is the half of $\#u$ in the perimeter of B.
- *Vertex visiting* is a point on the bargraph which lies on the x-axis.
- *Water cells* or *Water capacity* of B is the number of cells that keep the water after it falls through bargraph.

Before we start to discuss specific counting problem on bargraphs, we present several general methods and techniques that were used in deriving the results.

Wasp-waist decomposition: Bousquet-Mélou and Rechnitzer [30] and Prellberg and Brak [92] decompose the graphs into two smaller bargraphs at a special column. They called it the Wasp-waist decomposition, and it depends on the leftmost column of height 1. More precisely, any nonempty bargraph π is decomposed as either $\pi = 1$, $\pi = 1\pi''$, $\pi = \pi'$, $\pi = \pi'1$ or $\pi = \pi'1\pi''$, where the size of each column in π' is at least two and π', π'' are nonempty bargraphs.

Scanning-element algorithm or **Column by column**: The Scanning-element algorithm has been suggested by Firro and Mansour [53] to study the number of permutations of length n that satisfy a certain set of conditions. Then it is used by Heubach and Mansour in [63] to study the number of compositions of n according to a fixed statistic. The main idea of this algorithm is to scan each composition/bargraph letter by letter (column by column) either from left to right or from right to left. In each step, we try to see if our set with the letters that have been scanned is in bijection with another set with less scanned letters.

Adding a slice: Sometimes, instead of removing, it is more convenient to add a column to the structure, for instance, see Example III.22 in [54]. This method is introduced while studying some combinatorial problems which were suggested in statistical mechanics, see [98].

Maximal column: The main idea in this method is to consider the column of maximal size, which is to consider the set of all bargraphs of $\mathcal{B}$ such that each column has size at most d. Denote such set by $\mathcal{B}^{(d)}$. Then each bargraph $B \in \mathcal{B}^{(d)}$ can be written as $B^{(0)}dB^{(1)}d\cdots B^{(s-1)}dB^{(s)}$, for some $s \geq 0$, where $B^{(j)} \in \mathcal{B}^{(d-1)}$ for all $j = 0, 1, \ldots, s$. This factorization gives a recursive relation between the number of bargraphs of $\mathcal{B}^{(d)}$ and those bargraphs of $\mathcal{B}^{(d-1)}$. This method is used a lot in counting occurrences of "subword patterns", for example, see [63].

Note that instead of looking at the column with maximal size, we can consider the column with minimal size, which is to write each bargraph in $\mathcal{B}$ as $B^{(0)}1B^{(1)}1\cdots B^{(s-1)}1B^{(s)}$, for some $s \geq 0$, where each column of $B^{(j)} \in \mathcal{B}$ has size at least 2, for all $j = 0, 1, \ldots, s$. Clearly, considering the column with the minimal size is equivalent to Wasp-waist decomposition.

3. Statistics on bargraphs

By the obvious bijection between the set of bargraphs with n cells and compositions of n, we have that the number of bargraphs with n cells is given by 2^{n-1} (see [63]). Such a bijection implies that a given generating function for the number of compositions of n according to some statistics st_j (see [63]), is the same generating function for the number of bargraphs according to the area (number of cells) and statistics st_j. Therefore, from now on, we will not be interested in counting bargraphs according to the area and statistics st_j, unless these statistics are not considered on compositions. To give an example, the problem of counting the number of bargraphs according to the area and perimeter is not discussed in [63], however, in this section, we will present a complete answer for it (see Theorem 3.3).

3.1 Counting bargraphs

Let $S_{m,n}$ be the number of bargraphs from $(0,0)$ to $(m,0)$ with m horizontal steps and n up steps. Clearly, the perimeter of a bargraph in $S_{m,n}$ is $2m+2n$ (including the steps on the x-axis). Define the following generating function $f(x,y) = \sum_{n,m\geq 0} S_{m,n}x^m y^n$. Geraschenko [58] showed that $f = f(x,y)$ satisfies

$$f = y\left(\frac{x}{1-x} + \frac{1}{(1-x)^2}\cdot\frac{f}{1-\frac{xf}{1-x}}\right).$$

Using the Lagrange inversion formula [99], we have the following result.

Theorem 3.1. [58] *For all $n, m \geq 0$, $S_{m,n} = \sum_{i=0}^{n} N(n-1,i)\binom{m+2i-1}{2n-2}$, where $N(n-1,i)$ are the Narayana numbers (see [101]).*

Moreover, in [58] the above result is interpreted by constructing a bijection between Dyck paths and bargraphs, see Subsection 3.4.

In this section, we describe the time line of the research on counting bargraphs according to some statistics (see Table 1).

3.2 Perimeter

Using Wasp-waist decomposition, we can state the following result.

Year	Statistic, Reference	Theorem
1982	Perimeter, [45]	3.2
1990	Perimeter, Area [91]	3.2, 3.3
1993	Perimeter, Area [90]	3.2, 3.3
1995	Perimeter [92]	3.2
1997	Area [31,37,42]	3.8
2002	Edge visiting, Vertex visiting [64]	
2003	Number of cells in the rightmost column, Horizontal half-perimeter, Vertical half-perimeter, Site-perimeter [30]	3.4
2015	Horizontal and up steps [58] Height [25] Width [25] Levels, Position of the first level, Height of the first level, [15]	3.1 3.11 3.13 3.18
2016	Peaks, Position of the first peak, Height of the first peak, Distance between leftmost and rightmost peaks [14] Staircases [27] Descents, Up-steps, Levels [16]	3.17 3.15, 3.16
2017	Walls [12] Height of the first column, Double rises, Peaks/Valleys of width ℓ, Corners of type dh and uh, Horizontal segments, Horizontal segment of length ℓ, Length of the first descent, Number of columns of height h, Least column hight, Width of the leftmost horizontal segment, Occurrences of UHU, Length of the initial staircase, Area [42]	3.14 3.8
2018	Corners [82] Water Cells [85], [17] Peaks, Valleys, Semi-perimeter, initial u steps, final d steps, Sum of heights of valleys [43] Durfee squares, Grounded squares [7] Bargraph c of one column, Bargraph $11\cdots1$ of one row, Bargraph $c1$, Bargraph cd [79]	3.23 3.19, 3.20 3.7 3.26, 3.27 3.29
2019	Protected Cells [86] Interior vertices and edges [78] Inner site-perimeter [19,76] Border cells, Tangent cells [76] Perimeter [34,44] Depth [23]	3.21, 3.22 3.24, 3.25 3.10 3.9 8.1 3.28

Table 1: Time line of research for bargraphs.

Theorem 3.2. [30,31,92] *The generating function that counts all nonempty bargraphs is given by*

$$B(x,y)=\frac{1-x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x}$$

where x counts the number of horizontal steps and y counts the number of vertical up steps.

Thus, the generating function for nonempty bargraphs in terms of the semi-perimeter often called *the isotropic generating function*, is given by $B(x,x)$. Feretić, [49] derived again the expression for $B(x,y)$. Moreover, in [49] it is found the generating function for column-convex polyominoes according to the perimeter on the honeycomb lattice (A plane with hexagonal tiles is called a *hexagonal lattice* and is also known as the *honeycomb lattice*).

The asymptotics of the coefficient of x^n in $B(x,x)$ has been considered, and it is computed the dominant singularity ρ which is the positive root of $1-4x+2x^2+x^4=0$. Thus, by singularity analysis (for example, see [54]) we have

$$[x^n]B(x,x)\sim\frac{1}{2}\sqrt{\frac{1-\rho-\rho^3}{\pi\rho n^3}}\rho^{-n}\text{ with }\rho=\frac{1}{3}\left(-1-\frac{2^{8/3}}{(13+3\sqrt{3})^{1/3}}+2^{1/3}(13+3\sqrt{33})^{1/3}\right)\approx 0.295598\cdots. \quad (1)$$

Blecher, Brennan and Knopfmacher [16] by using Scanning-element method refined Theorem 3.2, by finding the generating function for bargraphs according to the number of horizontal steps, number of up steps and area, where it is shown that

Theorem 3.3. [16] *We have*

$$B_{\#h,\#u,area}(x,y,q)=\frac{y\sum_{i\ge0}\frac{x^{i+1}(y-1)^iq^{\binom{i+2}{2}}}{\prod_{j=1}^{i}(1-q^j)\prod_{j=1}^{i+1}(1-yq^j)}}{1-\sum_{i\ge0}\frac{x^{i+1}(y-1)^iq^{\binom{i+2}{2}}}{\prod_{j=1}^{i+1}(1-q^j)\prod_{j=1}^{i}(1-yq^j)}}.$$

Moreover, in [16], by differentiating $B_{\#h,\#u,area}(x,x,q)$ with respect to q and evaluating at $q=1$, it is found that the average semi-perimeter of a bargraph with n cells is given by $\frac{15n+29}{18}-2^{-n}(1+\frac{(-1)^n}{9})$. In the computation of the asymptotics, in [16] it is used singularity analysis (see for instance [54]) and showed that the average area for bargraphs of semi-perimeter n is asymptotic to $\frac{\sqrt{\pi}\rho^{3/2}(2-\rho-\rho^3)^2}{2(1+\rho^2)^2(1-\rho-\rho^3)^{3/2}}n^{3/2}$ as $n \mapsto \infty$.

The enumeration of bargraphs according to the perimeter (see Theorem 3.2) has been considered as "SOS" walks by Owczarek and Prellberg [90] (see also Penaud [91]), where has been derived the generating function for the number of bargraphs according to the perimeter and the area (see [37,45] to find the motivation on perimeter and the area statistics). In the case of counting column-convex and row-convex polyominoes according to the perimeter, we refer the reader to [10,29,31,32,38,39,41,46,49–52,71] and references therein.

3.3 Site-perimeter

A *staircase polyomino* is a polyomino such that its perimeter contains only up and horizontal steps. Delest, Gouyou-Beauchamps and Vauquelin [40] showed that the generating function for staircase polyominoes according to the site-perimeter is given by

$$\frac{x^2}{2}(1-x^2-2x^3+x^4-(1+x-x^2)\sqrt{(1+x+x^2)(1-3x+x^2)}).$$

The techniques used in deriving this generating function were quite close to the techniques used for deriving the generating function $B(x,x)$. Using the Wasp-waist decomposition of bargraphs, Bousquet-Mélou and Rechnitzerin [30] studied the number of bargraphs according to the site-perimeter.

Theorem 3.4. [30] *Let* $A(x)=\frac{1+2x^3-x^4-x^5-\sqrt{(1+2x^3-x^4-x^5)^2-4x^2(1+x-x^2)^2}}{2x^2(1+x-x^2)}$. *Then the generating function for the number of bargraphs according to the site-perimeter is given by*

$$\frac{-x^3\sum_{j\geq 0}\frac{A^jx^{\binom{j+5}{2}}}{(1+x-x^2)^j\prod_{i=1}^j(1-x^i)\prod_{i=1}^j(1-x^{i+2}A^2)}}{\sum_{j\geq 0}\frac{A^jx^{\binom{j+5}{2}}}{(1+x-x^2)^j\prod_{i=1}^j(1-x^i)\prod_{i=1}^j(1-x^{i+2}A^2)}\frac{(1-Ax^{j+1})(1-Ax^{j+2})+A^2x^{2j+4}(1-x)}{(1-Ax^j)(1-Ax^{j+1})}}.$$

Moreover, the number of bargraphs with site-perimeter n *is asymptotic to* $C\xi^{-n}n^{-3/2}$ *for some positive constant* C*, where* $\xi=0.45002\ldots$ *is the smallest positive root of the polynomial* $1-2x-2x^2+4x^3-x^4-x^5$.

By appending one or two columns at a time to a bargraph, in [30] it is shown the following result.

Theorem 3.5. [30] *The generating function* $B(s,x,y,q)$ *for the number of bargraphs according to the number of cells in the rightmost column, horizontal half-perimeter, vertical half-perimeter and site-perimeter (marked by* s,x,y,q*) satisfies*

$$B(s,x,y,q)=a(s)+b(s)B(1,x,y,q)+c(s)B(sq,x,y,q)+d(s)B(sq,x,y,q),$$

where

$$a(s)=\frac{sxyq^4}{1-syq^2},\quad b(s)=\frac{sxq((1-syq)(1-syq^2)+s^2xy^2q^4(1-q))}{(1-s)(1-syq)(1-syq^2)},\quad c(s)=\frac{sx^2yq^4(1-q)}{(1-yq)(1-syq)(1-syq^2)},$$
$$d(s)=-\frac{xq((1-yq)(1-q)(1+s^2yq^2)+sq((1-yq)(1+yq^2-2yq)+xyq^3(1-q)))}{(1-yq)(1-s)(1-syq^2)}.$$

In [30] is used the kernel method to solve the functional equation in the above theorem. It is shown the following result.

Theorem 3.6. [30] *The generating function for the number of bargraphs according to horizontal half-perimeter, vertical half-perimeter and site-perimeter (marked by* x,y,q*) is given by*

$$\frac{-yq^3\sum_{j\geq 0}\frac{A^jx^{2j}q^{3j}(1-q)^j(yq)^{\binom{j+2}{2}}}{(1-yq)^j(1+xq-xq^2)^j\prod_{i=1}^j(1-q^i)\prod_{i=1}^j(1-q^{i+2}y^2A^2)}}{\sum_{j\geq 0}\frac{A^jx^{2j}q^{3j}(1-q)^j(yq)^{\binom{j+2}{2}}}{(1-yq)^j(1+xq-xq^2)^j\prod_{i=1}^j(1-q^i)\prod_{i=1}^j(1-q^{i+2}y^2A^2)}\frac{(1-A(yq)^{j+1})(1-Aq(qy)^{j+1})+xA^2q^2(yq)^{2j+2}(1-q)}{(1-A(yq)^j)(1-A(yq)^{j+1})}},$$

where A *is the unique power series in* q *which satisfies*

$$(1-yq)(1-A)(1-Ayq^2)=-xq(((1-yq)(1-q)(1+A^2yq^2)+Aq((1-yq)(1+yq^2-2yq)+xy^2q^3(1-q)))).$$

3.4 Lattice paths

A *Motzkin path* of length n is a lattice path that starts at $(0, 0)$, ends at $(n, 0)$, remains weakly above the x-axis, and consists of up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal (or level) steps $H = (1, 0)$. A *Dyck path* of length $2n$ or semi-length n is a Motzkin path of length $2n$ that has no level steps. It is well-known that the number of Motzkin paths of length n and Dyck paths of semi-length n is given by the nth Motzkin number and nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, respectively. Given P any Dyck path, a *peak* in P is an occurrence of UD, a *valley* in P is an occurrence of DU, and a *return* in P is a down step that ends on the x-axis. The *height of a peak* in P is the maximal y-coordinate of its points. The *height of* P is the maximal y-coordinate it reaches.

Dyck paths: Following results obtained by Deutsch and Elizalde [43], we establish a bijection ϕ between Dyck paths and bargraphs, where the semi-length of a Dyck path becomes the semi-perimeter minus the number of peaks of the corresponding bargraph. Here is the construction. Let P be any Dyck path, define the height of each step of P to be the highest point of the y-coordinates of it. In this context, we denote the sequence of these y-coordinates by $YC(P)$. For example, Figure 5 presents the Dyck path $P = UDUUDUDDUDUD$ with $YC(P) = 111222211111$. We define $\phi(P)$ as

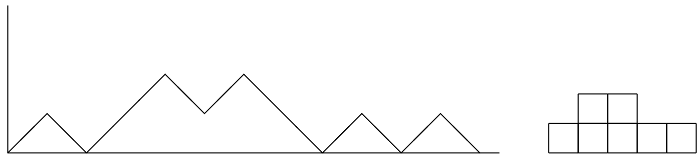

Figure 5: A Dyck path and its corresponding bargraph.

$c_1^{d_1}c_2^{d_2}\cdots c_m^{d_m}$, where d_i is the length of the ith maximal block of c_i consecutive letters (here c^d means the word $cc\cdots c$ with d occurrences of c). For instance, if P is the Dyck path in 5 then $Y(P) = 111222211111$, so $c_1 = 3$, $d_1 = 1$, $c_2 = 4$, $d_2 = 2$, $c_3 = 5$ and $d_3 = 1$, which implies $\phi(P) = 12211$ (see Figure 5).

Theorem 3.7. [43] *The map ϕ exhibited above is a bijection between Dyck paths and bargraphs. Moreover, if P is any Dyck path and $B = \phi(P)$ is its corresponding bargraph, then:*

- *semi-length of P=semi-perimeter of B - number of peaks in B;*
- *number of peaks in P=#h in B - number of valleys in B;*
- *sum of heights of peaks in P=area of B - sum of heights of valleys in B;*
- *height of P=height of B;*
- *Number of initial up steps in P=number of initial u steps in B;*
- *Number of final down steps in P=number of final d steps in B;*
- *number of returns in P=number of h steps at height 1 in B + $1 - \delta_{P \text{ of height one}}$, where δ_X is defined as 1 when X is true and 0 otherwise.*

As a corollary of the above result, it is shown that the number of bargraphs B having semi-perimeter of B-number of peaks in B=m is given by the mth Catalan number $\frac{1}{m+1}\binom{2m}{m}$. Moreover, in [43] it is shown that the generating function $G(x, q) = \sum_{n\geq1}\sum_{\pi\in B_n} q^{\text{number of peaks in }\pi}x^{\text{semi-perimeter of }\pi}$ satisfies $x(1-x)G^2(x, q) - (1 - 3x + x^2 + qx^3)G(x, q) + qx^2(1 - x) = 0$. Thus,

$$G(x, q) = \frac{qx^2(1-x)}{1-3x+x^2+qx^3}C\left(\frac{qx^3(1-x)^2}{(1-3x+x^2+qx^3)^2}\right),$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. In particular,

$$G(1/q, q) = \sum_{n\geq1}\sum_{\pi\in B_n} q^{\text{semi-perimeter of }\pi\text{-\#peaks in }\pi} = C(q) - 1 = \sum_{m\geq1}\frac{1}{m+1}\binom{2m}{m}q^m.$$

Note that Blecher et al. [25] established a bijection between Dyck paths avoiding $UUDD$ and bargraphs. Let $f_h(x)$ be the generating function for the number of Dyck paths avoiding $UUDD$ with height at most h. By the First return decomposition of a Dyck path, in [25] is shown that $f_h(x) = 1/(1 - x(f_{h-1}(x) - x))$ with $f_0(x) = 1$ and $f_1(x) = 1/(1 - x)$. By induction on h, we see that

$$f_h(x) = \frac{(1-x)U_{h-2}(t) - \sqrt{x}U_{h-3}(t)}{\sqrt{x}((1-x)U_{h-1}(t) - \sqrt{x}U_{h-1}(t))}, \tag{2}$$

where $t = (1+x^2)/(2\sqrt{x})$ and U_m is the mth Chebyshev polynomial of the second kind (which is defined via the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with the initial conditions $U_0(t) = 1$ and $U_1(t) = 2t$, see [95]).

Motzkin paths: Janse van Rensburg and Rechnitzer [64] established a bijection between Motzkin paths and bargraphs. Using the Wasp-waist decomposition of a bargraph (as a path from $(0,0)$ to $(m,0)$), therein it is found that the generating function $g(z,w) = B_{\#u+\#h+\#d,ev}(z,w)$ (ev is the number of horizontal steps on x-axis) is given by

$$g(z,w) = \frac{wz + z^2 g(z,1) + wz^3 g(z,1)}{1 - wz - wz^3 g(z,1)}.$$

Moreover, if $h(z,v) = B_{\#u+\#h+\#d,vv}(z,w)$ (vv is the number of vertices of the path on x-axis), then

$$h(z,v) = \frac{v^2 z + v^2 z^2 h(z,1) + v^3 z^3 h(z,1)}{1 - vz + v^2 z^3 h(z,1)}.$$

For motivation and more details on these generating functions, we refer the interested reader to [64].

Following [64], Deutsch and Elizalde [42] noticed a bijection between Motzkin paths and bargraphs as follows. Given a path in M, insert an up step at the beginning and a down step at the end, and then turn all the up steps $U = (1,1)$ into u, all the down steps $D = (1,-1)$ into d, and leave the horizontal steps $H = (1,0)$ unchanged as h. By using this bijection, it is derived the generating function $B(x,y)$ (as in Theorem 3.2). Moreover, it is shown the following result.

Theorem 3.8. [42] *Let $B_{st} = B_{\#h,\#u,st}(x,y,q)$ be the generating function for the number of bargraphs according to the number of horizontal steps, the number of up steps and occurrences of the statistic st.*

- *If st=height of the first column, then*

$$(1 - q(1 - x + y + xy) + q^2 y)B_s t^2 - q(1-y)(1 - x - qy - qxy)B_{st} + q^2 xy(1-y) = 0.$$

- *If st=number of double rises, then $xB_{st}^2 - (1 - x - qy - xy)B_{st} + xy = 0$. Moreover, there exists a bijection between bargraphs of semi-perimeter $n+1$ with no double rises and secondary structures on n vertices (*A secondary structure *is a simple graph with vertices $[n]$ such that $(i, i+1)$ is an edge for all i, every i is adjacent to at most one vertex j with $|j - i| > 1$, and there are no two edges $(i,k),(j,l)$ with $i < j < k < l$).*
- *If st=number of valleys of width ℓ, then $(1 - (1-q)(1-x)x^{\ell-1})B_s t^2 - (1 - x - y - xy - (1-q)x^{\ell+1}y)B_{st} + xy = 0.$*
- *If st=number of peaks of width ℓ, then $xB_s t^2 - (1 - x - y - xy - (1-q)x^{\ell+1}y)B_{st} + y(x - (1-q)(1-x)x^{\ell}) = 0.$*
- *If st=number of corners of type DH, then $qxB_{st}^2 - (1 - x - y + xy - qxy - xy)B_{st} + xy = 0.$*
- *If st=number of corners of type UH, then $xB_{st}^2 - (1 - x - y + xy - xy - qxy)B_{st} + qxy = 0.$*
- *If st=number of horizontal segments, then $qxB_{st}^2 - (1 - x - y + xy - 2qxy)B_{st} + qxy = 0.$*
- *If st=number of horizontal segments of length 1, then*

$$x(q + x - qx)B_{st}^2 - ((1-x)(1-y) - 2xy(q + x - qx))B_{st} + xy(q + x - qx) = 0.$$

- *If st=length of the first descent, then*

$$B_{st} = q(1 - x - y)\frac{1 - x - y - xy - \sqrt{(1-y)(1-x)^2 - y(1-y)(1+x)^2}}{2x(1 - x - qy)}.$$

- *If st=x-coordinate of the first descent, then*

$$B_{st} = qy\frac{1 + (1-2q)x - y - xy - \sqrt{(1-y)(1 - 2x - y - 2xy + x62 - x^2 y)}}{2(1 - q - (1-q)y + q(q-1)x + qxy)}.$$

- *If st_h=number of columns of height h, then $B_{st_h} = \frac{y}{x(1 - x - xB_{st_{h-1}})} - \frac{y(1+x)}{x}$ with*

$$B_{st_1} = \frac{(1-x)^2 - y - (2q-1)x^2 y - (1-x)\sqrt{(1-y)(1-x)^2 - y(1-y)(1+x)^2}}{2x(1 - qx)}.$$

- *If st=least column height, then*

$$B_{st} = q(1-y)\frac{1-x-y-xy-\sqrt{(1-y)(1-x)^2-y(1-y)(1+x)^2}}{2x(1-qx)}.$$

- *If st=width of the leftmost horizontal segment, then*

$$B_{st} = \frac{q(1-q)xy}{1-qx} + q\frac{(1-x)^2-y-(2q-1)x^2y-(1-x)\sqrt{(1-y)(1-x)^2-y(1-y)(1+x)^2}}{2x(1-qx)}.$$

- *If st=number of occurrences of* uhu*, then* $xB_{st}^2-(1-x-y-qxy)B_{st}+xy=0$.

- *If st=length of the initial staircase* $uhuh\dots$*, then* $B_{st} = \frac{q(A-(1-x+qx^2-q^2xy)\sqrt{B})}{2x(1-q^2x+q^2x^2-q^2xy+q^2(q^2-1)x^2y)}$*, where* $A=(1-x)^2-y+q(1-x+y)x^2-q^2x(1-y)y+(q^2+q^2y-1)x^2y+q(1-2q^2)x^3y$ *and* $B=(1-y)((1-x)^2-y(1+x)^2)$*. Moreover, the generating function for the number of bargraphs according to the number of horizontal steps, up steps, the number of odd-height and number of even-height columns is given in [42].*

- *If st=area=number of cells, then*

$$B_{st} = -y + \cfrac{y}{-qx + \cfrac{1}{1-y+\cfrac{y}{-q^2x+\cfrac{1}{1-y+\cfrac{y}{-q^3x+\cfrac{1}{\ddots}}}}}}.$$

3.5 Inner site-perimeter

Let B be a bargraph. The *inner site-perimeter* is the number of cells inside B that have at least one edge in common with an outside cell. A *border cell* of B is a cell inside of B that has at least one edge in common with an outside cell of B. Clearly, the inner site-perimeter of B is the number of border cells of B. A *tangent cell* of B is a cell inside of B which is not a border cell of B and that has at least one vertex in common with an outside cell of B. By decomposing each bargraph regrading to columns of size 1, Mansour [76] showed the following result.

Theorem 3.9. [76] *Let* $C(x,y,p,q)$ *be the generating function for the number of bargraphs according to the area, width, number of border cells and number of tangent cells marked by* x*,* y*,* p *and* q*, respectively. Then the generating function* $C(x,y,p,q)$ *satisfies*

$$C(x,y,p,q) = \frac{\overline{C}(x,pxy,p,q)}{1-pxy\overline{C}(x,pxy,p,q)},$$

where $\overline{C}(x,y,p,q)$ *satisfies*

$$\begin{aligned}\overline{C}(x,y,p,q) &= 1+p^2W(y)+p^2\left(\overline{C}(x,xy,p,q)-1-pW(y)\right)\\ &\quad+\frac{pxy\left(1+p^2W(y)+pq\left(\overline{C}(x,xy,p,q)-1-pW(y)\right)\right)^2}{1-pxy\left(1+pqW(y)+q^2\left(\overline{C}(x,xy,p,q)-1-pW(y)\right)\right)}\end{aligned}$$

and $W(y)=\frac{x^2y}{1-px}$.

By differentiating $C(x,y,p,q)$ with respect to p or q and evaluating at $p=1$ or at $q=1$, we obtain the total number of border and tangent cells.

Theorem 3.10. [19,76] *The average number of border cells (inner site-perimeter) / tangent cells over all bargraphs of* B_n *is given by* $\frac{27}{28}n+\frac{13}{49}+O(n/2^n)$ / $\frac{3n}{112}-\frac{71}{392}+O(1/2^n)$*, respectively.*

3.6 Height

Using the Wasp-waist decomposition of any bargraph, Blecher et al. [25], studied the generating function $H_h(x,y)$ for the number of bargraphs with height at most h, where x marks the total number of horizontal steps, y marks the total number of up steps.

Theorem 3.11. [25] *The generating function* $H_h(x,y)$ *is given by* $\frac{xy(t_+^h-t_-^h)}{x^2y(t_-^{h-1}-t_+^{h-1})+(1-x)((t_+-y-xy)t_+^{h-1}-(t_--y-xy)t_-^{h-1})}$*, where* $t_\pm = \frac{1-x+y+xy\pm\sqrt{(1-x+y+xy)^2-4y}}{2}$.

Using (1) and (2), in [25] is studied the average height of Dyck paths avoiding $UUDD$ of semi-length n, and it is shown the following result.

Theorem 3.12. [25] *The average height of Dyck paths avoiding $UUDD$ of semi-length n and the average height of bargraphs of semi-perimeter n is asymptotic to* $\frac{1}{2}\sqrt{\frac{n\pi}{\rho(1-\rho-\rho^3)}}$.

Note that to find the average of the statistic under consideration over all bargraphs with semi-perimeter n, as $n \mapsto \infty$, we do the following steps: (1) Find a formula for the generating function $B_{stat}(x, y, w)$ (not necessary to be explicit), (2) Find $\frac{\partial}{\partial w} B_{stat}(x, x, w) \mid_{w=1}$, (3) Using singularity analysis of (1), find the asymptotics of the coefficient of x^n in $\frac{\partial}{\partial w} B_{stat}(x, x, w) \mid_{w=1}$, then at the end (4) we divide what we got from Step 3 by (1) (the asymptotic formula of the number of bargraphs with semi-perimeter n, as $n \mapsto \infty$).

3.7 Width

By Theorem 3.2, we see that the generating function $B(xy, y)$ counts all the bargraphs according to the width (marked by x) and total semi-perimeter (marked by y). In [25] is studied the generating function $B(xy, y)$ and it is shown the following result.

Theorem 3.13. [25] *The average width of bargraphs of semi-perimeter n is asymptotic to* $\frac{(1-\rho)(1-2\rho-\rho^2)n}{\rho^2(1-\rho-\rho^3)}$, *where ρ is defined in* (1).

3.8 Walls

A *wall* in a bargraph is a subword consisting of a maximal number of adjacent up steps. If a wall consists of precisely r up steps it is called a *wall of size* r. In the latest case, it can neither preceded nor followed by another adjacent up step. Referring to Figure 3, there are 3 walls of size 1; 2 walls of size 2 and 1 wall of size 3. The main results obtained by Blecher, Brennan and Knopfmacher [12] include the generating functions for the number of bargraphs according to the number of walls of size r, horizontal steps, and vertical steps and asymptotics for the total number of walls of size r. In deriving their results, the authors used a variant of what is known as Wasp-waist decomposition. After they described the decompositions and turned the problem into generating functions, they stated the following result.

Theorem 3.14. [12] *The generating function for the total number of walls of size r over all bargraphs is given by*

$$\frac{2^{1-r}xy(-1+y)(1+x(-1+y)+y\sqrt{X})^r}{\sqrt{X}(-1+x-y-xy+\sqrt{X})},$$

where x counts horizontal steps, y counts vertical steps and $X = (1-y)(1+x^2(1-y)-y-2x(1+y))$. The average number of walls of size r is asymptotic to $\frac{2^{1-r}(1-\rho)\rho^2(1+\rho^2)^{1-r}\cdot n}{1-\rho-\rho^3}$ *as $n \to \infty$, where $\rho = 0.295598\ldots$ is the dominant singularity, which is the positive root of polynomial $1-4x+2x^2-x^4$.*

3.9 Descents, up-steps, levels

Blecher, Brennan and Knopfmacher [16] studied the generating function for the number of bargraphs according to h steps, u steps, number of descents (des), and area, and it is shown the following results.

Theorem 3.15. [16] *We have*

$$B_{\#h,\#u,des,area}(x,y,p,q) = \frac{y\sum_{i\ge1}\frac{x^i q^{\binom{i+1}{2}}(py-1)^{i-1}\prod_{j=1}^{i}(1-yq^j(p-1)/(py-1))}{\prod_{j=1}^{i}(1-q^j)\prod_{j=1}^{i}(1-yq^j)}}{1-\sum_{i\ge1}\frac{x^i(py-1)^{i-1}q^{\binom{i+1}{2}}\prod_{j=1}^{i}(1-yq^j(p-1)/(py-1))}{\prod_{j=1}^{i}(1-q^j)\prod_{j=1}^{i-1}(1-yq^j)}}.$$

Moreover, the average number of descents in bargraphs of semi-perimeter n is asymptotic to $\frac{\rho(1-\rho^2)}{2(1-\rho-\rho^3)}n$, *as $n \mapsto \infty$.*

By Theorem 3.15 it is shown that the average number of up steps in bargraphs of semi-perimeter n is asymptotic to $\frac{1-\rho-\rho^2-\rho^3}{2(1-\rho-\rho^3)}n$, as $n \mapsto \infty$.

Theorem 3.16. [16] *We have*

$$B_{\#h,\#u,level,area}(x,y,q,w) = \frac{y\sum_{i\ge1}\frac{x^iq^i}{1-yq^i}\prod_{j=1}^{i-1}(yq^j/(1-yq^j)-1/(1-q^j)+w)}{1-\sum_{i\ge1}\frac{x^iq^i}{1-q^i}\prod_{j=1}^{i-1}(yq^j/(1-yq^j)-1/(1-q^j)+w)}.$$

Moreover, the average number of levels in bargraphs of semi-perimeter n is asymptotic to $\frac{1-3\rho+\rho^2+\rho^3}{2(1-\rho-\rho^3)}n$, *as $n \mapsto \infty$.*

3.10 Peaks

We define a peak to be the set of horizontal steps in a sequence of the form uh^kd with $k \geq 1$. We define the height of the peak to be the y-coordinate of the horizontal step(s) of the peak (see [14]).

In all following theorems, related to above-mentioned statistics on peaks, ρ is defined in (1).

Theorem 3.17. [14] *Let ρ as in (1).*

- *The generating function for the number of bargraphs according to the number horizontal steps, vertical up steps, and the number of peaks is given by*

$$\frac{(1-x)^2 - y(1-wx^2) - \sqrt{((1-x)^2 - y(1-wx^2))^2 - 4wx^2y(1-x)^2}}{2x(1-x)},$$

  where x counts the number of horizontal steps, y counts the number of vertical up steps, and w counts the number of peaks. Moreover, the average number of peaks in a bargraph of semi-perimeter n is asymptotic to $\frac{\rho^2(1+\rho^2)n}{2(1-\rho)(1-\rho-\rho^3)}$.

- *The generating function for the number of bargraphs according to the number horizontal steps, vertical up steps, and the number of horizontal steps in all the peaks is given by*

$$\frac{(1-wx)(1-x)^2 - y(1-wx-wx^2+wx^3) - \sqrt{X}}{2x(1-wx)(1-x)},$$

  where $X = -4wx^2y(1-wx)(1-x)^3 + ((1-wx)(1-x)^2 - y(1-wx-wx^2+wx^3))^2$, x counts the number of horizontal steps, y counts the number of vertical up steps and w counts the number of horizontal steps in peaks. Moreover, the average number of horizontal steps in all the peaks in the bargraphs of semi-perimeter n is asymptotic to $\frac{\rho^2(1+\rho^2)n}{2(1-\rho)^2(1-\rho-\rho^3)}$.

- *The generating function for the number of bargraphs according to the number horizontal steps, vertical up steps, and the height of the first peak is given by*

$$\frac{2wxy}{(2-w)(1-x) - wy - wxy + w\sqrt{(1-y)((1-x)^2 - y(1+x)^2)}},$$

  where x counts the number of horizontal steps, y counts the number of vertical up steps and w counts the height of the first peak. Moreover, the average height of the first peak in the bargraphs of semi-perimeter n is asymptotic to $\frac{16(1-\rho)\rho^3}{(1-2\rho-\rho^3)^2}$.

- *The generating function for the number of bargraphs according to the number horizontal steps, vertical up steps, and the position of the first peak is given by*

$$\frac{xy(1-x) + x^2y(1-w)}{(1-x)(1-wx-y-xy+\frac{1}{2}(-1+x+y+xy+\sqrt{(1-x-y-xy)^2-4x^2y}))},$$

  where x counts the number of horizontal steps, y counts the number of vertical up steps and w counts the position of the first peak. Moreover, the average position of the first peak in the bargraphs of semi-perimeter n is asymptotic to $\frac{8\rho^4(3-10\rho+4\rho^2+\rho^4)}{(1-\rho)(1-2\rho-\rho^2)^4}$.

- *The generating function for the number of bargraphs according to the number horizontal steps, vertical up steps, and the distance between the leftmost peak and the rightmost peak is given by*

$$\frac{xy(1-x)}{(1-x)^2-y}\left(1 - \frac{2wx^2y(1+wx(y-1)+y-\sqrt{Y})}{(1-x)(1-x(2+w(y-1)+y-\sqrt{Y}))}\right),$$

  where $Y = (1-y)(1+w^2x^2(1-y)-y-2wx(1+y))$, x counts the number of horizontal steps, y counts the number of vertical up steps and w counts the height of the first peak. Moreover, the average of the distance between the leftmost peak and the rightmost peak in the bargraphs of semi-perimeter n is asymptotic to $\frac{4\rho^6(3-8\rho+4\rho^2-2\rho^3+3\rho^4)n}{(1-3\rho+\rho^2)(1-2\rho-\rho^2)^3(1-\rho-\rho^3)}$.

3.11 Levels

A *level* in a bargraph is a maximal sequence of two or more adjacent horizontal steps denoted by h^r where $r \geq 2$. It is preceded and followed by either an up step or a down step. The length of the level is the number r of horizontal steps in the sequence. The *height of a level* is the y-coordinate of the horizontal steps in the sequence. Using the Wasp-waist decomposition, we obtain the following theorem which is due to Blecher, Brennan and Knopfmacher [15].

Theorem 3.18. *The generating function for the number of bargraphs according to the number of horizontal steps, up steps and levels is given by*

$$B_{Levels}(x,y,w)=\frac{1-x-y-xy+2x^2y-2wx^2y-2x\sqrt{A}}{2x(1-x+wx)},$$

where $A=(1-x-y-xy+2x^2y-2wx^2y)^2-y(1-x(1-w))^2$*,* x *counts the number of horizontal steps,* y *counts the number of up steps and* w *counts the number of levels. Moreover, the average number of levels in bargraphs of semi-perimeter* n *is asymptotic to* $\frac{1-4\rho+4\rho^2-\rho^4}{2(1-\rho-\rho^3)}n$ *as* $n\mapsto\infty$ *(see (1)).*

Further, in [15] is obtained an explicit formula for the generating functions $B_{stat1}(x,y,w)$ and $B_{stat2}(x,y,w)$, where $stat1$ and $stat2$ represent statistics: the leftmost x-coordinate and the height of the first level, respectively.

3.12 Water cells

By a *water capacity* of a composition/bargraph we mean the number of cells which keep the water after it falls through it. Sometimes, such a number is refereed to as the number of *water cells*. For example, if we let water to flow through bargraph 276347175, see Figure 6, we find that its water capacity is 14.

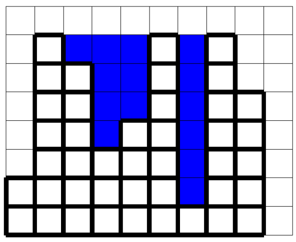

Figure 6: The water capacity of bargraph $w=276347175$.

Note that each composition with parts in $[d]$ can be decomposed as either π' or $\pi' d\pi''$ such that π' is a composition with parts in $[d-1]$ and π a composition with parts in $[d]$. Using such a decomposition, Mansour and Shattuck [85] studied the generating function for the number of bargraphs according to the number of water cells. More precisely, they showed the following result. Let $B_n^{(d)}$ be the set of bargraphs of n where the size of each column is at most d. Define

$$f_d(x,y;p,q)=1+\sum_{n\geq1}\left(\sum_{\pi\in B_n^{(d)}}y^{\nu(\pi)}p^{des(\pi)}q^{wc(\pi)}\right)x^n$$

to be the generating function for the number of bargraphs of $B_n^{(d)}$ with m columns, according to the number of descents and the number of water cells (marked by p and q, respectively).

Theorem 3.19. [85] *For all* $d\geq1$*,*

$$f_d(x,y;p,q)=\frac{1}{1-xy}+\sum_{k=2}^{d}x^ky\frac{a_{k-1}(x,yq;p,q)}{a_k(x,y;p,q)}\prod_{m=1}^{k-1}\frac{(1-x^my(1-p))(a_{m-1}(x,yq;p,q))^2}{(a_m(x,y;p,q))^2},$$

where $a_m(x,y;p,q)=1-xy\sum_{j=0}^{m-1}x^jq^{m-1-j}\prod_{i=1}^{j}(1-x^iy(1-p)q^{m-i})$.

The water capacity of bargraphs is also studied by Blecher, Brennan and Knopfmacher [17]. More precisely, therein it is shown the following result.

Theorem 3.20. [17] *The water capacity generating function for all bargraphs* B_n^k *is given by*

$$\sum_{r=1}^{k}z^t\Big(1+\frac{(x-z)z^r}{(x-z)-z(x^r-z^r)}\Big)\prod_{i=1}^{r-1}z^t\Big(1+\frac{(x-z)z^i}{(x-z)-z(x^i-z^i)}\Big)^2.$$

Moreover the mean water capacity in compositions of n *as* $n\to\infty$ *is given by*

$$\frac{n\log_2 n}{2}+\frac{n}{2}\Big(\frac{3\gamma-4}{\log 2}-\frac{3}{2}\Big)+\frac{n}{2\log 2}\delta_2(\log_2 n)+o(n),$$

where γ *is Euler's constant and* $\delta_2(x)$ *is a periodic function of period* 1*, mean zero and small amplitude, and it is given by the Fourier series* $\delta_2(x)=\sum_{k\neq0}(\chi_k+3)\Gamma(-1-\chi_k)e^{2k\pi ix}$*, where the complex numbers* χ_k *are given by* $\chi_k=e^{2k\pi ix}/\log 2$.

3.13 Protected cells

Assume it is given a bargraph B that contains m horizontal steps. It can be identified as a sequence of columns $t(B) = t_1t_2\ldots t_m$ such that the jth column from the left contains exactly t_j cells. For a cell s of B, we denote its left (respectively, right) neighbour by $L(s)$ (respectively, $R(s)$). Further, let m be the number of columns of B and let b be the maximal height of its columns. Then B defines uniquely a rectangle (m,b), denoted by $R(B)$. A cell s in $R(B)$ is called *a protected cell* (respectively, *empty cell*, *empty protected cell*) of B if and only if the cells s, $L(s)$ and $R(s)$ lie in $t(B)$ (respectively, if the cells s, $L(s)$ and $R(s)$ lie in $R(B)\setminus t(B)$). Consider the bargraph 24123612 given in Figure 7. It has 7 protected cells (cells marked with red lines), 21 empty cells and 13 empty protected cells (cells marked with blue lines). Clearly, the empty cells are those cells of rectangle not contained within the bargraph. In our example there $6\cdot 8-27=21$. Protected cells were

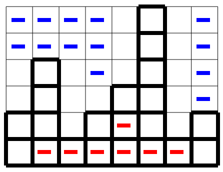

Figure 7: The bargraph $B = 24123612$ with protected cells and empty protected cells.

studied by Mansour, Schork and Yaqubi [86].

Theorem 3.21. [86] *The generating function for the total number of protected cells in all bargraphs according to the number of cells is given by* $\frac{x^3}{(1-2x)^2(1-x)(1-x^3)}$. *Moreover, as $n\to\infty$ the total number of protected cells in all bargraphs with n cells is given by* $\frac{n}{7}2^{n+1}$.

A d-bargraph is a bargraph where the height of all columns is at most d and which has at least one column of height d. The following result appeared in [86].

Theorem 3.22. [86] *Let q_i mark the number of empty cells in ith row of a given d-bargraph. The generating function for the number of d-bargraphs according to the number of cells, the number of columns, and the number of empty cells in each row is given by:*

$$1+\sum_{d\geq 1}\frac{x^dy}{\left(1-y\sum_{i=1}^{d}q_{i+1}q_{i+2}\ldots q_dx^i\right)\left(1-y\sum_{i=1}^{d-1}q_{i+1}q_{i+2}\ldots q_dx^i\right)}.$$

Moreover the generating function for the number of d-bargraphs according to the number of cells and the number of empty protected cells in the top row is given by

$$\frac{(1-(q-1)f_d(x))^2x^d}{\left(1-q\frac{x-x^d}{1-x}-x^d+(q-1)x^df_d(x)\right)\left(1-q\frac{x-x^d}{1-x}\right)},$$

where $f_d(x)=x+2x^2+\ldots+(d-1)x^{d-1}+(d-1)x^d+\ldots+2x^{2d-3}+x^{2d-2}$.

3.14 Corners

A *corner* in a bargraph is a point of intersection of an up (u) or down (d) step with a horizontal (h) step. Considering left-to-right orientation we distinguish four basic types of corners: up-horizontal (uh), horizontal-up (hu), down-horizontal (dh), and horizontal-down (hd). Let $cor_{uh}(\pi)$ denote the number of corners of type uh in a bargraph π, and similarly $cor_{hu}(\pi)$, $cor_{dh}(\pi)$, $cor_{hd}(\pi)$ for the corners of type hu, dh, hd, respectively. By symmetries, it is not hard to see that $cor_{uh}(\pi)=cor_{hu}(\pi)$ and $cor_{dh}(\pi)=cor_{hd}(\pi)$ (see [82]). Referring to the example in the Figure 7, where it is presented the bargraph $\pi=24123612$, we have that $cor_{uh}(\pi)=6=cor_{hu}(\pi)$ and $cor_{dh}(\pi)=2=cor_{hd}(\pi)$. Define

$$B(x,y;p,q)=\sum_{n\geq 0}\sum_{m=0}^{n}x^ny^m\sum_{\sigma\in\mathcal{B}_{n,m}}p^{cor_{uh}(\sigma)}q^{cor_{dh}(\sigma)}.$$

to be the generating function that counts bargraphs of $\mathcal{B}_n$ according to the number of columns, the number of corners of types uh and the number of corners of types dh (marked by y, p and q, respectively). By the Scanning-element algorithm, Mansour, Shabani and Shattuck [82] showed the following.

Theorem 3.23. [82] *The generating function $B(x,y;p,q)$ is given by*

$$\frac{1+p(q-1)y\sum_{m\geq1}\left((q-p)^{m-1}y^{m-1}\sum_{0<i_1<\cdots<i_m}\frac{x^{i_1+\cdots+i_m}}{\prod_{j=1}^{m}(1-(1-p)x^{i_j}y)}\right)}{1-py\sum_{m\geq1}\left((q-p)^{m-1}y^{m-1}\sum_{0<i_1<\cdots<i_m}\frac{x^{i_1+\cdots+i_m}}{\prod_{j=1}^{m}(1-(1-p)x^{i_j}y)}\right)}.$$

Moreover, the total number of corners of type uh (hu) in all bargraphs of $\mathcal{B}_n$, $n\geq1$, is given by $\frac{(3n+13)2^{n-2}-(-1)^n}{9}$.

Also, Mansour, Shabani and Shattuck [82] studied the joint distribution for corners of type uh^i in bargraphs of $\mathcal{B}_{n,m}$.

3.15 Interior vertices and edges

A vertex is called an *interior vertex* if it is adjacent to exactly four different cells of a bargraph B, otherwise, it is called a *boundary vertex*. A *horizontal / vertical* edge of a bargraph is called a *d-h-edge / d-v-edge* if it is formed from d interior vertices. Let $ehintd(B)$/$evintd(B)$ denote the number of horizontal/vertical edges in B formed from d interior vertices. By the Scanning-element algorithm, Mansour and Shabani [78] showed the following result.

Theorem 3.24. [78] *The generating function for the number of bargraphs in $\mathcal{B}_{n,m}$ according to the number interior vertices is given by*

$$\frac{1+\sum_{j\geq1}\frac{(1-q)^jq^{\binom{j}{2}}x^{\binom{j+1}{2}+j}y^j}{\prod_{i=0}^{j-1}(1-q^ix^{i+1})(1-q^{i+1}x^{i+1})}}{1-\sum_{j\geq0}\frac{(1-q)^jq^{\binom{j}{2}}x^{\binom{j+2}{2}}y^{j+1}}{(1-q^{j+1}x^{j+1})\prod_{i=0}^{j-1}(1-q^ix^{i+1})(1-q^{i+1}x^{i+1})}},$$

where x marks n, y marks m and q marks the number of interior vertices. Moreover, the total number of interior vertices in all bargraphs of $\mathcal{B}_n$ is given by $\frac{3n-11}{9}2^{n-2}+\frac{1}{18}(9+(-1)^n)$.

In [78] is also found an explicit formula for the generating function for the number of bargraphs in $\mathcal{B}_{n,m}$ according to the number of horizontal d-edges, $d=0,1,2$. This leads to the following result.

Theorem 3.25. [78] *Asymptotically, the total number of all horizontal 0-edges / 1-edges / 2-edges over all bargraphs of B_n is given by $\frac{13n}{12}2^n$ / $\frac{n}{21}2^n$ / $\frac{n}{28}2^n$ as $n\mapsto\infty$, and the total number of all vertical 0-edges / 1-edges / 2-edges over all bargraphs of $\mathcal{B}_n$ is given by $\frac{25n}{48}2^n$ / $\frac{n}{8}2^n$ / $\frac{13n}{336}2^n$ as $n\mapsto\infty$, respectively.*

3.16 Durfee squares

A bargraph π is said to contain an $s\times s$ square if π contains s consecutive columns each of which has at least s cells. *Durfee squares* are the largest squares that lie on the base of the bargraph representation of a given composition (in an analogy to the classical Durfee square of an integer partition which is the largest square in the Ferrers diagram of that partition, see [5]). We decompose any bargraph π using symbolic decomposition (see [54]) as $\pi=\pi_1\pi_1'\pi_2\pi_2'\ldots\pi_a'\pi_{a+1}$ where the compositions π_i' are singletons of size less than or equal to s and the π_i are compositions of length $0,1,2,\ldots,s$ with parts of any size greater than s. Using such a decomposition, one has the following result.

Theorem 3.26. [7] *The generating function for the compositions of n that avoid Durfee squares of size s is given by*

$$\frac{(1-x)^{s+1}-(1-x)x^{s^2}}{(1-x)^{s+1}-x(1-x)^s+x^{s^2+1}(1-x^{s-1})}.$$

Note that Archibald et. al. [7] considered the asymptotics for compositions that avoid Durfee squares of size s. Therein, it is studied the number of times an $s\times s$ *grounded squares* (namely the $s\times s$ squares, not necessarily maximal) occur on the bottom row of a bargraph.

Theorem 3.27. [7] *The generating function for the number of compositions of n with m parts according to the number of grounded $s\times s$ squares, counted by q, is given by*

$$F_s(x,y,q):=\frac{x-1}{\left(\frac{qy^sX^s}{qyx^s+x-1}-\frac{y^sX^s-1}{yx^s+x-1}\right)^{-1}-yx^s+xy}.$$

where $X=\frac{x^s}{1-x}$. Moreover, the total number of grounded $s\times s$ squares of all bargraphs in B_n is given by

$$\binom{n-s^2+s-1}{s-1}+\sum_{k=1}^{n-s^2}2^{k-2}(k+3)\binom{n-s^2+s-1-k}{s-1}.$$

From the last relation it follows that the average number of grounded $s\times s$ squares in a bargraph with n cells is asymptotic to $\frac{n+3-s-s^2}{2^{s^2-s+1}}$ as $n\to\infty$.

3.17 Depth

Let B be any bargraph. The *depth* of a cell c in B, denoted by $dep(c)$, is the minimum number of horizontal or up steps to exit B starting from c. The *depth* of B is defined as $dep(B) = \max_c dep(c)$, where the maximum is over all cells c of B. An alternative conception of the depth of B is given by the size of its Durfee square, see Section 3.16. Equivalently, the depth of B can be understood as the number of times the inner site-perimeter can be recursively removed from B until nothing remains. For instance, the depth of the composition 3454312 is three. Blecher et. al [23] characterized the set of bargraphs with depth at most r in terms of "patterns". In particular, they showed that a composition (bargraph) σ satisfies $dep(\sigma) \geq r$ if and only if it contains $\tau^{(r)} := r(r+1)\cdots(2r-2)(2r-1)(2r-2)\cdots(r+1)r$. To enumerate the compositions that avoid (i.e., do not contain) $\tau^{(r)}$ for any given r, Blecher et. al [23] defined a directed graph and then by using the transfer matrix found a determinantial formula. In particular, they showed the following result.

Theorem 3.28. [23] *The generating function for the number of compositions of n that avoid $\tau^{(2)} = 232$ is given by*

$$\frac{(1-x)^2 + x^3(1-x) + x^5}{(1-2x)(1-x) + x^3(1-2x) + x^5 - x^6}$$

and the generating function for the number of compositions of n that avoid $\tau^{(3)} = 34543$ is given by

$$\frac{(1-x)^4 + x^5(1-x)^3 + x^9(1-x)^2 + x^{13} - x^{14} + x^{16}}{(1-2x)((1-x)^3 + x^5(1-x)^2 + x^9(1-x) + x^{13}) + x^{16}(1-x-x^2)}.$$

In general, they presented an explicit formula for the transfer matrix (the adjacent matrix of the directed graph).

3.18 Counting bargraphs

Let B and C be any two bargraphs. Following [79], we say that a vertex (x, y) of B is a C*-vertex* if C lies entirely in B when positioned starting at (x, y). Let $C(B)$ denote the number of C-vertices of B, so $C(B)$ is the number of ways C can be positioned within B so that its vertices coincide with those contained on or within B. We point out that the definition of C-vertices is related to *the generalized factor order in words* (see, e.g., [69]), where we recall that the word $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ is a generalized factor order of the word $\pi = \pi_1\pi_2\cdots\pi_n$ at position j if $\sigma_s \leq \pi_{j-1+s}$ for all $s = 1, 2, \ldots, m$. Thus our $C = C_1C_2\cdots C_m$-vertex at (a, b) in bargraph $B = B_1B_2\cdots B_n$ corresponds to the word $(C_1+b)(C_2+b)\cdots(C_m+b)$ which is a generalized factor order of the word $B_{a+1}B_{a+2}\cdots B_{a+m}$. For instance, if $C = 12$, then the bargraph 23213 contains C three times, namely, the C-vertices are $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(3, 0)$, which correspond to the factor 12 at position 1, the factor 23 at position 1, the factor 12 at position 2 and the factor 12 at position 1, respectively.

Let C be a fixed bargraph. Mansour and Shabani [79] studied the generating function $F_C(x, y, q)$ for the number of bargraphs with n cells (marked by x) and m columns (marked by y) according to the number of C-vertices (marked by q), namely

$$F_C(x, y, q) = \sum_{n\geq 0}\sum_{B\in\mathcal{B}_n} x^n y^{\#\text{columns in } B} q^{C(B)}.$$

In [79] are studied the cases when $C = m$ is one column, $C = 1^c = 11\cdots 1$ is one row, $C = c1$, and $C = cd$.

Theorem 3.29. [79]

(1) Let C be a bargraph with c cells and one column. Then the generating function $F_C(x, y, q)$ is given by

$$\frac{1}{1 - \frac{x-x^c}{1-x}y - \frac{x^c yq}{1-xq}}.$$

(2) Let $C = 1^c$ with $c \geq 1$. Then the generating function $F_C(x, y, q)$ is given by

$$\alpha(x,y,q) - \cfrac{1/q^{c-1}}{qxy - \cfrac{1}{\alpha(x,qxy,q) - \cfrac{1/q^{c-1}}{(qx)^2y - \cfrac{1}{\alpha(x,(qx)^2y,q) - \cfrac{1/q^{c-1}}{(qx)^3y - \ddots}}}}},$$

where $\alpha(x, y, q) = \sum_{j=0}^{c-1} \frac{x^j y^j (1-q^{j-c+1})}{(1-x)^j}$.

(3) The generating function $F_{c1}(x, y, q)$ is given by

$$\frac{\sum_{j\geq 0} \frac{x^{c\binom{j+1}{2}} q^{c\binom{j}{2}} y^j (1-q)^j}{\prod_{i=0}^{j-1}(1-x^{i+1}q^i)(1-(xq)^{i+1})}}{1 - \sum_{j\geq 0} \frac{x^{c\binom{j+1}{2}} q^{c\binom{j}{2}} (1-q)^j y^j \left(\frac{y((x^{j+1}q^j)^c - x^{j+1}q^j)}{x^{j+1}q^j - 1} + \frac{q^{1+cj} x^{c(1+j)} y}{1-(qx)^{j+1}}\right)}{\prod_{i=0}^{j-1}(1-x^{i+1}q^i)(1-(xq)^{i+1})}}.$$

(4) Let $c \geq d \geq 1$ and $C = cd$. Then the generating function $F_{cd}(x, y, q)$ is given by

$$\frac{\sum_{j\geq 0} \alpha(q^j x^j) \prod_{i=0}^{j-1} \beta(q^i x^i)}{1 - \sum_{j\geq 0} \alpha'(q^j x^j) \prod_{i=0}^{j-1} \beta(q^i x^i)},$$

where

$$\alpha(u) = 1 - (1 - y(qxu)^{d-1}) \frac{(1-q)x^{c-d+1} yu^{c-d+1}}{q^{d-1}(1-xu)(1-qxu)}, \qquad \beta(u) = \frac{(1-q)x^{c-d+1} yu^{c-d+1}}{q^{d-1}(1-xu)(1-qxu)},$$

$$\alpha'(u) = \sum_{j=1}^{c-1} x^j yu^j + \frac{(1-q)x^c yu^c}{(1-xu)(1-qxu)} \sum_{j=1}^{d-1} x^j y + \frac{qx^c yu^c}{1-qxu} - \beta(u) \sum_{j=1}^{d-1} q^j x^{2j} yu^j.$$

Note the total number occurrences of C in a bargraph has been considered in [79], where C is either $C = c$, $C = 11\cdots 1$, $C = c1$, or $C = cd$. We remark that Blecher and Mansour [27] established the generating function that counts the number of times the staircase $C = 1^+2^+\cdots m^+$ (which is bargraph with m columns such that the jth column has at least j cells) fits inside a bargraph.

4. Statistics on words

We continue with another important combinatorial family - *words*. As described in the book [63], many statistics have been considered in words over a finite alphabet, with a special attention given to those statistics characterized in terms of "patterns". In this section, we do not look at such statistics and instead we focus on statistics that are not easily formulated in terms of patterns, but can be seen as geometric properties when the word is presented as a bargraph. In this regard we emphasize that these statistics have not been considered in [63]. First we recall a definition of k-ary words. A *word* w of length n over the alphabet $[k]$ is an element of $[k]^n$. It is also called a *k-ary word of length* n. For instance, the words of length two over the alphabet $[3] = \{1, 2, 3\}$ are 11, 12, 13, 21, 22, 23, 31, 32 and 33. A word can be represented by a bargraph such that the height of the ith column in the bargraph equals the number of cells in the corresponding part of that word. Therefore these bargraphs have column heights less than or equal to k.

Let $W_{st_1,\ldots,st_d;k}(x, x_1, \ldots, x_d)$ be the generating function $F_{\#col,st_1,\ldots,st_d}(x, x_1, \ldots, x_d)$ where $\#col$ is the number of columns in the bargraphs, and $\mathcal{F}$ is the combinatorial class of all k-ary words represented as bargarphs (including the empty word), see Section 2.

In this section we describe, the time line of the research on counting k-ary words represented as bargraphs according to some statistics (see Table 2).

Year	Statistic, Reference	Theorem
2015	Height [25]	3.11, 3.12
2017	Perimeter [21]	4.1, 4.2, 4.3
	Site-perimeter [22]	4.4, 4.5
	Shedding light [6]	4.8
2018	Water cells [18]	4.6, 4.7

Table 2: Time line of research for words.

4.1 Perimeter

Using the Wasp-waist method, Blecher et. al [21] showed the following result.

Theorem 4.1. [21] *The generating function $W_k(x, q)$ for the number of k-ary words of length n according to the perimeter satisfies*

$$W_{per;k+1}(x, q) = \frac{xq^4 + q^2(1 + xq^2)W_{per;k}(x, q)}{1 - xq^2 - xq^2 W_{per;k}(x, q)}$$

with $W_{per;1}(x, q) = \frac{q^4 x}{1-xq^2}$, where the statistic per denotes the perimeter.

By the Scanning-element algorithm, the authors gave another form for the generating function $W_{per;k}(x, q)$.

Theorem 4.2. [21] *We have*

$$W_{per;k}(x, q) = \frac{1 + xq^2 \sum_{j=1}^{k}(q^{2j} - 1) + \sum_{j=2}^{k}(xq^2)^j \gamma_{k,j;1}(q)}{1 - kxq^2 - \sum_{j=2}^{k}(xq^2)^j \gamma_{k,j;2}(q)},$$

where $\gamma_{k,j;d}(q) = \sum_{1\le i_1<i_2<\cdots<i_j\le k-1} \prod_{s=d}^{j}(q^{2i_s}-1)$ *and the statistic* per *denotes the perimeter.*

Moreover, in [21], the authors presented another form for the generating function $W_{per;k}(x,q)$ by solving system of k linear equation. After, they found three different forms for the generating function $W_{per;k}(x,q)$, they calculated the average of the perimeter over all k-ary words of length n. By finding the coefficient of x^n in $\frac{\partial}{\partial q}W_{per;k}(x,q)\mid_{q=1}$ and then dividing it by k^n (the number of k-ary words of length n), it is shown the following result.

Theorem 4.3. [21] *For* $n\ge 2$, *the average perimeter per words of length* n *over the alphabet* $[k]$ *is* $\frac{1}{3k}(2k^2+3k+1+(k^2+6k-1)n)$.

4.2 Site-perimeter

The site-perimeter has been investigated also for the case of k-ary words. By the Wasp-waist decomposition (see [30,92]), we split the bargraph of any k-ary word into two smaller bargraphs at column of height 1 as follows: any nonempty bargraph π can be decomposed as $\pi=1$, $\pi=1\pi''$, $\pi=\pi'$, $\pi=\pi'1$, $\pi=\pi'1\pi''$, where the size of each column in π' is at least two and π',π'' are nonempty bargraphs. Blecher et. al. [22] derived a functional equation for the generating function that counts the number of k-ary words according to the size of the rightmost part, the number of parts, and the site-perimeter. In particular, it is shown the following result.

Theorem 4.4. [22] *Let* $|p|,|x|<1$ *and*

$$F(s)=\frac{(1+px-p^2x)(1-p^2s+p^2s^2)+(p^5x^2-p^3x+p^2x-1)s}{(1-s)(1-sp^2)},\qquad G(s)=-\frac{x^2p^4s}{(1-ps)(1-sp^2)},$$

$$H_0(s)=-\frac{sxp^4(1-(sp^2)^k)}{1-sp^2},\qquad H_3(s)=\frac{xp^{2k+3}s^{k+1}(1-p^2+xp^4)}{(1-p^2)(1-sp^2)},$$

$$H_1(s)=-\frac{pxs}{1-s}-\frac{x^2p^4s^2}{(1-s)(1-ps)}-\frac{x^2p^4s^2(p^2-1-s^{k-1}(1-s)p^{2k})}{(1-s)(1-p^2)(1-sp^2)}.$$

Define $A_m(s)=\sum_{j\ge0}(-1)^j\frac{H_m(p^js)\prod_{i=0}^{j-1}G(p^is)}{\prod_{i=1}^{j-1}F(p^is)}$, *where* $m=0,1,3$. *Then*

$$W_{sp;k}(x,p)=-\frac{A_0(r_+)A_3(r_-)-A_0(r_-)A_3(r_+)}{A_1(r_+)A_3(r_-)-A_1(r_-)A_3(r_+)},$$

where the statistic sp *denotes the site-perimeter and* $r_\pm=\frac{g\pm\sqrt{g^2-4p^2(xp^2-px-1)^2}}{2p^2(xp^2-px-1)}$ *with* $g=x^2p^5+xp^4-2xp^3-p^2+xp^2-1$.

Differentiating the generating function in Theorem 4.4 with respect to p and evaluating at $p=1$, gives the following result.

Theorem 4.5. [22] *Let* $n\ge2$. *The average site perimeter of a* k*-ary word of length* n *is given by*

$$\frac{2+3k+16k^2+3k^3}{12k^2}n+\frac{-2+k+8k^2+5k^3}{6k^2}.$$

4.3 Water cells

The counting of water cells was extended to the case of k-ary words. By considering the Maximal column factorization of k-ary word, the following result was found by Blecher, Brennan and Konpfmacher [18].

Theorem 4.6. [18] *We have*

$$W_{wc;k}(x,q)=1+x\sum_{j=1}^{k}\left(1+\frac{x(1-q)}{1-x-q+q^jx}\right)\prod_{i=1}^{j-1}\left(1+\frac{x(1-q)}{1-x-q+q^ix}\right)^2,$$

where wc *marks the number of water cells.*

By differentiating $W_{wc;q}(x,q)$ with respect to q and evaluating at $q=1$, we obtain the average of the water cells over all k-ary words of length n.

Theorem 4.7. [18] *The total number of water cells of all* k*-ary words of length* n *is given by*

$$\frac{k^n}{2}((n+4)(k-1)-4kH_{k-1})+(k-1)^n(n+2k-2)+\sum_{i=1}^{k-2}\frac{i^n(i(n-2)-kn)}{i-k},$$

and the average number of water cells for k*-ary words of length* n *is* $\frac{k-1}{2}n-4k(H_k-1)+O(n(k-1)^n/k^n)$, *where* H_n *is the* n*th Harmonic number.*

4.4 Shedding light

Assume it is given a bargraph. We let the positive x, y-axis represent the East and the North, respectively. Further, suppose that there is a light source at infinity in the North-West direction that sheds parallel light rays onto the bargraph. Clearly, the ray of light will hit and also will miss some of the cells of a given bargraph. A cell is called *Shedding light cell* or *a lit cell* if the ray of light hits the edge facing north or the edge facing west or both. For example, Figure 8 presents the bargraph 24123612 with its lit cells.

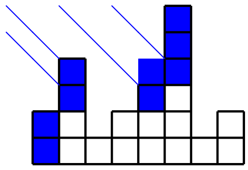

Figure 8: The word 24123612 and its 9 lit (blue) cells.

The statistic *lit* has been studied by Archibald et. al. [6]. The generating functions techniques and matrix algebra were used to study this statistic. Further a modification of an *adding a slice* method leads to the following result.

Theorem 4.8. [6] *We have* $W_{lit;k}(x,q) = \frac{1+\frac{x(q-1)}{\gamma}+x(q-1)\sum_{a=2}^{k}\left[\frac{A_a}{\gamma^a}+\frac{B_a}{\gamma^{a-1}}\right]}{1-\frac{x}{\gamma}\sum_{a=1}^{k}\frac{A_a}{\gamma^{a-1}}}$, *where* lit *marks the number of lit cells, and*

$$A_a = \gamma^{a-1}\sum_{j=0}^{a-1}\left(\frac{-x}{\gamma}\right)^j\prod_{m=0}^{j}(1-mq)\binom{a-1}{j},$$

$$B_a = q\gamma^{a-1}\sum_{l=0}^{a-1}\sum_{j=0}^{a-1-l}\frac{l}{a-l}\binom{a-1}{l,j,a-1-j-l}\frac{A_l x^j (q-1)^{a-1-j-l}}{\gamma^{l+j}}\prod_{m=0}^{j-1}(1+mq).$$

and $\gamma = 1+x-xq$. *The average number of lit cells is asymptotic to* $\eta_k n$, *where* $\eta_k = \frac{1}{k}+\frac{1}{k}\sum_{j=1}^{k-1}\frac{(k-j)\binom{k}{j}(-1+j)!}{k^j(1+j)}$.

5. Statistics on set partitions

Being closely related to the well known families of numbers, namely Stirling numbers of the second kind and Bell numbers, the combinatorial family of *set partitions* and statistics related to it, always attracted a lot of research interest. Many statistics in set partitions over finite alphabet, characterized in terms of "patterns" and "avoidance" were deeply described in the book [73]. Here we review the geometrically described statistics on set partitions when the latest are formulated as bargraphs. First we recall some definitions related to set partitions. A *partition of a set* or a *set partition* is a collection of non-empty, mutually disjoint subsets, called *blocks*, whose union is the set. Let $\mathcal{P}_{n,k}$ denote the set of partitions of $[n] = \{1,2,\ldots,n\}$ having k blocks and $\mathcal{P}_n = \bigcup_{k=0}^{n}\mathcal{P}_{n,k}$. Note that $|\mathcal{P}_{n,k}| = S(n,k)$ is the classical Stirling number of the second kind and the number of set partitions of $[n]$ is given by B_n the nth Bell number (see [73, 77]). Further, let $\pi = B_1/B_2/\cdots/B_k \in \mathcal{P}_{n,k}$. Then π is said to be in *standard form* if the blocks B_i are labeled such that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. A set partition π in standard form can be expressed equivalently by the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$ wherein $i \in B_{\pi_i}$ for all i, which we will denote also by π. This sequence satisfies the *restricted growth property* (see [73, 77]), meaning that the first occurrence of ℓ always precedes the first occurrence of $\ell+1$, that is, $\pi_{i+1} \le \max(\pi_1\pi_2\cdots\pi_i)+1$ for all $1 \le i \le n-1$. For example, if $\pi = \{1,6\},\{2,7,9\},\{3,4,8\},\{5\},\{10,11\} \in \mathcal{P}_{11,5}$, then $\pi = 12334123255$. We represent the restricted growth sequences associated with members of $\mathcal{P}_{n,k}$ as bargraphs. For example, the bargraph representation of $\pi = 12334123255$ is given in Figure 9 below. Table 3 shows the time line of the

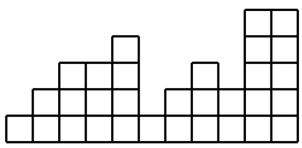

Figure 9: Bargraph representation of the set partition 12334123255.

research on set partitions.

Year	Statistic, Reference	Theorem
2017	Area, Up steps [84]	5.1, 5.2
2018	Interior vertices [74]	5.6
	Corners [82]	5.9, 5.10
	Water Cells [85]	5.7, 5.8
2019	1×2 rectangles [33]	5.11
	Perimeter [75]	5.3, 5.4
	Site-perimeter [75]	5.5

Table 3: Time line of research for set partitions.

In this section, we define $P_{st_1,\dots,st_d;k}(x, x_1, \dots, x_d) = F_{\#col,st_1,\dots,st_d}(x, , x_1, \dots, x_d)$, where $\#col$ is the number of columns in the bargraphs, and $\mathcal{F}$ is the combinatorial class of all set partitions with exactly k blocks represented as bargarphs, see Section 2. Moreover, we define $P_{st_1,\dots,st_d}(x, y, x_1, \dots, x_d) = \sum_{k\geq 0} P_{st_1,\dots,st_d;k}(x, x_1, \dots, x_d)y^k$.

Throughout this section, we present several basic facts.

The ordinary and exponential generating function for the Stirling numbers of the second kind is

$$\sum_{n\geq k} S(n,k)x^n = \frac{x^k}{(1-x)\cdots(1-kx)} \text{ and } \sum_{n\geq k} S(n,k)x^n = \frac{(e^x-1)^k}{k!}.$$

The number of partitions of $[n]$ is given by the n-Bell number, with exponential generating function $\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{e^x-1}$. Asymptotically as $n \to \infty$ we have

$$B_n \sim n! \frac{e^{e^\kappa - 1}}{\kappa^n \sqrt{2\pi\kappa(\kappa+1)e^\kappa}} \tag{3}$$

where κ is the positive root of $\kappa e^v = n + 1$, from which we get $\kappa \equiv \kappa(n) = \log n - \log\log n + O\left(\frac{\log\log n}{\log n}\right)$. In order to obtain asymptotic estimates for the averages, we need an extension of (3), namely:

$$B_{n+h} = B_n \cdot \frac{(n+h)!}{n!\kappa^h} \left(1 + O\left(\frac{\log n}{n}\right)\right),$$

uniformly for $h = O(\log n)$, as given in [35].

5.1 Area and up steps

The next results are due to Mansour and Shattuck. It studies the number of set partitions according to the number of cells (area) and the number of up steps.

Theorem 5.1. [84] *Let $n \geq k \geq 1$. The sum of the areas of all partitions of $[n]$ having k blocks is given by*

$$\binom{k+1}{2} S(n,k) + \sum_{j=1}^{k} \sum_{i=0}^{n-k-1} j^i \binom{j+1}{2} S(n-i-1,k).$$

Moreover, the average of area over all partitions of $[n]$ is asymptotic to $\frac{n(n+1+\kappa)}{2\kappa}$.

Theorem 5.2. [84] *If $n \geq k \geq 1$, then the total number of up steps within all members of $\mathcal{P}_{n,k}$ is given by*

$$kS(n,k) + \binom{k}{3} S(n-1,k) + \sum_{j=1}^{k} \sum_{i=0}^{n-k-2} \frac{j^{i+1}(j^2-1)}{6} S(n-i-2,k).$$

Moreover, the average of up steps over all partitions of $[n]$ is asymptotic to $\frac{n(n+1)-\kappa^2}{6\kappa}$.

5.2 Perimeter

The perimeter statistic has been investigated also for the case of set partitions. Mansour [75], by using the Scanning-element algorithm, has shown the following result.

Theorem 5.3. [75] *Let $1 \leq k \leq n$. The generating function for the total of the half of the perimeter over all set partitions of $[n]$ with exactly k blocks is given by*

$$\frac{x^k}{6\prod_{j=1}^{k}(1-jx)} \sum_{i=1}^{k} \frac{12 + 3(i^2 - 5i + 2)x - i(i-1)(2i-7)x^2}{1 - ix}.$$

By translating this ordinary generating function to exponential generating function, we have the following.

Theorem 5.4. [75] *The exponential generating function for the total of the half of the perimeter over all set partitions of* $[n]$ *(marked by* x*) with exactly* k *blocks (marked by* y*) is given by*

$$\frac{y}{36}\int_0^x ((6t-5)y^2e^{3t}+9(6t+1)ye^{2t}+9(y^2+4t+8)e^t-y(4y+9))e^{ye^t-y}dt.$$

Moreover, the total of the half of the perimeter over all set partitions of $[n+1]$ *with exactly* k *blocks is given by*

$$\frac{n+4}{6}S_{n+2,k}-\frac{5}{36}S_{n+3,k}+\frac{36n+53}{36}S_{n+1,k}+\frac{1}{4}S_{n+1,k-2}-\frac{n}{6}S_{n,k}-\frac{1}{9}S_{n,k-3}-\frac{1}{4}S_{n,k-2},$$

and the total of the half of the perimeter over all set partitions of $[n+1]$ *is given by*

$$\frac{n+4}{6}B_{n+2}-\frac{5}{36}B_{n+3}+\frac{18n+31}{18}B_{n+1}-\frac{6n+13}{36}B_n,$$

where $S_{n,k}$ *denotes the Stirling number of the second kind and* B_n *denotes the* n*th Bell number.*

5.3 Site-perimeter

The site-perimeter statistic has been investigated also for the case of set partitions. Mansour [75], by using the Scanning-element algorithm, has obtained the following result.

Theorem 5.5. [75] *The exponential generating function* $\frac{\partial^2}{\partial x^2}R(x,y)$ *for the total of the site-perimeter over all set partitions of* $[n]$ *(marked by* x*) with exactly* k *blocks (marked by* y*) is given by*

$$\frac{y}{144}e^{y(e^x-1)}\left(y^2(9y+16)-12(4y^3+3y^2-24x-72)e^x+72y(y^2+12x+14)e^{2x}+4y^2(102x+5)e^{3x}+3y^3(12x-11)e^{4x}\right).$$

Moreover, the total of the site-perimeter over all set partitions of $[n+2]$ *with exactly* k *blocks is given by*

$$\begin{aligned}(\frac{n}{4}+\frac{109}{72})S_{n+3,k}-\frac{11}{48}S_{n+4,k}+(\frac{4n}{3}+\frac{65}{16})S_{n+2,k}+\frac{1}{2}S_{n+2,k-2}+(\frac{n}{4}+\frac{47}{72})S_{n+1,k}\\ -\frac{1}{3}S_{n+1,k-3}-\frac{3}{4}S_{n+1,k-2}+\frac{n}{6}S_{n,k}+\frac{1}{16}S_{n,k-4}+\frac{1}{9}S_{n,k-3},\end{aligned}$$

and the total of the site-perimeter over all set partitions of $[n+2]$ *is given by*

$$(\frac{n}{4}+\frac{109}{72})B_{n+3}-\frac{11}{48}B_{n+4}+(\frac{4n}{3}+\frac{73}{16})B_{n+2}+(\frac{n}{4}-\frac{31}{72})B_{n+1}+(\frac{n}{6}+\frac{25}{144})B_n,$$

where $S_{n,k}$ *denotes the Stirling number of the second kind and* B_n *denotes the* n*th Bell number.*

5.4 Interior vertices

A vertex in a bargraph π is called an *interior* if it is adjacent to exactly four different cells in π, otherwise it is called a *boundary vertex*. For instance, Figure 9 presents a bargraph with 8 interior vertices. Mansour [74], counted set partitions of $\mathcal{P}_{n,k}$ according to the number of interior vertices. In particular, it is shown the following result.

Theorem 5.6. *Fix* $k\geq 1$*. Then the generating function for the number of set partitions of* $\mathcal{P}_{n,k}$ *according to the number of interior vertices is given by*

$$P_{iv;k}(x,q)=x^kq^{\binom{k-1}{2}}\prod_{j=1}^{k}\frac{1+x\sum_{a=1}^{j-2}\alpha_a(x)(q^{a+1-j}-1)}{1-x\left(q^{j-1}+\sum_{a=1}^{j-1}\alpha_a(x)\right)-x^2q^{j-1}\left(\sum_{a=1}^{j-1}\alpha_a(x)(q^{a-j}-1)\right)},$$

where iv *denotes the number of interior vertices, and*

$$\alpha_a(x)=\sum_{m=1}^{a}x^{m-1}\left(q^{\binom{m-1}{2}}(1-q)^{m-1}\sum_{k=0}^{a-m}q^k\binom{a-2-k}{m-2}_q\binom{m-1+k}{m-1}_q\right).$$

In particular, the exponential generating function $\sum_{k\geq0}\sum_{n\geq k}\sum_{\pi\in\mathcal{P}_{n,k}}iv(\pi)\frac{x^n}{n!}y^k$ *for the total number of interior vertices in all set partitions of* $\mathcal{P}_{n,k}$ *is given by*

$$E(x,y)=\frac{(3x-1)y^2}{9}e^{2x+ye^x-y}-\frac{(6x+13)y}{36}e^{x+ye^x-y}+\frac{6x+19}{36}e^{ye^x-y}+\frac{4y^3+9y^2-6}{36}\int_0^x e^{ye^t-y}dt+\frac{4y^2+13y-19}{36}.$$

Moreover, the total number of interior vertices in set partitions of $[n+1]$ *with exactly* k *blocks is given by*

$$\frac{4n+1}{12}S(n+2,k)-\frac{18n-1}{36}S(n+1,k)+\frac{n}{6}S(n,k)-\frac{1}{9}S(n+3,k)+\frac{1}{9}S(n,k-3)+\frac{1}{4}S(n,k-2).$$

and the total number of interior vertices in set partitions of $[n+1]$ *is given by*

$$\frac{4n+1}{12}B(n+2)-\frac{18n-1}{36}B(n+1)+\frac{6n+13}{36}B(n)-\frac{1}{9}B(n+3),$$

where $S(n,k)$ *is the Stirling number of the second kind and* $B(n)$ *is the* n*th Bell number.*

5.5 Water cells

In order to find the distribution of water cells on set partitions, Mansour and Shattuck [85] determined the distribution on words of the form $k\pi'(k+1)$ and of the form $k\pi'$, where π' is a k-ary word. This leads to the following result.

Theorem 5.7. [85] *For all* $k\geq 1$*, we have*

$$P_{des,wc;k}(x,p,q)=x^k\frac{\prod_{j=0}^{k-1}(1-x(1-p)q^j)^{k-1-j}\prod_{m=1}^{k-1}f_m(xq;p,q)}{f_k(x;p,q)\prod_{m=1}^{k-1}(f_m(x;p,q))^2},$$

where des, wc *denotes the number of descents and number of water cells, respectively, and*

$$f_m(x;p,q)=1-xq^{m-1}-\sum_{j=0}^{m-2}xq^j\prod_{s=1}^{m-1-j}(1-x(1-p)q^{s+j}).$$

In particular, for $p=1$*, we have*

$$P_{des,wc;k}(x,1,q)=x^k\left(\frac{1-([k]_q-1)x}{1-[k]_qx}\right)\prod_{j=1}^{k-1}\frac{1-([j]_q-1)x}{(1-[j]_qx)^2}.$$

As consequence of Theorem 5.7, in [85] it is found the total number of water cells over all set partitions of $\mathcal{P}_{n,k}$.

Theorem 5.8. [85] *Let* $n\geq k\geq 2$*. The total number of water cells over all set partitions of* $\mathcal{P}_{n,k}$ *is given by*

$$\binom{k}{2}\sum_{i=k}^{n-1}k^{n-i-1}S(i,k)+\sum_{j=1}^{k-1}\sum_{i=k}^{n-1}\left(\binom{j-1}{2}-1\right)j^{n-i-1}S(i,k).$$

Moreover, the total number of water cells of all set partitions of $[n]$ *is given by*

$$\begin{aligned}&-\frac{3}{4}B_{n+2}+\frac{2n+11}{4}B_{n+1}-\frac{6n+9}{4}B_n+\sum_{j=0}^{n}\binom{n+1}{j}B_j\widetilde{B}_{n-j}\\&\quad=\frac{n+1}{4\kappa^2}\left((2n+11)\kappa-3n-6+4\kappa C-6\kappa^2\right)B_n(1+O(\log n/n)),\end{aligned}$$

where $C=0.5963473622\cdots$.

Moreover, the study of water cells on non-crossing set partitions and non-nesting set partitions (see [73]) has been investigated in [85].

5.6 Corners

Let $f_k^{(r)}(x,y;q_1,\ldots,q_a)$ be the generating function for the number of set partitions of $\mathcal{P}_{n,k}$ having k blocks in which the element n belongs to the rth leftmost block, according to the area and the number of corners of type uh^i for $1\leq i\leq a$ (marked by y and the q_i, respectively). Define $f_k(x,y;q_1,\ldots,q_a)=\sum_{r=1}^k f_k^{(r)}$. By writing recurrence relations for $f_k^{(r)}(x,y;q_1,\ldots,q_a)$, Mansour, Shabani and Shattuck [82] stated the following formula.

Theorem 5.9. [82] *For all* $k\geq 1$*,*

$$f_k(x,y;q_1,\ldots,q_a)=L_kL_{k-1}\cdots L_2\left(\sum_{j=1}^{a-1}(xy)^jq_1q_2\cdots q_j+\frac{(xy)^aq_1q_2\cdots q_a}{1-xy}\right),$$

where

$$L_j=\frac{xy^j(1-\beta_j)}{1-xy^j-x(1-xy^j\beta_j)\left(\frac{y-y^j}{1-y}+\sum_{m=1}^{j-2}(-x)^m\sum_{2\leq i_1<\cdots<i_m\leq j-1}y^{\sum_{s=1}^m i_s}\frac{y-y^{i_1}}{1-y}\prod_{s=1}^m\beta_{i_s}\right)}$$

for $2\leq j\leq k$ *and* $\beta_r=\sum_{i=0}^{a-1}x^iy^{ri}q_1q_2\cdots q_i(1-q_{i+1})$.

Moreover, if $q_\ell = q$ and $q_j = 1$ for all $j \neq \ell$, then Theorem 5.9 leads to an explicit formula for the total number of the corners of type uh^ℓ.

Theorem 5.10. [82] *The total number of corners of type uh^ℓ where $\ell \geq 1$ in all set partitions of $\mathcal{P}_{n,k}$ is given by*

$$kS(n-\ell+1,k)+\sum_{j=2}^{k}\sum_{i=1}^{n-k-\ell} j^{i-1}\binom{j}{2}S(n-i-\ell,k).$$

Similarly, in [82] it is shown that the total number of corners of type $h^\ell d$ where $\ell \geq 1$ in all set partitions of $\mathcal{P}_{n,k}$ is given by

$$S(n-\ell,k-1)+\binom{k+1}{2}S(n-\ell,k)+\sum_{j=2}^{k}\sum_{i=1}^{n-k-\ell} j^{i-1}\binom{j}{2}S(n-i-\ell,k).$$

Moreover, therein, are given combinatorial proofs for the total number of corners of types uh^ℓ and $h^\ell d$.

5.7 1×2 rectangles

A 1×2 *rectangle* is a rectangle that is composed of two horizontally adjacent cells. Cakić, Mansour and Shabani [33], by using the Scanning-element algorithm, have shown the following result.

Theorem 5.11. *Let $R(x,y;q;v)=\sum_{k\geq 0}(\sum_{a=1}^{k} R_k(x;q|a)v^a)y^k$, where $R_k(x;q|a)$ is the generating function enumerating the set partitions of $\mathcal{P}_{n,k}$ such that the element n belongs to the block a, according to the number of 1×2 rectangles. Then*

$$R(x,y;q;v)=1+xR(x,vy;q;q)+xvyR(x,vy;q;q)+\frac{x(R(x,y;q;vq)-R(x,vy;q;q))}{1-v}+\frac{x(vqR(x,y;q;1)-R(x,y;q;vq))}{1-vq}.$$

Moreover, the exponential generating function $E(x,y)$ for the total number of 1×2 rectangles over all set partitions of $\mathcal{P}_{n,k}$ satisfies

$$\frac{\partial}{\partial x}E(x,y)=\frac{y}{36}e^{ye^x-y}(4y^2(3x-1)e^{3x}+9y(6x-1)e^{2x}+36xe^x+4y^2+9y).$$

The total number of 1×2 rectangles over all set partitions of $\mathcal{P}_{n+1,k}$ is given by

$$\frac{4n+1}{12}S_{n+2,k}-\frac{1}{9}S_{n+3,k}+\frac{18n+1}{36}S_{n+1,k}+\frac{n}{6}S_{n,k}+\frac{1}{9}S_{n,k-3}+\frac{1}{4}S_{n,k-2}$$

and the total number of 1×2 rectangles over all set partitions of $[n+1]$ is given by

$$\frac{4n+1}{12}B_{n+2}-\frac{1}{9}B_{n+3}+\frac{18n+1}{36}B_{n+1}+\frac{6n+13}{36}B_n.$$

The average of 1×2 rectangles in all the set partitions of $[n]$ is asymptotic to

$$\frac{n^2}{3(\log(n)-\log\log n)}\left(1-\frac{1}{3(\log n-\log\log n)}\right)\left(1+O\left(\frac{\log n}{n}\right)\right).$$

5.8 2×2 squares

Archibald et. al [8], using Column by column method from left to right, determined the generating function for the number of set partitions of $[n]$ with exactly k blocks according to the number of squares of size two. This leads to exact and asymptotic formulas for the average number of 2×2 squares over all set partitions of $[n]$.

Theorem 5.12. [8] *The total number of squares of size 2 in all set partitions of $[n]$ is given by*

$$\frac{19-18n}{36}B_n+\frac{4n-3}{12}B_{n+1}-\frac{1}{9}B_{n+2}+\frac{6n+7}{36}B_{n-1}.$$

Moreover, the average of the number of squares of size 2 over all set partitions of $[n]$ is asymptotic to $\frac{n^2}{3\log n}$, as $n\mapsto\infty$.

6. Statistics on permutations

Permutations is the combinatorial family we discuss in what follows. For the non-geometric statistics in permutations we refer to the following book [28], in which "patterns" and "pattern avoidance" are the main characters. As it is common for this review we again focus on statistics that are formulated via geometric properties when one presents a permutation as

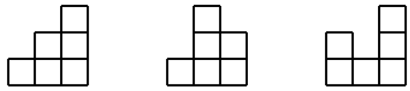

Figure 10: Bargraph representation of $123, 132, 213 \in \mathcal{S}_3$.

a bargraph. We point out that none of these statistics has been considered in [28]. We first recall some definitions related to permutations.

A *permutation* of $[n] := \{1, 2, \dots, n\}$ is a bijection from $[n]$ to itself. We denote the permutations of $[n]$ by $\pi = \pi_1\pi_2 \dots \pi_n$ and the set of all permutations of $[n]$ by $\mathcal{S}_n$. For example $\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$. As in the case of other combinatorial families, a permutation $\pi = \pi_1\pi_2 \dots \pi_n$ can also be expressed as bargraph, in a such way that we have π_j cells in the jth column of corresponding bargraph. For example, the first three permutations of $\mathcal{S}_3$ are shown in the Figure 10.

Table 4 shows the time line of the research on permutations.

Year	Statistic, Reference	Theorem
2016	Site-perimeter [13]	6.3
2018	Descent and up steps [83]	6.1
2019	Water cells [26]	6.4

Table 4: Time line of research for permutations.

6.1 Descents and up steps

Let $n \geq 1$ and $1 \leq i \leq \lfloor (n+1)/2 \rfloor$. We denote by $\mathcal{A}_{n,i}$ the set of permutations of the multiset $12 \cdots (n-i)(n-i+1)^i$ such that no two of the letters $n-i+1$ are adjacent. For example, we have $\mathcal{A}_{4,2} = \{1323, 2313, 3123, 3132, 3213, 3231\}$. Note that for $i = 1$, $\mathcal{A}_{n,i}$ coincides with $\mathcal{S}_n$ - the usual set of permutations of $[n]$.

A *descent* in a word $w = w_1w_2 \cdots w_n \in \mathcal{P}_n$ is an index $i \in [n-1]$ such that $w_i > w_{i+1}$. Let $des(w)$ denote the number of descents in w, which is equivalently the number of runs of down steps in bargraph representation of the word w. Note that $up(w) = \#u(w)$ and $area(w)$ is simply the sum of the entries of w. For example, if $w = 12334123255$ (see Figure 9) then the descents are at indexes $5, 8$, thus $des(w) = 2$, $up(w) = 9$. We have the following results, due to Mansour and Shattuck [83], regarding the up step statistic, where in the second one, are given the bounds for the up step statistic on members of $\mathcal{A}_{n,i}$.

Theorem 6.1. [83] *The number of up steps within the bargraph representations of all members of $\mathcal{A}_{n,i}$ is given by*

$$\frac{1}{3}(n-i)!\binom{n-i+1}{i}\left[\binom{n+2}{2} - \binom{i}{2}\right], 1 \leq i \leq \lfloor (n+1)/2 \rfloor.$$

In particular, the number of up steps within all bargraphs of $\mathcal{S}_n$ is given by $(n-1)!\binom{n+2}{3}$ for $n \geq 1$.

Theorem 6.2. [83] *Let $n \geq 1$ and $1 \leq i \leq \lfloor (n+1)/2 \rfloor$.*

1. *The maximum number of up steps in the bargraph representation of a member of $\mathcal{A}_{n,i}$ is given by $m(m+1) - \binom{i}{2}$ if $n = 2m$ and by $(m+1)^2 - \binom{i}{2}$ if $n = 2m+1$. Furthermore, these maximum values are attained by $\frac{2(m!)^2}{i!}$ members of $\mathcal{A}_{n,i}$ if $n = 2m$ and by $\frac{m!(m+1)!}{i!}$ members if $n = 2m+1$.*
2. *The minimum number of up steps in the bargraph representation of a member of $\mathcal{A}_{n,i}$ is given by $n + \binom{i-1}{2}$, and it is attained by $2^{n-2i+1}(i-1)!$ members of $\mathcal{A}_{n,i}$.*

6.2 Site-perimeter

Blecher, Brenan and Knopfmacher [13] considered the site-perimeter for the case of permutations when the latest are presented as bargraphs. Actually, therein the authors only considered the total of site-perimeter over all permutations of S_n, while the generating function for the number of permutations of S_n according to the site-perimeter statistic still remains open. The main method for obtaining such a result is to construct the set of permutations of S_{n+1} by raising the permutation from S_n by 1 and then appending the letter 1 at every position between the letters. Among other results in [13] it is shown the following result.

Theorem 6.3. [13] *Let $n \geq 2$. The average site-perimeter for permutations of S_n is given by $\frac{3n^2+29n+14}{12}$.*

6.3 Water cells

Let cap_n be the total capacity (water cells) of all permutations of S_n. Note that each permutation in S_{n+1} can be obtained from a permutation in S_n by adding one to each element and insert 1 somewhere between the letters. By using this simple fact, Blecher et. al [26] showed the following result.

Theorem 6.4. *For all $n \geq 1$, $cap_n = \frac{n!}{2}(n(n+7) - 4(n+1)H_n)$, where $H_n = \sum_{j=1}^{n} \frac{1}{j}$ the nth Harmonic number. Thus, the average $\frac{cap_n}{n!}$ is asymptotic to $\frac{n^2}{2} - 2n \ln n + (7/2 - 2\gamma)n - 2\ln n$ when $n \mapsto \infty$.*

A permutation π in S_n is said to have exactly *d-dam* if there exist only d disjoint connected areas of water containment, that is, the water cells in π exist in exactly d different connected areas. The width of each connected area is called *dam width*. In [26] it is shown that the total capacity of all one-dam permutations in S_N with dam width p, $1 \leq p \leq n-2$, is given by $\frac{pn!2^{n-2-p}}{(p+2)(n-2-p)!}$. Asymptotics expressions for the total capacity of all one-dam permutations in S_n and the number of permutations in S_n with one dam are studied in [26].

6.4 Pushes

We recall that *a weak left-to-right maximum* is a part which is greater than or equal to all parts to its left. Assume that the leftmost element in the permutation, which is not a weak left-to-right maximum, occurs in position i and has height $v(i)$. We call a *push* the process of shifting all cells which are to the left of i and of height greater than $v(i)$ for one position to the right. For example, by 4 pushes the word 322132, becomes 232132, 223132, 122332, and at end 122233. Blecher et. al. [24] showed the following result.

Theorem 6.5. [24] *Let H_n be the nth Harmonic number. Then*

- *the average number of pushes over all permutations of S_n is given by $n - H_n$.*
- *the average number of cells that do not move in all permutations of S_n is given by $(n+1)(H_{n+1} - 1)$.*
- *the average number of fixed cells over all members of S_n is given by $\frac{1}{6}(n+1)(2n+1)$.*

Note that in [24], is also considered the case of multipermutations. For the case of k-ary words, we refer the reader to Archibald et. al. [9].

7. Statistics on integer partitions

The last family under consideration is the family of *integer partitions*. A very good research reference for integer partitions is [3] while for an introductory reference we refer to [5]. In this section we will consider several geometrically motivated statistics in integer partitions when the latest are represented as bargraphs, using the same approach as explained in the case of integer compositions. To do this, we first begin by recalling some definitions related to integer partitions. A *partition* of a positive integer n is a finite non-increasing sequence of positive integers, whose sum is n. The number of partitions of n is denoted by $P(n)$ (where we define $P(0) = 1$). For example, integer partitions of 4 are the following: $4, 31, 22, 211, 1111$.

A standard result is

$$\sum_{j\geq 0} P(j)x^j = \frac{1}{\prod_{j\geq 1}(1-x^j)}.$$

Many studies had their focus on the number of integer partitions satisfying certain conditions (for example, see [1–4, 59, 94, 100, 101] and references therein). For instance, if $Q(n)$ is the number of partitions of n with distinct parts where $Q(0) = 1$, we have the following generating function

$$\sum_{j\geq 0} Q(j)x^j = \prod_{j\geq 0}(1+x^j).$$

In [20] is defined the statistic *corner* for integer partitions. This is related both to descents and the number of occurrences of a part (of fixed size). We describe corners in terms of the associated Ferrers diagrams.

The *Ferrers diagram* is a useful tool for visualizing integer partitions, but not only. It is constructed by stacking left-justified rows of cells, such that the number of cells in the ith row correspond to the size of the ith part. By rotating a given Ferrers diagram by 90^0 counterclockwise gives its corresponding bargraph, see left and right side of Figure 11.

7.1 Perimeter

It is not hard to see that the generating function $P(t,s,x)$ of the partitions of a nonnegative integer according to the sum of parts, the number of parts and the largest part (marked by x,t,s, respectively) is given by

$$P(t,s,x) = 1 + \sum_{i\geq 1} \frac{ts^i x^i}{\prod_{j=1}^{i}(1-tx^j)}$$

(see [96, Sequence A211978]). By setting $s=t=q$, we obtain that the generating function for the partitions according to the sum of parts and the semi-perimeter (of corresponding bargraphs) is given by

$$P(q,q,x) = 1 + \sum_{i\geq 1} \frac{q^{i+1} x^i}{\prod_{j=1}^{i}(1-qx^j)}.$$

The sum of the semi-perimeters of the bargraphs of the partitions of n is also discussed in [96, Sequence A211978].

7.2 Corners

A *corner* of a partition π is a point of degree two in the corresponding Ferrers diagram. Let $cor(\pi)$ denote the number of corners of π. For example, if $\pi = 4422111$, then $cor(\pi)=6$, see Figure 11. Further let $cor_k(\pi)$ denote the number of corners at line $y=k$ in the Ferrers diagram of π, where the topmost horizontal line of the Ferrers diagram corresponds to the line $y=0$. For example, for the partition illustrated below, $cor_0(\pi)=2$ (corners E and F), $cor_4(\pi)=1$ (corner B), and so on. Blecher et.al. [20] defined several types of corners. A *corner is of type* (a,b) if it is at the bottom-right of a specific maximal $a\times b$ rectangle (where a and b are the height and the length of rectangle, respectively). For such a maximal rectangle there are no cells below it and no cells to its right. Thus corner B, is at the bottom-right of the 2 by 1 rectangle, at the cell marked by X. So that B is a $(2,1)$ corner. Therein were only considered corners at the bottom-right extremities of such b by a rectangles. So for convenience, the 3 corners D, E and F at levels $x=0$ and $y=0$ are ignored. Thus, the partition in Figure 11 has corners C of type $(2,2)$, B of type $(2,1)$ and A of type $(3,1)$.

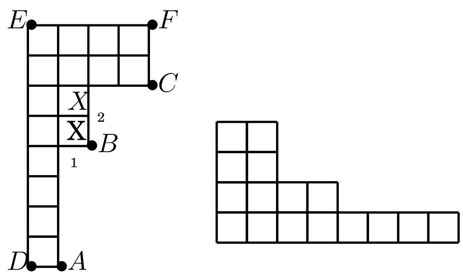

Figure 11: The 6 corners of $\pi = 4422111$, with A, B and C of type (a,b).

One easily finds that a corner of type (a,b) in the Ferrers diagram of π corresponds to consecutive parts c, $c-b$ in the partition π such that the multiplicity of c is exactly a and the last occurrence of c is followed by a part of size $c-b$. Other types of considered corners are of type $(a+,b)$; these are corners of type (j,b) for any $j\geq a$, and similarly, corners of type $(a+,b+)$. The size of corners of type (a,b) is defined to be $a+b$.

We summarize the main results due to Blecher et. al. [20] via the following theorem.

Theorem 7.1. [20] *We have*

Type of corner	*Generating function for the total number of such corners*	*Main term asymptotics for the average number of such corners*
All corners	$-3+\frac{3-2x}{(1-x)\prod_{j\geq1}(1-x^j)}$	$\frac{\sqrt{6n}}{\pi}$
(a,b)	$\frac{x^{ab}\prod_{i=1}^{a}(1-x^i)}{\prod_{i\geq1}(1-x^i)\prod_{i=b+1}^{b+a+1}(1-x^i)}$	$\frac{a!b!\sqrt{6n}}{\pi(a+b+1)!}$
$(a+,b)$	$\frac{x^{ab}\prod_{i=1}^{a}(1-x^i)}{\prod_{i\geq1}(1-x^i)\prod_{i=b}^{a+b}(1-x^i)}$	$\frac{a!(b-1)!\sqrt{6n}}{\pi(a+b)!}$
$(a+,b+)$	$\frac{x^{ab}\prod_{i=1}^{a-1}(1-x^i)}{\prod_{i\geq1}(1-x^i)\prod_{i=b}^{a+b-1}(1-x^i)}$	$\frac{(a-1)!(b-1)!\sqrt{6n}}{\pi(a+b-1)!}$
size m	$\frac{1}{\prod_{l=1}^{\infty}(1-x^l)}\sum_{p=1}^{m-1}\frac{x^{p(m-p)}\prod_{i=1}^{p}(1-x^i)}{\prod_{i=m-p+1}^{m+1}(1-x^i)}$	$\frac{1}{m+1}\sum_{a=1}^{m-1}\binom{m}{a}^{-1}\frac{\sqrt{6n}}{\pi}$

8. Extensions, generalizations and connections on bargraphs

As we have seen from the previous sections, the interest for studying bargaph is growing. Besides bargraphs coming from the representation of well known of combinatorial families, recently appeared some extensions of them. We end this paper by presenting several such extensions related to the study of bargraphs.

- An x-bargraph is a bargraph remaining after removing the edges on x-axis, see Mansour [76]. Therein, it is shown that the generating function for the number of x-bargraphs according to the number of border cells (inner site-perimeter) is given by

$$\frac{1+t-3t^2+2t^3-\sqrt{1-2t-9t^2+6t^3+9t^4-12t^5+4t^6}}{2t}.$$

- *A cylindrical lattice* is a lattice with vertex set $\{W(a,b) : (a,b) \in \mathbb{Z}_m \times \mathbb{Z}\}$, whose edges are formed by transformations:

$$W(a,b) \mapsto W(a+1,b+1); W(a,b) \mapsto W(a-1,b+1),$$

naturally known as counterclockwise step and a clockwise step, respectively. For more on cylindrical lattices see [56]. Let $m > 1$ be an arbitrary fixed integer. Consider the map $W : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^3$ defined by $W(a,b) = \Big(\cos\frac{2\pi a}{m}, \sin\frac{2\pi a}{m}, b\Big)$. By this map the lattice $\mathbb{Z} \times \mathbb{Z}$ simply wraps around the cylinder. To be more precise, we view the image of W as a cylindrical lattice; which we call *the m-cylinder*.

A *circular bargraph* is a self-avoiding walk in a cylindrical lattice, starting at the origin and ending with its first return to the xy-plane, which coincides with the starting point. In an analogy to regular bargraphs, there are three types of steps: an up-step $(0,0,1)$, a down-step $(0,0,-1)$, and a horizontal-step $(0,1,0)$. Also it is natural to require that an up-step cannot directly follow a down-step, and other way around, as well as that all horizontal steps must lie strictly above the xy-plane. A circular bargraph can be presented as a sequence of columns $\pi = \pi_1\pi_2\cdots\pi_m$ such that the horizontal step of the jth column (from the left) lies on the horizontal line $y = \pi_j$. It is straightforward to see that the number of horizontal steps of the circular bargraph is m. Cakić, Mansour and Shabani [34], showed the following result.

Theorem 8.1. [34] *The generating function for the number of circular bargraph according to the number of cells, number of columns, and the perimeter is given by*

$$\frac{1+\frac{x^2}{(1-x)(1-q^2x)}\sum_{i\ge0}\frac{(q^2-1)^{i+1}q^{2(i+2)}y^{i+2}x^{\binom{i+2}{2}}}{(1-x^{i+2})\prod_{j=1}^{i}(1-q^2x^j)(1-x^j)}}{1-\sum_{i\ge0}\frac{q^{2(i+1)}(q^2-1)^iy^{i+1}x^{\binom{i+2}{2}}}{\prod_{j=1}^{i}(1-q^2x^j)\prod_{j=1}^{i+1}(1-x^j)}}.$$

Moreover, the total length of the perimeter over all circular bargraphs with n cells is given by $\frac{5}{18}\left((3n+1)2^n-(-1)^n\right)+\frac{1}{2}-n$.

- A *superdiagonal bargraph* is a bargraph such that the highest side of each column is placed above the diagonal $y = x$. Deutsch, Munarini and Rinaldi [44] showed that the generating function for the number of superdiagonal bargraphs according to the semi-perimeter is given by $C(x^2/(1-x))$, where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers. Moreover, it is shown that there exists a bijection between the superdiagonal bargraphs of semi-perimeter n and the Motzkin paths of length n with no level steps at even levels. In addition, it is shown that the number of all superdiagonal bargraphs with n cells and with k columns is $\binom{n-\binom{k}{2}-1}{k-1}$.

- A *symmetric bargraph* is a bargraph which is invariant under reflection along a vertical line. Deutsch and Elizalde [42] showed that the generating function for the number of symmetric bargraphs according to the number horizontal steps and up steps is given by

$$(1+x)\frac{\sqrt{(1-y^2)((1-x^2)^2-y^2(1+x^2)^2)}-1+x^2+y^2+2x^2y+x^2y^2}{2x(1-x^2-y-x^2y)}.$$

- A *weakly alternating bargraph* is a bargraph of the form $u^{i_1}h^{j_1}d^{k_1}h^{\ell_1}u^{i_2}h^{j_2}d^{k_2}h^{\ell_2}\cdots u^{i_m}h^{j_m}d^{k_m}$, where $m \ge 1$ and $i_r, j_t, k_r, \ell_t \ge 1$ for all r. A *strictly alternating bargraph* is a weakly alternating bargraph with no consecutive horizontal two steps. In [42] it is proved that the generating function for the weakly and strong alternating bargraphs according to number of horizontal steps and up steps is given by

$$\frac{(1-x)^2-(1-2x)y-\sqrt{((1-x)^2-y)((1-x)^2-y(1-2x)^2)}}{2x(1-x)}$$

and
$$\frac{1-y+x^2y-\sqrt{(1-y)^2-x^2y(2+2y-x^2y)}}{2x},$$
respectively. Moreover, it is shown that the generating function for the number of strong alternating bargraphs is related to counting Motzkin paths. In particular, it is shown that there is a bijection between strong alternating bargraphs and Motzkin paths that maps the statistic number of horizonal steps and up steps to the statistics number of up steps+number of down steps+1 and number of horizontal steps+2.

- A *t-bargraph* is a bargraph that besides cells of size 1×1, also includes *d-cells* (cells of size $1\times d$ with $2\leq d\leq t$) such that these cells touch the x-axis. Mansour and Shabani [81] (also, Mansour and Shabani [80] for $t=2$), found the generating function for the number of t-bargraphs according to the number of cells and the perimeter. Moreover, an explicit formula for the total length of the perimeter over all t-bargraphs with n-cells has been derived.

- A *bargraph and a topological index*. Recently, Elumalai and Mansour [48], connected bargraphs with a well known topological index - the general zeroth-order Randić index. Therein, they studied ${}^0R_\alpha(G)$ for the case when G is a graph corresponding to a bargraph and where ${}^0R_\alpha(G)$ is defined as ${}^0R_\alpha(G)=\sum_{u\in V(G)}deg(u)^\alpha$, where $deg(u)$ denotes the degree of the vertex u and α is a non-zero real number. It is shown that the expected value of the general zeroth-order Randić index over all bargraphs with n cells is asymptotic to $\frac{n}{3}(3^{\alpha+1}+2^\alpha+3\cdot 2^{2\alpha-1})$ as $n\to\infty$.

References

[1] S. Ahlgren, J. Lovejoy, The arithmetic of partitions into distinct parts, *Mathematika* **48** (2001) 191–202.
[2] N. Alon, Restricted integer partition functions, *Integers* **13** (2013) Art# A16.
[3] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, 1976; reprinted, Cambridge University Press, Cambridge, 1998.
[4] G. E. Andrews, Two theorems of Euler and a general partition theorem, *Proc. Amer. Math. Soc.* **20** (1969) 499–502.
[5] G. E. Andrews, K. Eriksson, *Integer Partitions*, Cambridge University Press, Cambridge, 2004.
[6] M. Archibald, A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, Shedding light on words, *Appl. Anal. Discrete Math.* **11** (2017) 216–231.
[7] M. Archibald, A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, Durfee squares in compositions, *Diskret. Mat.* **30**(3) (2018) 3–13.
[8] M. Archibald, A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, Two by two squares in set partitions, *Math. Slovaca*, To appear.
[9] M. Archibald, A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, Pushes in words - a primitive sorting algorithm, Preprint.
[10] A. Bacher, Average site perimeter of directed animals on the two-dimensional lattices, *Discrete Math.* **312** (2012) 1038–1058.
[11] G. Barequet, G. Rote, M. Shalah, $\lambda>4$: An improved lower bound on the growth constant of polyominoes, *Commun. ACM* **59**(7) (2016) 88–95.
[12] A. Blecher, C. Brennan, A. Knopfmacher, Walls in bargraphs, *Online J. Anal. Comb.* **12** (2017) Art# 6.
[13] A. Blecher, C. Brennan, A. Knopfmacher, The site-perimeter of permutations, *Util. Math.* **99** (2016) 619–635.
[14] A. Blecher, C. Brennan, A. Knopfmacher, Peaks in bargraphs, *Trans. Roy. Soc. South Afr.* **71** (2016) 97–103.
[15] A. Blecher, C. Brennan, A. Knopfmacher, Levels in bargraphs, *Ars Math. Contemp.* **9** (2015) 287–300.
[16] A. Blecher, C. Brennan, A. Knopfmacher, Combinatorial parameters in bargraphs, *Quaest. Math.* **39** (2016) 619–635.
[17] A. Blecher, C. Brennan, A. Knopfmacher, The water capacity of integer compositions, *Online J. Anal. Comb.* **13** (2018) Art# 6.
[18] A. Blecher, C. Brennan, A. Knopfmacher, Capacity of words, *J. Combin. Math. Combin. Comput.* **107** (2018) 249–258.
[19] A. Blecher, C. Brennan, A. Knopfmacher, The inner site-perimeter of compositions, *Quaest. Math.*, DOI: 10.2989/16073606.2018.1536088, In press.
[20] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, Counting corners in partitions, *Ramanujan J.* **39** (2016) 201–224.
[21] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, The perimeter of words, *Discrete Math.*, **340** (2017) 2456–2465.
[22] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, The site-perimeter of words, *Trans. Combin.* **6**(2) (2017) 37–48.
[23] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, The depth of compositions, *Math. Comput. Sci.*, DOI: 10.1007/s11786-019-00421-8, In press.
[24] A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, M. Shattuck, Pushes in permutations, *J. Combin. Math. Combin. Comput.*, To appear.
[25] A. Blecher, C. Brennan, A. Knopfmacher, H. Prodinger, The height and width of bargraphs, *Discrete Appl. Math.* **180** (2015) 36–44.
[26] A. Blecher, C. Brennan, A. Knopfmacher, M. Shattuck, Capacity of permutations, *Ann. Math. Inform.*, DOI: 10.33039/ami.2019.03.004, In press.
[27] A. Blecher, T. Mansour, Counting staircases in integer compositions, *Online J. Anal. Comb.* **11** (2016) Art# 7.
[28] M. Bona, *Combinatorics of Permutations*, (Discrete Mathematics and its Applications), Chapman & Hall/CRC, Boca Raton, 2012.
[29] M. Bousquet-Mélou, A method for the enumeration of various classes of column-convex polygons, *Discrete Math.* **154** (1996) 1–25.
[30] M. Bousquet-Mélou, A. Rechnitzer, The site-perimeter of bargraphs, *Adv. in Appl. Math.* **31** (2003) 86–112.
[31] M. Bousquet-Mélou, R. Brak, Exactly solved models, In: A. J. Guttmann (Ed.), *Polygons, Polyominoes and Polycubes*, Vol. 775, (Lecture Notes in Physics), Springer, Dordrecht, 2009, pp. 43–78.
[32] R. Brak, A. J. Guttmann, Exact solution of the staircase and row-convex polygon perimeter and area generating function, *J. Phys. A* **23** (1990) 4581–4588.
[33] N. Cakić, T. Mansour, A. Sh. Shabani, Counting 1x2 rectangles in set partitions, *J. Difference Equ. Appl.* **25** (2019) 708–715.
[34] N. Cakić, T. Mansour, A. Sh. Shabani, Circular Bargraphs, *Appl. Math. Comput.* **347** (2019) 803–807.
[35] E. R. Canfield, Engel's inequality for Bell numbers, *J. Combin. Theory Ser. A* **72** (1995) 184–187.
[36] A. Conway, Enumerating $2D$ percolation series by the finite-lattice method: theory, *J. Phys. A* **28** (1995) 335–349.
[37] A. R. Conway, M. Delest, A.J. Guttmann, On the number of three choice polygons, *Math. Comput. Modelling* **26** (1997) 51–58.
[38] M. P. Delest, Generating functions for column-convex polyominoes, *J. Combin. Theory Ser. A* **48** (1988) 12–31.
[39] M. P. Delest, S. Dulucq, Enumeration of directed column–convex animals with given perimeter and area, *Croat. Chem. Acta* **66** (1993) 59–80.
[40] M. P. Delest, D. Gouyou-Beauchamps, B. Vauquelin, Enumeration of parallelogram polyominoes with given bond and site perimeter, *Graphs Combin.* **3** (1987) 325–339.
[41] M. P. Delest, G. Viennot, Algebraic languages and polyominoes enumeration, *Theoret. Comput. Sci.* **34** (1984) 169–206.
[42] E. Deutsch, S. Elizalde, Statistics on bargraphs viewed as cornerless Motzkin paths, *Discrete Appl. Math.* **221** (2017) 54–66.
[43] E. Deutsch, S. Elizalde, A bijection between bargraphs and Dyck paths, *Discrete Appl. Math.* **251** (2018) 340–344.
[44] E. Deutsch, E. Munarini, S. Rinaldi, Skew Dyck paths, area, and superdiagonal bargraphs, *J. Stat. Plan. Inference*, **140** (2010) 1550–1562.
[45] D. Dhar, Equivalence of the two-dimensional directed-site animal problem to Baxter's hard-square latticegas model, *Phys. Rev. Lett.* **49** (1982) 959–962.
[46] E. Duchi, S. Rinaldi, An object grammar for column-convex polyominoes, *Ann. Comb.* **8** (2004) 27–36.
[47] H. E. Dudeney, *The Canterbury Puzzles and Other Curious Problems*, Dover, New York, 1958.
[48] S. Elumalai, T. Mansour, On the general zeroth-order Randić index of bargraphs, *Discrete Math. Lett.* **2** (2019) 6–9.
[49] S. Feretić, A perimeter enumeration of column-convex polyominoes, *Discrete Math. Theoret. Comput. Sci.* **9** (2007) 57–84.

[50] S. Feretić, A new way of counting the column-convex polyominoes by perimeter, *Discrete Math.* **180** (1998) 173–184.
[51] S. Feretić, The column-convex polyominoes perimeter generating function for everybody, *Croat. Chem. Acta* **69** (1996) 741–756.
[52] S. Feretić, D. Svrtan, On the number of column-convex polyominoes with given perimeter and number of columns, In: A. Barlotti, M. Delest, R. Pinzani (Eds.), *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics*, Firenze, 1993, pp. 201–214.
[53] G. Firro, T. Mansour, Three-letter-pattern-avoiding permutations and functional equations, *Electron. J. Combin.* **13** (2006) Art# R51.
[54] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.
[55] S. Flesia, D. S. Gaunt, C. E. Soteros, S. G. Whittington, Statistics of collapsing lattice animals, *J. Phys. A* **27** (1994) 5831–5846.
[56] M. Fulmek, Nonintersecting lattice paths on the cylinder, *Sém. Lothar. Combin.* **52** (2004) Art# B52b.
[57] M. Gardner, More about the shapes that can be made with complex dominoes (Mathematical Games), *Sci. Am.* **203**(5) (1960) 186–201.
[58] A. Geraschenko, An investigation of skyline polynomials, (2015), Preprint, Available at http://people.brandeis.edu/gessel/47a/geraschenko.pdf.
[59] J. W. L. Glaisher, A theorem in partitions, *Messenger Math.* **12** (1883) 158–170.
[60] S. W. Golomb, *Polyominoes: Puzzles, Patterns, Problems, and Packings*, 2nd ed., Princeton Univ. Press, Princeton, 1994.
[61] G. Grimmett, *Percolation*, Springer, New York, 1989.
[62] A. J. Guttmann, *Polygons, Polyominoes and Polycubes*, Springer, Berlin, 2009.
[63] S. Heubach, T. Mansour, *Combinatorics of Compositions and Words*, (Discrete Mathematics and its Applications), Chapman & Hall/CRC, Boca Raton, 2009.
[64] E. J. Janse van Rensburg, P. Rechnitzer, Exchange symmetries in Motzkin path and bargraph models of copolymer adsorption, *Electron. J. Combin.* **9** (2002) Art# R20.
[65] I. Jensen, *Series for lattice animals or polyominoes*, 2009, Available at https://web.archive.org/web/20070612141716/http://www.ms.unimelb.edu.au/~iwan/animals/Animals_ser.html
[66] I. Jensen, Enumerations of lattice animals and trees, *J. Stat. Phys.* **102** (2001) 865–881.
[67] I. Jensen, Counting polyominoes: a parallel implementation for cluster computing, In: P. M. A. Sloot, D. Abramson, A. V. Bogdanov, J. J. Dongarra, A. Y. Zomaya, Y. E. Gorbachev (Eds.), *Computational Science, ICCS 2003, Lecture Notes in Computer Science*, vol. 2659, Springer, Berlin, 2003.
[68] I. Jensen, A. J. Guttmann, Statistics of lattice animals (polyominoes) and polygons, *J. Phys. A* **33** (2000) 257–263.
[69] S. Kitaev, J. Liese, J. Remmel, B. Sagan, Rationality, irrationality, and Wilf equivalence in generalized factor order, *Electron. J. Combin.* **16** (2009) Art# R22.
[70] D. A. Klarner, R. L. Rivest, A procedure for improving the upper bound for the number of n-ominoes, *Canad. J. Math.* **25**:3 (1970) 585–602.
[71] K. Lin, Perimeter generating function for row-convex polygons on the rectangular lattice, *J. Phys. A* **23** (1990) 4703–4705.
[72] P. A. MacMahon, Memoir on the theory of the compositions of numbers, *Philos. Trans. Royal Soc. Lond. A* **184** (1893) 835–901.
[73] T. Mansour, *Combinatorics of Set Partitions*, (Discrete Mathematics and its Applications), Chapman & Hall/CRC, Boca Raton, 2012.
[74] T. Mansour, Interior vertices in set partitions, *Adv. in Appl. Math.* **101** (2018) 60–69.
[75] T. Mansour, The perimeter and the site-perimeter of set partitions, *Electron. J. Combin.* **26** (2019) Art# P2.30.
[76] T. Mansour, Border and tangent cells in bargraphs, *Discrete Math. Lett.* **1** (2019) 26–29.
[77] T. Mansour, M. Schork, *Commutation Relations, Normal Ordering, and Stirling Numbers*, (Discrete Mathematics and its Applications), Chapman & Hall/CRC, Boca Raton, 2015.
[78] T. Mansour, A. Sh. Shabani, Interior vertices and edges in bargraphs, *Notes Number Theory Discrete Math.* **25** (2019) 181–189.
[79] T. Mansour, A. Sh. Shabani, Bargraphs in bargraphs, *Turkish J. Math.* **42** (2018) 2763–2773.
[80] T. Mansour, A. Sh. Shabani, The perimeter of 2-compositions, *Quaest. Math.*, DOI: 10.2989/16073606.2019.1648324, In press.
[81] T. Mansour, A. Sh. Shabani, t-bargraphs and t-compositions, *Quaest. Math.*, DOI: 10.2989/16073606.2019.1691080, In press.
[82] T. Mansour, A. Sh. Shabani, M. Shattuck, Counting corners in compositions and set partitions presented as bargraphs, *J. Difference Equ. Appl.* **24** (2018) 849–1022.
[83] T. Mansour, M. Shattuck, Combinatorial parameters on bargraphs of permutations, *Trans. Comb.* **7** (2018) 1–16.
[84] T. Mansour, M. Shattuck, Bargraph statistics on words and set partitions, *J. Difference Equ. Appl.* **23** (2017) 1025–1046.
[85] T. Mansour, M. Shattuck, Counting water cells in bargraphs of compositions and set partitions, *Appl. Anal. Discrete Math.* **12** (2018) 413–438.
[86] T. Mansour, M. Schork, D. Yaqubi, Protected cells in bargraphs, *Australas. J. Combin.* **74** (2019) 169–178.
[87] G. E. Martin, *Polyominoes: A Guide to Puzzles and Problems in Tiling*, 2nd ed., Mathematical Association of America, Washington, 1996.
[88] A. Owczarek, Effect of stiffness on the pulling of an adsorbing polymer from a wall: an exact solution of a partially directed walk model, *J. Phys. A: Math. Theor.* **43** (2010) Art# 225002.
[89] A. Owczarek, Exact solution for semi-flexible partially directed walks at an adsorbing wall, *J. Stat. Mech.* (2009) Art# P11002.
[90] A. Owczarek, T. Prellberg, Exact solution of the discrete (1+1)-dimensional SOS model with field and surface interactions, *J. Stat. Phys.* **70** (1993) 1175–1194.
[91] J.-G. Penaud, Animaux dirigés diagonalement convexes et arbres ternaires, *Technical Report* 90-62, LaBRI, Universite Bordeaux 1, 1990.
[92] T. Prellberg, R. Brak, Critical exponents from nonlinear functional equations for partially directed cluster models, *J. Stat. Phys.* **78** (1995) 701–730.
[93] D. H. Redelmeier, Counting polyominoes: yet another attack, *Discrete Math.* **36** (1981) 191–203.
[94] J. B. Remmel, Bijective proofs of some classical partition identities, *J. Combin. Theory Ser. A* **33** (1982) 273–286.
[95] T. Rivlin, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, Wiley, New York, 1990.
[96] N. J. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org, 2010.
[97] D. Stauffer, A. Aharony, *An Introduction to Percolation Theory*, 2nd ed., Taylor and Francis, London, 1992.
[98] H. Temperley, Combinatorial problems suggested by the statistical mechanics of domains and of rubberlike molecules, *Phys. Rev.* **103** (1956) Art# 116.
[99] H. S. Wilf, *Generatingfunctionology*, 3rd ed., A. K. Peters/CRC Press, Wellesley, 2005.
[100] H. S. Wilf, Identically distributed pairs of partition statistics, *Sém Lothar. Combin.* **44** (2000) Art# B44c.
[101] H. S. Wilf, Three problems in combinatorial asymptotics, *J. Combin. Theory Ser. A* **35** (1983) 199–207.